Algebraic problems and exercises for high school

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Sets, sets operations ■ Relations, functions ■ Aspects of combinatorics

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Foreword

In this book, you will find algebraic exercises and problems, grouped by chapters, intended for higher grades in high schools or middle schools of general education. Its purpose is to facilitate training in mathematics for students in all high school categories, but can be equally helpful in a standalone work. The book can also be used as an extracurricular source, as the reader shall find enclosed important theorems and formulas, standard definitions and notions that are not always included in school textbooks.

Contents

Notations

$=$	equality;
$\neq$	inequality;
$\in$	belongs to;
$\notin$	doesn't belong to;
$\subseteq$	subset;
$\supseteq$	superset;
$\cup$	union;
$\cap$	intersection;
$\emptyset$	empty set;
$\vee$	(or) disjunction;
$\wedge$	(and) conjunction;
$p \overset{def}{\Longleftrightarrow} q$	p equivalent with q;
$\mathbb{N} = \{0, 1, 2, 3, 4, \dots\}$	natural numbers
$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$	integer numbers set;
$\mathbb{Q} = \{\frac{m}{n} \middle\vert\, m, n \in \mathbb{Z}, n \neq 0\}$	rational numbers set;
$\mathbb{R}$	real numbers set;
$\mathbb{C} = \{a + bi \vert a, b \in R, i^2 = -1\}$	complex numbers set;

$A_+ = \{x \in A \vert x > 0\}, A \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}\}$;

$A_- = \{y \in A \vert y < 0\},\ A \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}\}$;

$A^* = \{z \in A \vert z \neq 0\} = A \setminus \{0\},\ A \in \{\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}\}$;

$\lvert x \rvert$	absolute value of $x \in \mathbb{R}$;
$[x]$	integer part of x $\in$ R;
$\{x\}$	fractional part of $x \in \mathbb{R}$, $0 \leq \{x\} < 1$;
(a, b)	pair with the first element a and the second element b (also called "ordered pair");

(a, b, c)	triplet with corresponding elements a, b, c;
$A \times B = \{a, b \mid a \in A, b \in B\}$	Cartesian product of set A and set B;
$A \times B \times C = \{a, b, c \mid a \in A, b \in B, c \in C\}$	Cartesian product of sets A, B, C;
E	universal set;
$P(E) = \{X \mid X \subseteq E\}$	set of parts (subsets) of set E;
$A = B \overset{def}{\Leftrightarrow} (\forall)\, x \in E (x \in A \Leftrightarrow x \in B)$	equality of sets A and B;
$A \subseteq B \overset{def}{\Leftrightarrow} (\forall)\, x \in E (x \in A \Leftrightarrow x \in B)$	A is included in B;
$A \cup B = \{x \in E \mid x \in A \vee x \in B\}$	union of sets A and B;
$A \cap B = \{x \in E \mid x \in A \wedge x \in B\}$	intersection of sets A and B;
$A \setminus B = \{x \in E \mid x \in A \wedge x \notin B\}$	difference between sets A and B;
$A \Delta B = (A \setminus B) \cup (B \setminus A)$	symmetrical difference;
$\complement_{\varepsilon}(A) = \bar{A} = E \setminus A$	complement of set A relative to E;
$\alpha \subseteq A \times B$	relation α defined on sets A and B;
$f: A \to B$	function (application) defined on A with values in B;
$D(f)$	domain of definition of function f:
$E(f)$	domain of values of function f.

1. Sets. Operations with sets

1.1. Definitions and notations

It is difficult to give an account of the axiomatic theory of sets at an elementary level, which is why, intuitively, we shall define **a set** as a collection of objects, named **elements** or **points** of the set. A set is considered defined if its elements are given or if a property shared by all of its elements is given, a property that distinguishes them from the elements of another set. Henceforth, we shall assign capital letters to designate sets: $A, B, C, \dots, X, Y, Z$, and small letters for elements in sets: $a, b, c, \dots, x, y, z$ etc.

If a is an element of the set A, we will write $a \in A$ and we will read "a belongs to A" or "a is an element of A". To express that a is not an element of the set A, we will write $a \notin A$ and we will read "a does not belong to A".

Among sets, we allow the existence of a particular set, noted as $\emptyset$, called the **empty set** and containing no element.

The set that contains a sole element will be noted with $\{a\}$. More generally, the set that doesn't contain other elements except the elements $a_1, a_2, \dots, a_n$ will be noted by $\{a_1, a_2, \dots, a_n\}$.

If A is a set, and all of its elements have the quality P, then we will write $A = \{x | x \text{ verifies } P\}$ or $A = \{x | P(x)\}$ and we will read: "A consists of only those elements that display the property P (for which the predicate $P(x)$ is true)."

We shall use the following notations:

$\mathbb{N} = \{0, 1, 2, 3, \dots\}$ – the natural numbers set;

$\mathbb{N}^* = \{0, 1, 2, 3, \dots\}$ – the natural numbers set without zero;

$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ – the integer numbers set;

$\mathbb{Z} = \{\pm 1, \pm 2, \pm 3 \dots\}$ – the integer numbers set without zero;

$\mathbb{Q} = \{\frac{m}{n} \big| m, n \in \mathbb{Z}, n \in \mathbb{N}^*\}$ – the rational numbers set;

$\mathbb{Q}^*$ = the rational numbers set without zero;

$\mathbb{R}$ = the real numbers set;

$\mathbb{R}^*$ = the real numbers set without zero;

$\mathbb{R}_+ = \{x \in \mathbb{R} | x \geq 0\}$; $\mathbb{R}_+^* = \{x \in \mathbb{R} | x > 0\}$;

$\mathbb{C} = \{a + bi | a, b \in R, i^2 = -1\}$ = the complex numbers set;

$\mathbb{C}^*$ = the complex numbers set without zero;

$m \in \{1, 2, \dots, n\} \Leftrightarrow m = \overline{1, n}$;

$D(a) = \{c \in \mathbb{Z}^* | a \vdots c\}$ = the set of all integer divisors of number $a \in \mathbb{Z}$;

$n(A) = |A|$ = the number of the elements of finite set A.

Note. We will consider that the reader is accustomed to the symbols of Logic: *conjunction* ∧ (…and…), *disjunction* ∨ (…or…), *implication* ⟹, *existential quantification* (∃) and *universal quantification* (∀).

Let A and B be two sets. If all the elements of the set A are also elements of the set B, we then say that **A is included in B**, or that **A is a part of B**, or that **A is a subset of the set B** and we write $A \subseteq B$. So

$$A \subseteq B \Leftrightarrow (\forall)x(x \in A \Longrightarrow x \in B).$$

The properties of inclusion

a. $(\forall)\ A, A \subseteq A$ (*reflexivity*);

b. $(A \subseteq B \wedge B \subseteq C) \Longrightarrow A \subseteq C$ (*transitivity*);

c. $(\forall)A, \emptyset \subseteq A$.

If A is not part of the set B, then we write $A \nsubseteq B$, that is, $A \nsubseteq B \Leftrightarrow (\exists)x(x \in A \wedge x \notin B)$.

We will say that the **set A is equal to the set B**, in short $A = B$, if they have exactly the same elements, that is

$$A = B \ \Leftrightarrow (A \subseteq B \wedge B \subseteq A).$$

The properties of equality

Irrespective of what the sets A, B and C may be, we have:

a. $A = A$ (*reflexivity*);

b. $(A = B) \Longrightarrow (B = A)$ (*symmetry*);

c. $(A = B \wedge B = C) \Longrightarrow (A = C)$ (*transitivity*).

With $P(A)$ we will note **the set of all parts of set A**, in short

$$X \in P(A) \Leftrightarrow X \subseteq A.$$

Obviously, $\emptyset, A \in P(A)$.

The **universal set**, the set that contains all the sets examined further, which has the containing elements' nature one and the same, will be denoted by E.

Operations with sets

Let A and B be two sets, $A, B \in P(E)$.

1. **Intersection.**

$$A \cap B = \{x \in E | x \in A \wedge x \in B\},$$

i.e.

$$x \in A \cap B \Leftrightarrow (x \in A \wedge x \in B), \qquad (1)$$

$$x \notin A \cap B \Leftrightarrow (x \notin A \wedge x \notin B), \qquad (1')$$

2. **Union**.

$$A \cup B = \{x \in E | x \in A \vee x \in B\},$$

i.e.

$$x \in A \cup B \Leftrightarrow (x \in A \vee x \in B), \qquad (2)$$

$$x \notin A \cup B \Leftrightarrow (x \notin A \vee x \notin B), \qquad (2')$$

3. **Difference.**

$$A \setminus B = \{x \in E | x \in A \wedge x \notin B\},$$

i.e.

$$x \in A \setminus B \Leftrightarrow (x \in A \wedge x \notin B), \qquad (3)$$

$$x \notin A \setminus B \Leftrightarrow (x \notin A \vee x \in B), \qquad (3')$$

4. **The complement** of a set. Let $A \in P(E)$. The difference $E \setminus A$ is a subset of E, denoted $C_E(A)$ and called "the complement of A relative to E", that is

$$C_E(A) = E \setminus A = \{x \in E | x \notin A\}.$$

In other words,

$$x \in C_E(A) \Leftrightarrow x \notin A, \qquad (4)$$

$$x \notin C_E(A) \Leftrightarrow x \in A. \qquad (4')$$

Properties of operations with sets

$A \cap A = A, A \cup A = A$ (**idempotent laws**)

$A \cap B = B \cap A, A \cup B = B \cup A$ (**commutative laws**)

$(A \cap B) \cap C = A \cap (B \cap C)$;

$(A \cup B) \cup C = A \cup (B \cup C)$ (**associativity laws**)

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$;

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ (**distributive laws**)

$A \cup (A \cap B) = A$;

$A \cap (A \cup B) = A$ (**absorption laws**)

$C_E(A \cup B) = C_E(A) \cap C_E(B)$;

$C_E(A \cap B) = C_E(A) \cup C_E(B)$ (**Morgan's laws**)

Two "privileged" sets of E are $\emptyset$ and E.

For any $A \in P(E)$, we have:

$$\emptyset \subseteq A \subseteq E,$$

$$A \cup \emptyset = A, \qquad A \cap \emptyset = \emptyset, \qquad C_E(\emptyset) = E,$$

$$A \cup E = E, \qquad A \cap E = A, \qquad C_E(E) = \emptyset,$$

$$A \cup C_E(A) = E, \qquad A \cap C_E(A) = \emptyset,$$

$$C_E\left(C_E(A)\right) = A \text{ (\textbf{principle of reciprocity})}.$$

Subsequently, we will use the notation $C_E(A) = \bar{A}$.

5. **Symmetric difference.**

$$A\Delta B = (A \setminus B) \cup (B \setminus A).$$

Properties. Irrespective of what the sets A, B and C are, we have:

a. $A\Delta A = \emptyset$;

b. $A\Delta B = B\Delta A$ (**commutativity**);

c. $A\Delta\emptyset = \emptyset\Delta A = A$;

d. $A\Delta\,(A\Delta B) = B$;

e. $(A\Delta B)\Delta C = A\Delta(B\Delta C)$ (**associativity**);

f. $A \cap (B\Delta C) = (A \cap B)\Delta(A \cap C)$;

g. $A\Delta B = (A \cup B) \setminus (A \cap B)$.

6. **Cartesian product**. Let x and y be two objects. The set $\{\{x\},\{x,y\}\}$, whose elements are the sets $\{x\}$ and $\{x,y\}$, is called an **ordered pair** (or an **ordered couple**) with the first component x and the second component y and is denoted as (x,y). Having three objects x,y and z, we write $(x,y,z)=((x,y),z)$ and we name it an **ordered triplet**.

Generally, having n objects $x_1,x_2,\dots,x_n$ we denote

$$(x_1,x_2,\dots,x_n)=\left(\dots\left((x_1,x_2)x_3\right)\dots x_n\right)$$

and we name it an **ordered system** of n elements (or a **cortege** of length n).

We have

$$(x_1,x_2,\dots,x_n)=(y_1,y_2,\dots,y_n)$$
$$\Leftrightarrow (x_1=y_1\wedge x_2=y_2\wedge\dots\wedge x_n=y_n).$$

Let $A,B\in P(E)$. The set

$$A\times B=\{(a,b)|a\in A\wedge b\in B\}$$

is called a **Cartesian product** of sets A and B. Obviously, we can define

$$A\times B\times C=\{(x,y,z)|x\in A\wedge y\in B\wedge z\in C\}.$$

More generally, the Cartesian product of the sets $A_1,A_2,\dots,A_n$

$$A_1\times A_2\times\dots\times A_n=\{(x_1,x_2,\dots,x_n)|x_i\in A_i, i=\overline{1,n}\}.$$

For $A=B=C=A_1=A_2=\ldots=A_n$, we have

$$A\times A\stackrel{\text{def}}{=}A^2, A\times A\times A\stackrel{\text{def}}{=}A^3, \underbrace{A\times A\times\dots\times A}_{n\text{ ori}}\stackrel{\text{def}}{=}A^n.$$

For example, $\mathbb{R}^3=\{(x,y,z)|x,y,z\in\mathbb{R}\}$. The subset

$$\Delta=\{(a,a)|a\in A\}\subseteq A^2$$

is called the **diagonal** of set A^2.

Examples.

1. Let $A=\{1,2\}$ and $B=\{1,2,3\}$. Then

$$A\times B=\{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3)\}$$

and

$$B\times A=\{(1,1),(1,2),(2,1),(2,2),(3,1),(3,2)\}.$$

We notice that $A\times B\neq B\times A$.

2. The Cartesian product $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ can be geometrically represented as the set of all the points from a plane to wich a rectangular systems of coordinates xOy has been assigned, associating to each element $(x, y) \in \mathbb{R}^2$ a point $P(x, y)$ from the plane of the abscissa x and the ordinate y.

Let $A = [2; 3]$ and $B = [1; 5], (A, B \subseteq \mathbb{R})$. Then $A \times B$ can be geometrically represented as the hatched rectangle $KLMN$ (see *Figure 1.1*), where $K(2,1), L(2,5), M(3,5), N(3,1)$.

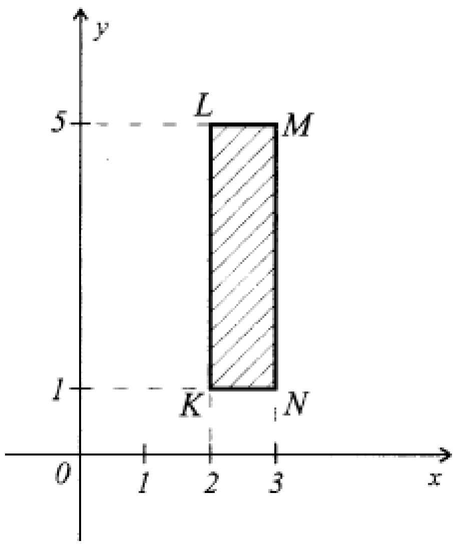


Figure 1.1.

The following properties are easily verifiable:

a. $(A \subseteq C \wedge B \subseteq D) \Longrightarrow A \times B \subseteq C \times D$;

b. $A \times (B \cup C) = (A \times B) \cup (A \times C)$,

$A \times (B \cap C) = (A \times B) \cap (A \times C)$;

c. $A \times B = \emptyset \Leftrightarrow (A = \emptyset \vee B = \emptyset)$,

$A \times B \neq \emptyset \Leftrightarrow (A \neq \emptyset \wedge B \neq \emptyset)$.

7. The intersection and the union of a family of sets. A family of sets is a set $\{A_i | i \in I\} = \{A_i\}_{i \in I}$ whose elements are the sets $A_i, i \in I, A_i \in P(E)$. We say that $A_i, i \in I, A_i \in P(E)$ is a family of sets indexed with the set I.

Let there be a family of sets $\{A_i | i \in I\}$. Its union (or the union of the sets $A_i, i \in I$) is the set

$$\bigcup_{i \in I} A_i = \{x \in E | (\exists) i \in I: x \in A_i\}.$$

The intersection of the given family (or the intersection of the sets A_i, $i \in I$) is the set

$$\bigcap_{i \in I} A_i = \{x \in E | x \in A_i, (\forall) i \in I\}.$$

In the case of $I = \{1,2, \dots, n\}$, we write

$$\bigcup_{i \in I} A_i = A_1 \cup A_2 \cup \ldots \cup A_n = \bigcup_{i=1}^{n} A_i,$$

$$\bigcap_{i \in I} A_i = A_1 \cap A_2 \cap \ldots \cap A_n = \bigcap_{i=1}^{n} A_i.$$

8. **Euler-Wenn Diagrams.** We call **Euler Diagrams** (in USA – Wenn's diagrams) the figures that are used to interpret sets (circles, squares, rectangles etc.) and visually illustrate some properties of operations with sets. We will use the Euler circles.

Example. Using the Euler diagrams, prove Morgan's law.

$$C_E(A \cap B) = C_E(A) \cup C_E(B).$$

Solution.

In fig. 1.2.a, the hatched part is $A \cap B$; the uncrossed one (except $A \cap B$) represents $C_E(A \cap B)$.

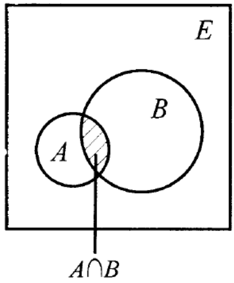


Figure 2.2.a

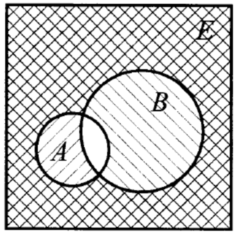


Figure 3.2.b

In fig. 1.2b the side of the square hatched with " \\\\ " is equal to $C_E(A)$ and the one hatched with " //// " is equal to $C_E(B)$. All the hatched side forms $C_E(A) \cup C_E(B)$ (the uncrossed side is exactly $A \cap B$).

From these two figures it can be seen that $C_E(A \cap B)$ (the uncrossed side of the square in fig. 1.2.a coincides with $C_E(A) \cup C_E(B)$ (the hatched side of fig. 1.2.b), meaning

$$C_E(A \cap B) = C_E(A) \cup C_E(B).$$

1.2. Solved exercises

1. For any two sets A and B, we have

$$A \cap B = A \setminus (A \setminus B)$$

Solution. Using the definitions from the operations with sets, we gradually obtain:

$$x \in A \setminus (A \setminus B) \overset{(3)}{\Leftrightarrow} (x \in A \wedge x \notin (A \setminus B)) \Leftrightarrow$$

$$\overset{(3')}{\Leftrightarrow} (x \in A \wedge (x \notin A \vee x \in B)) \Leftrightarrow$$

$$\Leftrightarrow ((x \in A \wedge x \notin A) \vee (x \in A \wedge x \in B)) \Leftrightarrow$$

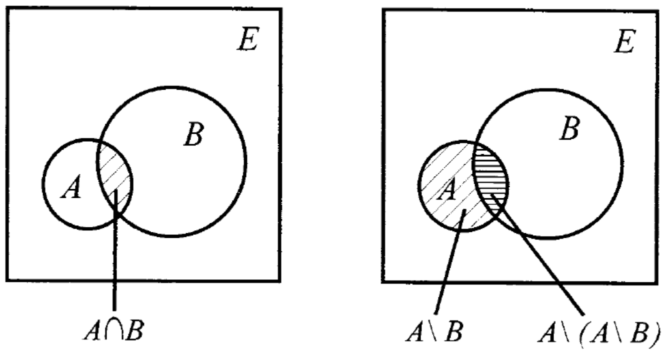
$$\Leftrightarrow (x \in A \wedge x \in B) \overset{(1)}{\Leftrightarrow} x \in A \cap B.$$

From this succession of equivalences it follows that:

$$A \setminus (A \setminus B) \subseteq A \cap B \text{ şi } A \cap B \subseteq A \setminus (A \setminus B),$$

which proves the required equality.

Remark. The equality can also be proven using Euler's diagrams.



So $A \cap B = A \backslash (A \backslash B)$.

2. Whatever $A, B \subseteq E$ are, the following equality takes place:
$(A \cap \overline{B}) \cup (\overline{A} \cap B) = (A \cup B) \cap (\overline{A \cap B})$.

Solution. **The analytical method**. Using the definitions from the operations with sets, we obtain:

$$x \in (A \cap \overline{B}) \cup (\overline{A} \cap B) \overset{(2)}{\Leftrightarrow} (x \in (A \cap \overline{B}) \vee x \in (\overline{A} \cap B)) \Leftrightarrow$$

$$\overset{(1)}{\Leftrightarrow} ((x \in A \wedge x \in \overline{B}) \vee (x \in \overline{A} \wedge x \in B)) \Leftrightarrow$$

$$\Leftrightarrow ((x \in A \vee x \in \overline{A}) \wedge (x \in A \vee x \in B) \wedge$$

$$\wedge (x \in \overline{B} \vee x \in \overline{A}) \wedge (x \in \overline{B} \vee x \in B)) \overset{(2),(4')}{\Longleftrightarrow}$$

$$\Leftrightarrow (x \in (A \cup B) \wedge (x \notin A \vee x \notin B)) \overset{(1')}{\Leftrightarrow}$$

$$\overset{(1')}{\Leftrightarrow} (x \in (A \cup B) \wedge x \notin (A \cap B)) \Leftrightarrow$$

$$\overset{(4)}{\Leftrightarrow} (x \in (A \cup B) \wedge x \in (\overline{A \cap B})) \overset{(1)}{\Leftrightarrow} x \in (A \cup B) \cap (\overline{A \cap B}).$$

This succession of equivalences proves that the equality from the enunciation is true.

The graphical method. Using Euler's circles, we have

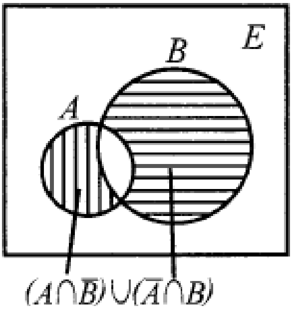


Figure 4.3.a

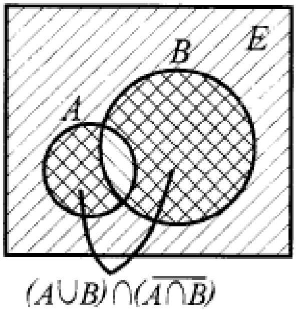


Figure 5.3.b

In fig. 1.3.a, we have $(A \cap \overline{B}) \cup (\overline{A} \cap B)$, which represents the hatched side of the square. From fig. 1.3.b it can be seen that $(A \cap \overline{B}) \cup (\overline{A} \cap B) = (A \cup B) \cap (\overline{A \cap B})$.

3. For every two sets $A, B \subseteq E$, the equivalence stands true:

$A \setminus B = B \setminus A \Leftrightarrow A = B.$

Solution. Let $A\backslash B = B\backslash A$. We assume that $A \neq B$. Then there is $a \in A$ with $a \notin B$ or $b \in B$ with $b \notin A$.

In the first case we obtain $a \in A\backslash B$ and $a \notin B\backslash A$, which contradicts the equality $A\backslash B = B\backslash A$. In the second case, we obtain the same contradiction.

So, if $A\backslash B = B\backslash A \Longrightarrow A = B$.

Reciprocally, obviously.

4. Sets $A = \{1,2,3,4,5,6,7,8,9,10\}, B = \{2,4,6,8,10\}$ and $C = \{3,6,9\}$ are given. Verify the following equalities:

a) $A\backslash(B \cup C) = (A\backslash B) \cap (A\backslash C)$;

b) $A\backslash(B \cap C) = (A\backslash B) \cup (A\backslash C)$.

Solution. a) We have $B \cup C = \{2,3,4,6,8,9,10\}$, $A\backslash(B \cup C) = \{1,5,7\}$, $A\backslash B = \{1,3,5,7,9\}$, $A\backslash C = \{1,2,4,5,7,8,10\}$, $(A\backslash B) \cap (A\backslash C) = \{1,5,7\} = A\backslash(B \cup C)$.

b) For the second equality, we have

$B \cap C = \{6\}$, $A\backslash(B \cap C) = \{1,2,3,4,5,6,7,8,9,10\}$, $(A\backslash B) \cup (A\backslash C) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} = A\backslash(B \cap C)$.

5. Determine the sets *A* and *B* that simultaneously satisfy the following the conditions:

1) $A \cup B = \{1,2,3,4,5\}$;

2) $A \cap B = \{3,4,5\}$;

3)$1 \notin A\backslash B$;

4) $2 \notin B\backslash A$.

Solution. From 1) and 2) it follows that $\{3,4,5\} \subseteq A \subseteq A \cup B$ and $\{3,4,5\} \subseteq B \subseteq A \cup B$. From 3) it follows that $1 \notin A$ or $1 \in B$. If $1 \notin A$, then from $A \cup B = \{1,2,3,4,5\}$ it follows that $1 \in B$. But, if $1 \in B$, then $1 \notin A$, because, on the contrary it would follow that $1 \in A \cap B = \{3,4,5\}$. So $1 \in B$ and $1 \notin A$ remain. Similarly, from 4) it follows that $2 \notin B$ and so $2 \in A$. In other words, $\{3,4,5\} \subseteq A \subseteq \{2,3,4,5\}$ and $\{3,4,5\} \subseteq B \subseteq \{1,3,4,5\}$ with $2 \in A \cup B$, $1 \in A \cup B$ and that is why $A = \{2,3,4,5\}$, and $B = \{1,3,4,5\}$.

Answer: $A = \{2,3,4,5\}$, $B = \{1,3,4,5\}$.

6. The sets $A = \{11k + 8 \mid k \in \mathbb{Z}\}$, $B = \{4m \mid m \in \mathbb{Z}\}$ and $C = \{11(4n + 1) - 3 \mid n \in \mathbb{Z}\}$, are given. Prove that $A \cap B = C$.

Solution. To obtain the required equality, we will prove the truth of the equivalence:

$x \in A \cap B \Leftrightarrow x \in C$.

Let $x \in A \cap B$. Then $x \in A$ and $x \in B$ and that is why two integer numbers $k, m \in \mathbb{Z}$, exist, so that $x = 11k + 8 = 4m \Leftrightarrow 11k = 4(m - 2)$. In this equality, the right member is divisible by 4, and 11, 4 are prime. So, from $11k \vdots 4$ it follows that $k \vdots 4$, giving $k = 4t$ for one $t \in \mathbb{Z}$. Then

$$x = 11k + 8 = 11 \cdot 4t + 8 = 11 \cdot 4t + 11 - 3 = 11(4t + 1) - 3$$

which implies $x \in C$, in other words, we have proven the implication

$$x \in A \cap B \Rightarrow e \in C. \quad (1)$$

Similarly, let $y \in C$. Then $s \in \mathbb{Z}$ exists with

$$y = 11(4s+1) - 3 = 11 \cdot 4s + 11 - 3 = 11 \cdot 4s + 8 = 4(11s+2)$$

Taking $4s = u \in \mathbb{Z}$ and $11s + 2 = v \in \mathbb{Z}$, we obtain

$$y = 11u + 8 = 4v \in A \cap B,$$

which proves the truth of the implication

$$y \in C \Rightarrow y \in A \cap B. \quad (2)$$

From (1) and (2) the required equality follows.

7. The following sets are given

$$A = \left\{ x \in I\!R \;\middle|\; \begin{cases} 2x \le 4x - 6 \\ 4x - 11 < 2x + 1 \end{cases} \right\}$$

and $B = A \cap \mathbb{N}$.

Prove that:

a) all X sets with $B \cup X = \{3,4,5,6,7,8,9\}$;

b) all $Y = \{y \in \mathbb{Z} \mid y^2 \in B \cup X\}$, so that $B \cap Y = \{3\}$.

Solution. We determine the set A:

$$\begin{cases} 2x \le 4x - 6, \\ 4x - 11 < 2x + 1 \end{cases} \Leftrightarrow \begin{cases} 2x \ge 6, \\ 2x < 12 \end{cases} \Leftrightarrow \begin{cases} x \ge 3, \\ x < 6 \end{cases} \Leftrightarrow x \in [3;6).$$

Then $B = (3;6) \cap IN = \{3,4,5\}$.

a) All possible subsets of B are

$$\emptyset, \{3\}\{4\}\{5\}\{3,4\}\{3,5\}\{4,5\}\{3,4,5\} = B$$

The required X sets are such that $X \cup B = \{3,4,5,6,7,8,9\}$ and will thus be like $X = C \cup \{6,7,8,9\}$, where $C \in P(B)$, namely, the sets required at point a) are:

$X_1\{6,7,8,9\}$, $X_2\{3,6,7,8,9\}$, $X_3\{4,6,7,8,9\}$, $X_4\{5,6,7,8,9\}$, $X_5\{3,4,6,7,8,9\}$, $X_6\{3,5,6,7,8,9\}$, $X_7\{4,5,6,7,8,9\}$, $X_8\{3,4,5,6,7,8,9\}$..

b) Because $y \in \mathbb{Z}$, then $y^2 \in I\!N$ and vice versa. Considering that $y^2 \in B \cup X = \{3,4,5,6,7,8,9\}$, we obtain $y^2 \in \{4,9\}$, namely $y^2 \in \{-3,-2,2,3\} = M$. The parts of the set M are:

$\emptyset$, $\{-3\}$, $\{-2\}$, $\{2\}$, $\{3\}$, $\{-3,-2\}$, $\{-3,2\}$, $\{-3,3\}$, $\{-2,2\}$, $\{-2,3\}$, $\{2,3\}$, $\{-3,-2,2\}$, $\{-3,-2,3\}$, $\{-3,2,3\}$, $\{-2,2,3\}$, M.

From the condition $B \cap Y = \{3\}$ it follows that Y is one of the sets

$Y_1 = \{3\}$, $Y_2 = \{-3,3\}$, $Y_3 = \{-2,3\}$, $Y_4 = \{2,3\}$, $Y_5 = \{-3,-2,3\}$, $Y_6 = \{-3,2,3\}$, $Y_7 = \{-2,2,3\}$, $Y_8 = M = \{-3,-2,2,3\}$.

Answer: a) $X \in \{X^1, X^2, X^3, X^4, X^5, X^6, X^7, X^8\}$;

b) $Y \in \{Y^1, Y^2, Y^3, Y^4, Y^5, Y^6, Y^7, Y^8\}$.

8. Determine $A, B, C \subseteq T$ and $A \,\Delta\, B$, if

$T = \{1,2,3,4,5,6\}$, $A \,\Delta\, C = \{1,2\}$, $B \,\Delta\, C = \{5,6\}$, $A \cap C = B \cap C = \{3,4\}$.

Solution. From $A \cap C = B \cap C = \{3,4\}$, it follows that $\{3,4\} \subseteq A \cap B \cap C$.

We know that

$A \,\Delta\, C = (A \setminus C) \cup (C \setminus A) = (A \cup C) \setminus (A \cap C)$,

$A \,\Delta\, C = (B \setminus C) \cup (C \setminus B) = (B \cup C) \setminus (B \cap C)$.

Then:

$1 \in A \,\Delta\, C \Leftrightarrow (1 \in A \cup C \wedge 1 \notin A \cap C) \Leftrightarrow \big((1 \in A \vee 1 \in C) \wedge 1 \notin A \cap C\big)$.

The following cases are possible:

a) $1 \notin A$ and$1 \in C$;

b) $1 \in A$ and $1 \notin C$

(case no. 3, $1 \in A$ and $1 \in C \Rightarrow 1 \in A \cap C = \{3,4\}$, is impossible).

In the first case $1 \notin A$ and $1 \in C$, from $B \,\Delta\, C = \{5,6\}$ it follows that $1 \in B$, because otherwise $1 \notin B$ and $1 \in C \Rightarrow 1 \in C \setminus B \subseteq B \Delta\, C = \{5,6\}$. So, in this case we have $1 \in B \cap C = \{3,4\}$which is impossible, so it follows that $1 \in A$, $1 \notin C$. Similarly, we obtain $2 \in A$ and $2 \notin B$ and $2 \notin C$, $5 \in B$ and $5 \notin A$, $5 \notin C$ and $6 \in B$, $6 \notin A$, $6 \notin A$.

In other words, we have obtained:

$A = \{1,2,3,4\}$, $B = \{3,4,5,6\}$, $C = \{3,4\}$ and $A \Delta B = \{1,2,5,6\}$.

Answer: $A = \{1,2,3,4\}$, $B = \{3,4,5,6\}$, $C = \{3,4\}$ and $A \Delta B = \{1,2,5,6\}$.

9. The sets $A = \{1,2\}$, $B = \{2,3\}$ are given. Determine the following sets:

a) $A \times B$; b) $B \times A$; c) A^2;
d) B^2; e) $(A \times B) \cap (B \times A)$; f) $(A \cup B) \times B$;
g) $(A \times B) \cup (B \times B)$.

Solution. Using the definition for the Cartesian product with two sets, we obtain:

a) $A \times B = \{(1,2),(1,3),(2,2),(2,3)\}$;
b) $B \times A = \{(2,1),(2,2),(3,1),(3,2)\}$;
c) $A^2 = \{(1,1),(1,2),(2,1),(2,2)\}$;
d) $B^2 = \{(2,2),(2,3),(3,2),(3,3)\}$;
e) $(A \times B) \cap (B \times A) = \{(2,2)\}$;
f)$A \cup B = \{1,2,3\}$; $(A \cup B) \times B = \{(1,2),(1,3),(2,2)$ $(2,3),(3,2),(3,3)\}$;
g) $(A \times B) \cup (B \times B) = \{(1,2),(1,3),(2,2),(2,3),$ $(3,2),(3,3)\} = (A \cup B) \times B$.

10. The sets $A = \{1,2,x\}$, $B = \{3,4,y\}$ are given. Determine x and y, knowing that $\{1,3\} \times \{2,4\} \subseteq A \times B$.

Solution. We form the sets $A \times B$ and $\{1,3\} \times \{2,4\}$:

$A \times B = \{(1,3),(1,4),(1,y),(2,3),(2,4),(2,y),(x,3),(x,4),(x,y)\}$;

$C = \{1,3\} \times \{2,4\} = \{(1,2),(1,4),(3,2),(3,4)\}$

Because $\{1,3\} \times \{2,4\} \subseteq A \times B$, we obtain

$(1,2) \in C \Rightarrow (1,2) \in A \times B \Rightarrow (1,y) = (1,2) \Rightarrow y = 2;$
$(3,4) \in C \Rightarrow (3,4) \in A \times B \Rightarrow (3,4) = (x,4) \Rightarrow x = 3.$

For $x = 3$ and $y = 2$, we have $(3,2) \in A \times B$.

Answer: $x = 3$, $y = 2$.

11. If $A \supseteq B$, then $A \times B = ((A \setminus B) \times B) \cup B^2$.
Prove.
Solution. $B \supseteq A \Rightarrow (A \setminus B) \cup B = A$ and that is why
$A \times B = ((A \setminus B) \cup B) \times B = ((A \setminus B) \times B) \cup (B \times B)$
$= ((A \setminus B) \times B) \cup B^2$
(we have used the equality $(A \cup B) \times C = (A \times C) \cup (B \times C)$).

12. How many elements does the following set have:

$$A = \left\{ x \in \mathbf{Q} \middle| x = \frac{n^2+1}{2n^2+n+1},\ n = \overline{1, 1000} \right\}?$$

Solution. The set A has as many elements as the fraction $(n^2+1)\ /\ (2n^2+n+1)$ has different values, when n takes the values $1,2,3, \ldots, 1000$. We choose the values of n for which the fraction takes equal values.

Let $m, l \in \mathbb{N}^*$, $m < 1$ with

$$\frac{m^2+1}{2m^2+m+1} = \frac{l^2+1}{2l^2+l+1}.$$

Then

$$(m^2+1)(2l^2+l+1) = (l^2+1)(2m^2+m+1) \Leftrightarrow$$

$$\Leftrightarrow (m-l)(m+l-ml+1) = 0 \overset{m \neq l}{\Leftrightarrow} m + l - ml + 1 =$$

$$= 0 \Leftrightarrow m(l-1) = l + 1 \Leftrightarrow m = \frac{l+1}{l-1} \Leftrightarrow$$

$$\Leftrightarrow m = \frac{(l-1)+2}{l-1} \Leftrightarrow m = 1 + \frac{2}{l-1}.$$

But $m \in \mathbb{N}^*$ and consequently

$$m \in \mathbb{N}^* \Leftrightarrow \frac{2}{l-1} \in \mathbb{N}^* \Leftrightarrow 2 \vdots (l-1) \Leftrightarrow \left[\begin{array}{l} l-1=1, \\ l-1=2 \end{array} \right. \Leftrightarrow \left[\begin{array}{l} l=2, \\ l=3. \end{array} \right.$$

For $l = 2$, we obtain $m = 3$, and for $l = 3$, we have $m = 2$. Considering that $m < l$, we obtain $m = 2$ and $l = 3$. So, only for $n = 2$ and $n = 3$, the same element of the set A: $x = 5\ /\ 11$ is obtained.

Answer: the set A has 999 elements, namely $n(A) = 999$.

13. Determine the integer numbers x, y that make the following statement true $(x-1)\cdot(y-3) = 13$.

Solution. We write

$$A = \{(x,y) \in \mathbf{Z}^2 | P(x) \colon (x-1)\cdot(y-3) = 13\}$$

As 13 is a prime number, and $x, y \in \mathbb{Z}$, the following cases are possible:

$$\begin{cases} x-1=1, \\ y-3=13, \end{cases} \begin{cases} x-1=-1, \\ y-3=-13, \end{cases} \begin{cases} x-1=13, \\ y-3=1, \end{cases} \begin{cases} x-1=-13, \\ y-3=-1, \end{cases}$$

meaning the proposition $P(x)$ is true only in these situations:

$$\begin{cases} x=2, \\ y=16, \end{cases} \begin{cases} x=0, \\ y=-10, \end{cases} \begin{cases} x=14, \\ y=4, \end{cases} \begin{cases} x=-12, \\ y=2. \end{cases}$$

14. Determine the set

$$A = \{x \in I\!R | \sqrt{a+x} + \sqrt{b+x} + \sqrt{c+x} = 0, \quad a,b,c \in I\!R\}.$$

Solution. Because:

$$\sqrt{a+x} \geq 0, \sqrt{b+x} \geq 0, \sqrt{c+x} \geq 0,$$

it follows that that equality

$$\sqrt{a+x} + \sqrt{b+x} + \sqrt{c+x} = 0$$

is possible, if and only if we simultaneously have:

$$a+x = b+x = c+x = 0 \Leftrightarrow x = -a = -b = -c.$$

Then:

a) if at least two of the numbers a, b and c are different, we cannot have the equality:

$$\sqrt{a+x} + \sqrt{b+x} + \sqrt{c+x} = 0,$$

namely in this case $A = \emptyset$;

b) if $a = b = c$, then $x = -a$ and $A = \{-a\}$.

Answer: 1) for $a = b = c$, we have $A = \{-2\}$; 2) if at least two of the numbers a, b and c are different, then $A = \emptyset$.

15. Determine the set

$A = \{x \in \mathbf{Z} | \min(x + 2, 4 - x/3) \geq 1\}.$

Solution. Possible situations:

$x + 2 \leq 4 - x/3$ or $x + 2 > 4 - x/3$.

We examine each case:

1) $x + 2 \leq 4 - x/3 \Leftrightarrow 3x + 6 \leq 12 - x \Leftrightarrow 4x \leq 6 \Leftrightarrow x \leq 3/2.$

In this case we have:

$\min(x + 2, 4 - x/3) \geq 1 \Leftrightarrow x + 2 \geq 1 \Leftrightarrow x \geq -1.$

All integer numbers x for which $-1 \leq x \leq 3/2$ applies, are: $-1,0,1$.

2) $x + 2 > 4 - x/3 \Leftrightarrow x > 3/2.$

Then

$\min(x + 2, 4 - x/3) \geq 1 \Leftrightarrow 4 - x/3 \geq 1 \Leftrightarrow 12 - x \geq 3 \Leftrightarrow x \leq 9.$

All integer numbers x for which $3/2 < x \leq 9$ applies, are: 2,3,4,5,6,7,8,9.

We thus obtain:

$A = \{-1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}.$

Answer: $A = \{-1,0,1,2,3,4,5,6,7,8,9\}$.

16. Determine the values of the real parameter m for which the set $A = \{x \in \mathbb{R} | (m - 1)x^2 - (3m + 4)x + 12m + 3 = 0\}$ has:

a) one element;

b) two elements;

c) is null.

Solution. The set A coincides with the set of the solutions of the square equation

$$(m - 1)x^2 - (3m + 4)x + 12m + 3 = 0 \qquad (1)$$

and the key to the problem is reduced to determining the values of the parameter $m \in \mathbb{R}$ for which the equation has one solution, two different solutions or no solution.

a) The equation (1) has one (two equal) solution, if and only if $D=0$ for $a=m-1\neq 0$ or if $a=m-1=0$.

We examine these cases:

1) $D=(3m+4)^2-4(m-1)(12m+3)=-39m^2+60m+28=0\Leftrightarrow$

$$\Leftrightarrow 39m^2-60m-28=0\Leftrightarrow \left[\begin{array}{l} m=\dfrac{30-2\sqrt{498}}{39}, \\ m=\dfrac{30+2\sqrt{498}}{39}. \end{array}\right.$$

2) For $m=1$, the equation (1) becomes $-5x+15=0\Leftrightarrow x=3$. So, the set A consists of one element for

$$m\in\left\{\frac{30-2\sqrt{498}}{39},1,\frac{30+2\sqrt{498}}{39}\right\}$$

b) The equation (1) has two different roots, if and only if $D>0$, namely

$$D>0\Leftrightarrow 39m^2-60m-28<0\Leftrightarrow m\in\left(\frac{30-2\sqrt{498}}{39},\frac{30+2\sqrt{498}}{39}\right)$$

c) The equation (1) doesn't have roots $D<0\Leftrightarrow 39m^2-60m-28>$

$$>0\Leftrightarrow m\in\left(-\infty,\frac{30-2\sqrt{498}}{39}\right)\bigcup\left(\frac{30+2\sqrt{498}}{39},+\infty\right).$$

Answer:

a) $m\in\left\{\dfrac{30-2\sqrt{498}}{39},1,\dfrac{30+2\sqrt{498}}{39}\right\}$;

b) $m\in\left\{\dfrac{30-2\sqrt{498}}{39},\dfrac{30+2\sqrt{498}}{39}\right\}$;

c) $m\in\left(-\infty,\dfrac{30-2\sqrt{498}}{39}\right)\bigcup\left(\dfrac{30+2\sqrt{498}}{39},+\infty\right)$.

17. Let the set $A=\{3,4,5\}$ and $B=\{4,5,6\}$. Write the elements of the sets $A^2\cap B^2$ and $\big(A\setminus(B\setminus A)\big)\times(B\cap A)$.

Solution. a) For the first set we have:

$A^2 = \{(3,3),(3,4),(3,5),(4,3),(4,4),(4,5),(5,3),(5,4),(5,5)\}$,
$B^2 = \{(4,4),(4,5),(4,6),(5,4),(5,5),(5,6),(6,4),(6,5),(6,6)\}$,
$A^2 \cap B^2 = \{(4,4),(4,5),(5,4),(5,5)\}$.

b) For the second set we have:

$B \setminus A = \{6\}$, $A \setminus (B \setminus A) = A$, $B \cap A = \{4,5\}$.

Then

$A \times (B \cap A) = \{(3,4),(3,5),(4,4),(4,5),(5,4),(5,5)\}$.

Answer:

$A^2 \cap B^2 = \{(4,4),(4,5),(5,4),(5,5)\}$;

$A \setminus (B \setminus A) \times (B \cap A) = \{(3,4),(3,5),(4,4),(4,5),(5,4),(5,5)\}$

18. Given the sets $A = \{1,2,3,4,5,6,7\}$ and $B = \{2,3,4\}$, solve the equations $A \,\Delta\, X = A \setminus B$ and $(A \,\Delta\, Y) \Delta\, B = C_A(B)$.

Solution. In order to solve the enunciated equations, we use the properties of the symmetrical difference: $A \,\Delta (A \,\Delta\, B) = B$, its associativity and commutativity .

a) $A \,\Delta\, X = A \setminus B \Rightarrow A \,\Delta\, (A \,\Delta\, X) = A \,\Delta\, (A \cap B) \Rightarrow X = A \,\Delta\, (A \setminus B)$.

But

$A \setminus B = \{1,5,6,7\}$, $A \setminus (A \setminus B) = \{2,3,4\}$, $(A \setminus B) \setminus A = \varnothing$

Consequently,

$$X = A \,\Delta\, (A \setminus B) = (A \setminus (A \setminus B)) \cup ((A \setminus B) \setminus A) =$$
$$= A \setminus (A \setminus B) = \{2,3,4\} = B.$$

b) $(A \,\Delta\, Y) \,\Delta\, B = A \setminus B \Leftrightarrow (A \,\Delta\, B) \,\Delta\, Y = A \setminus B \Leftrightarrow$
$\Leftrightarrow (A \,\Delta\, B) \,\Delta\, ((A \,\Delta\, B) \,\Delta\, Y) = (A \,\Delta\, B) \,\Delta\, (A \setminus B) \Rightarrow$
$\Leftrightarrow Y = (A \,\Delta\, B) \,\Delta\, (A \setminus B)$.

We calculate

$A \,\Delta\, B = (A \setminus B) \cup (B \setminus A) = \{1,5,6,7\} \cup \varnothing = \{1,5,6,7\}$.

Thus $Y = (A \setminus B) \,\Delta\, (A \setminus B) = \varnothing$.

Answer: $X = B, Y = \emptyset$.

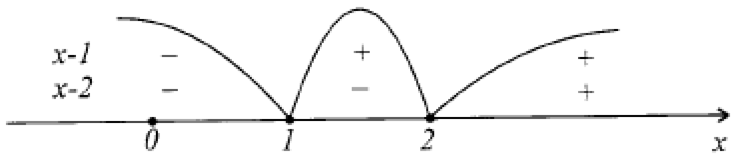
19. The following sets are given

$A = \{x \in \mathbb{R} \mid |x-1| + |2-x| > 3+x\}$, $B = \{x \in \mathbb{R} \mid (x^2-4) \times$
$\times (x+3)(x+2)^2 \leq 0\}$.

Determine the sets $A \cup B$, $A \cap B$, A, B, $A \setminus B$, $B \setminus A$, $(A \cup B) \setminus (B \setminus A)$ and $\bar{A} \times (B \setminus A)$.

Solution. 1) We determine the sets A and B.

a) $x \in A \Leftrightarrow |x-1| + |2-x| > 3+x \Leftrightarrow |x-1| + |x-2| > 3+x \quad (*)$



The inequation $(*)$ is equivalent to the totality of the three systems of inequations:

$$(*) \Leftrightarrow \left[\begin{array}{l} \begin{cases} x \in (-\infty, 1), \\ 1 - x + 2 - x > 3 + x, \end{cases} \\ \begin{cases} x \in [1;2), \\ x - 1 + 2 - x > 3 + x, \end{cases} \\ \begin{cases} x \in [2, +\infty), \\ x - 1 + x - 2 > 3 + x, \end{cases} \end{array}\right. \Leftrightarrow \left[\begin{array}{l} \begin{cases} x \in (-\infty, 1), \\ x < 0, \end{cases} \\ \begin{cases} x \in [1;2), \\ x < -2, \end{cases} \\ \begin{cases} x \in [2, +\infty), \\ x > 6 \end{cases} \end{array}\right. \Leftrightarrow$$

$$\Leftrightarrow \left[\begin{array}{l} x \in (-\infty, 0), \\ x \in \varnothing, \\ x \in (6, +\infty) \end{array}\right. \Leftrightarrow x \in (-\infty, 0) \cup (6, +\infty).$$

So

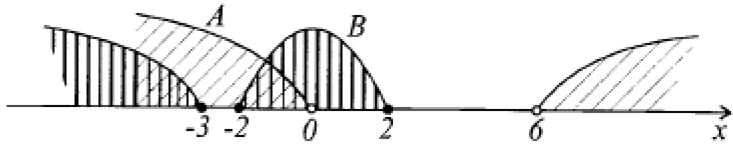
$A = (-\infty, 0) \cup (6, +\infty)$.

b) $x \in B \Leftrightarrow (x^2-4)(x+3)(x+2)^2 \leq 0 \Leftrightarrow (x+2)^3(x+3)(x-2) \leq$
$\leq 0 \Leftrightarrow x \in (-\infty, -3] \cup [-2; 2]$.

In other words,

$B = (-\infty, -3] \cup [-2; 2]$.

2) We determine the required sets with the help of the graphical representation



a) $A \cup B = (-\infty, 2] \cup (6, +\infty)$

b) $A \cap B = (-\infty, -3] \cup [-2; 0)$

c) $\overline{A} = C_{\mathbb{R}}(A) = [0; 6]$

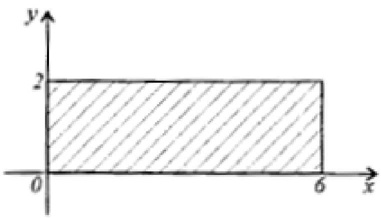
d) $\overline{B} = C_{\mathbb{R}}(B) = (-3, -2) \cup (2, +\infty)$

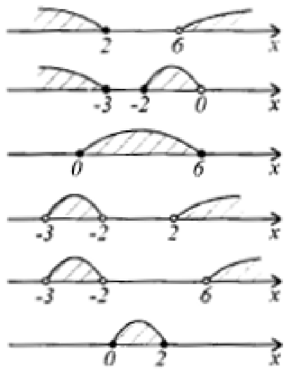
e) $A \setminus B = (-3, -2) \cup (6, +\infty)$

f) $B \setminus A = [0; 2]$

g) $(A \cup B) \setminus (B \setminus A) = \varnothing$

h) $\overline{A} \times (B \setminus A) = \{(x, y) \in \mathbb{R}^2 | x \in [0; 6], y \in [0; 2]$





20. 40 students have a mathematics test paper to write, that contains one problem, one inequation and one identity. Three students have successfully solved only the problem, 5 only the inequation, 4 have proven only the identity, 7 have solved not only the problem, 6 students – not only the inequation, and 5 students have proven not only the identity. The other students have solved everything successfully. How many students of this type are there?

Solution. Let A be the set of students who have correctly solved only the problem, B – only the inequation, C – that have proven only the identity, D – the set of students that have solved only the problem and the inequation, E – the set of students that have solved only the problem and have proven the identity, F – the set of students who have solved only the inequation and have proven the identity, and G – the set of students that have successfully solved everything.

From the given conditions, it follows that $n\,(A) = 3$, $n\,(B) = 5$, $n\,(C) = 4$, $n\,(D) = 8$, $n\,(E) = 7$, $n\,(F) = 9$.

But, as each of the students who have written the test paper have solved at least one point of the test correctly and, because the sets A, B, C, D, E, F, G have only elements of the null set in common, the union of the sets A, B, C, D, E, F, G is the set of students that have written the paper.

Consequently,

$$n(A \cup B \cup C \cup D \cup E \cup F \cup G) = n(A) + n(B) + n(C) +$$
$$+ n(D) + n(E) + n(F) + n(G).$$

So

$$n(G) = n(A \cup B \cup C \cup D \cup E \cup F \cup G) -$$
$$- n(A) - n(B) - n(C) - n(D) - n(E) - n(F) = 40-3-5-4-8-7-9 = 4.$$

Answer: 4 students out of all that have taken the test have solved everything successfully.

21. (Mathematician Dodjson's problem)

In a tense battle, 72 out of 100 pirates have lost one eye, 75 – one ear, 80 – one hand and 85 – one leg. What is the minimum number of pirates that have lost their eye, ear, hand and leg at the same time?

Solution. We note with A the set of one-eyed pirates, with B – the set of one eared pirates, with C – the set of pirates with one hand and with D – the set of the pirates with one leg.

The problems requires to appreciate the set $A \cap B \cap C \cap D$.

It is obvious that the universal set E is composed from the set $A \cap B \cap C \cap D$ and the number of pirates that have kept either both eyes, or both ears, or both hands, or both legs.

Consequently,

$$E = (A \cap B \cap C \cap D) \cup \overline{A} \cup \overline{B} \cup \overline{C} \cup \overline{D}.$$

It follows that the set E is not smaller (doesn't have fewer elements) than the sum of the elements of the sets $\bar{A}, \bar{B}, \bar{C}, \bar{D}$ and $A \cap B \cap C \cap D$ (equality would have been the case only if the sets A, B, C and D, two by two, do not intersect).

But,

$$n(\overline{A}) = 30,\ n(\overline{B}) = 25,\ n(\overline{C}) = 20,\ \text{şi}\ n(\overline{D}) = 15.$$

Substituting, we have $n(E) = 100$ namely $100 \leq n(A \cap B \cap C \cap D) + 30 + 25 + 20 + 15$.

Consequently, $n(A \cap B \cap C \cap D) \geq 100 - 30 - 25 - 20 = 10$.

Therefore, no less than 10 pirates have lost their eye, ear, hand and leg at the same time

Answer: No less than 10 pirates.

22. Of a total of 100 students, 28 study English, 8 – English and German, 10 - English and French, 5 – French and German, 3 students - all three languages. How many study only one language? How many not even one language?

Solution. Let A be the set of students that study English, B - German, C - French.

Then the set of students attending both English and German is $A \cap B$, English and French - $A \cap C$, French and German - $B \cap C$, English, German and French - $A \cap B \cap C$, and the set of students that study at least one language is $A \cup B \cup C$.

From the conditions above, it follows that the students who only study English form the set $A \setminus (A \cap B) \cup (A \cap C)$, only German - $B \setminus (A \cap B) \cup (B \cap C)$, only French - $C \setminus (A \cap C) \cup (B \cap C)$.

But, because $A \cap B \subseteq A$, we have $n(A\backslash((A \cap B) \cup (A \cap C))) = n(A) - n((A \cap B) \cup (A \cap C)) = n(A) - (n(A \cap B) + n(A \cap C) - n(A \cap B \cap C)) = n(A) - n(A \cap B) - n(A \cap C) + n(A \cap B \cap C)$.

Similarly, $n\left(B\backslash\big((A \cap B) \cup (B \cap C)\big)\right) = n(B) - n(A \cap B) - n(B \cap C) + n(A \cap B \cap C); n(C\backslash\big((A \cap C) \cap (B \cap C)\big) = n(C) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$.

Let D be the set of students who only study one language, then

$$n(D) = n(A \setminus ((A \cap B) \cup (A \cap C))) + n(B \setminus ((A \cap B) \cup (B \cap C))) + \\ + n(C \setminus ((A \cap C) \cup (B \cap C))).$$

Consequently,

$$n(D) = n(A) - n(A \cap B) - n(A \cap C) + n(A \cap B \cap C) + \\ + n(B) - n(A \cap B) - n(B \cap C) + n(A \cap B \cap C) + \\ + n(C) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C) = \\ 2n(A \cap C) - 2n(B \cap C) + 3n(A \cap B \cap C) = \\ -2n(A \cap C) - 2n(B \cap C) + 3n(A \cap B \cap C) =$$

$= 28 + 30 + 42 - 2.8 - 2.10 - 2.5 + 3.3 = 63. n(D) = 63.$

The number of students who don't study any language is equal to the difference between the total number of students and the number of students that study at least one language, namely $n(H) = n(T) - n(A \cup B \cup C)$, where H is the set of students that don't study any language and T the set of all 100 students.

From problem 20 we have

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - \\ n(A \cap C) - n(B \cap C) - n(A \cap B) + n(A \cap B \cap C) = \\ = 28 + 30 + 42 - 8 - 10 - 5 + 3 = 80.$$

So $n(H) = 100 - 80 = 20$.

Answer: 63 students study only one language, 20 students don't study any language.

1.3. Suggested exercises

1. Which of the following statements are true and which are false?

a) $x \in \{x\}$;
b) $x = \{x\}$;
c) $x \neq \{x\}$;
d) $\varnothing \in \{\varnothing\}$;
e) $\varnothing = \{\varnothing\}$;
f) $\varnothing \in \{\varnothing\}$;
g) $\varnothing = \{a\}$;
h) $\varnothing \in \{a\}$;
i) $\varnothing \subseteq \{a\}$;
j) $\{x\} \subseteq \{x\}$;
k) $\varnothing \subseteq \{\varnothing\}$;
l) $\{1,3,3\} = \{1,\{2,3\},3\}$;
m) $\{1,2,3,4,5\} = \{4,1,3,5,2,4,5\}$;
n) $\{3-1,\ 6+3\} = \{2,\ 5+3\}$;
o) $\{a+a\} = \{2a\}$, $a \in \mathbb{R}$.

2. Which of the following statements are true and which are false (A,B and C are arbitrary sets)?

a) $(A \in B$ şi $B \in C) \Rightarrow A \in C$;
b) $(A \subseteq B$ şi $B \in C) \Rightarrow A \in C$;
c) $(A \neq B$ şi $B \neq C) \Rightarrow A \neq C$;
d) $(A \cap B \subseteq \overline{C}$ şi $A \cup C \subseteq B) \Rightarrow A \cap C = \varnothing$;
e) $(A \subseteq (\overline{B \cup C})$ şi $B \subseteq (\overline{A \cup C})) \Rightarrow B = \varnothing$;
f) $(A \subseteq B$ şi $B \subseteq C$ şi $C \subseteq A) \Rightarrow A = B = C$;
g) $P(A \cup B) = \{A_1 \cup B_1 | A_1 \in P(A),\ B_1 \in P(B)\}$;
h) $P(A \cap B) = P(A) \cap P(B)$;
i) $A \subseteq \varnothing \Rightarrow A = \varnothing$;
j) $A \subseteq B \cup C \Rightarrow A \cap \overline{B} \subseteq C$;
k) $E \subseteq A \Rightarrow A = E$;
l) $A \subseteq B \Rightarrow B \cup C \subseteq A \cup C$;
m) $A \subseteq \overline{B} \Rightarrow B \subseteq \overline{A}$;
n) $A \subseteq \overline{B} \Rightarrow (A \cap B = \varnothing$ şi $A \cup B = E)$.

3. Let $A = \{x \in \mathbb{Q} \,|\, x^2 - 12x + 35) = 0\}$,
$B = \{x \in \mathbb{Q} \,|\, x^2 + 2x + 35) = 0\}$;$C = \{x \in \mathbb{Q} \,|\, x^2 + 2x + 35) = 0\}$

a) Determine the sets A, B and C.

b) State if the numbers $1/5, 5, 7, 1/2$ belong to these sets or not.

4. Determine the sets:

$A = \{x \in I\!N^* | x = 2n,\ n = \overline{1,9}\}$,

$B = \{y \in I\!N^* | y = 4m + 6n,\ m = \overline{1,3},\ n = \overline{-1,0}\}$.

5. Let $A = \{x \in \mathbb{N} \,|\, \mathrm{x} = 4\mathrm{n} + 6\mathrm{m}, \mathrm{n}, \mathrm{m} \in \mathbb{N}^*\}$;

a) Write three elements belonging to set A.

b) Determine if 26, 28, 33 belong to A.

6. Indicate the characteristic properties of the elements belonging to the sets: $A = \{4, 7, 10\}$, $B = \{3, 6, 12\}$, $C = \{1, 4, 9, 16, 25\}$, $D = \{1, 8, 27, 64, 125\}$.

7. How many elements do the following sets have?

$A = \{x \in \mathbf{Q} | x = 3n/(n+2),\ n = \overline{1,50}\}$,

$B = \{y \in \mathbf{Q} | y = (n-1)/(2^{n+1}),\ n = \overline{1,10}\}$,

$C = \{z \in I\!R | z = (an+b)/(cn+d),\ a, b, c, d \in I\!R,\ cd > 0,\ n = \overline{1,p}\}$?

8. Let there be the set $A = \{-3, -2, -1, 1, 2, 3\}$. Determine the subsets of A:

$A_1 = \{x \in A | P(x):\ 2x + 1 = x\}$,

$A_2 = \{y \in A | \mathbf{Q}(y):\ |y| = y\}$,

$A_3 = \{z \in A | R(x):\ |z| = -z\}$.

9. Determine the sets:

a) $A = \{x \in \mathbf{Z} | \min(x+1,\ 4 - 0.5x) \geq 1\}$;

b) $B = \{x \in \mathbf{Z} | \max(x-2,\ 13-2x) \leq 6\}$;

c) $C = \{x \in I\!N^* | \min(3x-1,\ 2x+10) \leq 20\}$;

d) $D = \{x \in I\!N^* | \max(20-x,\ (45-2x)/3) > 13\}$;

e) $E = \{x \in \mathbf{Z} | \min(2x+7,\ 16-3x) > 0\}$;

f) $F = \{x \in \mathbf{Z} | \max(x-1,\ 1-x) \leq 4\}$;

g) $G = \{x \in \mathbb{R} |\min(x-1, (13-x)/2) < 3\}$;
h) $H = \{x \in \mathbb{R} |\max(x+1, 7-x) > 5\}$;
i) $I = \{x \in \mathbf{Z} |\max(x+1, 4-0.5x) \leq 1\}$;
j) $J = \{x \in \mathbb{N}^* |\min(20-x, (45-2x)/3) \geq 20\}$;
k) $K = \{x \in \mathbf{Z} |\min(x+2, 10-x) > -2\}$;
l) $L = \{x \in \mathbf{Z} ||x-4| < 8\}$.

10. Compare ($\subset, \supset, =, \not\subset, \not\supset$) the sets A and B, if:

a) $A = \left\{x \in \mathbf{Q} \middle| x = \dfrac{2n+1}{n+4},\ n \in \mathbb{N}^*\right\}$, $B = \{x \in A | x < 2\}$;
b) $A = \left\{x \in \mathbf{Q} \middle| x = \dfrac{3n^2+1}{n^2+n},\ n \in \mathbb{N}^*\right\}$, $B = \{x \in A | 2 \leq x < 3\}$;
c) $A = \{x \in \mathbf{Z} | x^2 + 5x + 10 = n^2,\ n \in \mathbb{N}\}$, $B = \{-6, -3, -2, 1\}$;
d) $A = \{x \in \mathbf{Z} | x^2 + 3x - 3 = n^2,\ n \in \mathbb{N}\}$, $B = \{-7, -4, 1, 4\}$;
e) $A = \{x \in \mathbf{Z} | x^2 + 11x + 20 = n^2,\ n \in \mathbb{N}\}$, $B = \{-16, 5\}$;
f) $A = \{x \in \mathbf{R} | |x-1| + |x-2| > 5\}$, $B = \{x \in \mathbb{R} | (5/(x-4)) > -1\}$;
g) $A = \{x \in \mathbf{R} | |x| + |1-x| \geq 2\}$, $B = \{x \in \mathbb{R} | 4x^2 - 4x - 3 \geq 0\}$;
h) $A = \{x \in \mathbf{R} | 4x^2 - 4x - 3 \geq 0\}$, $B = \{x \in \mathbb{R} | (3/(x+1)) < 1\}$.

11. Let $A = \{1,2,3,4,5,6,7\}, B = \{5,6,7,8,9,10\}$. Using the symbols $\cup, \cap, \setminus, C$ (complementary), express (with the help of A, B and $\mathbb{N}^*$) the sets:

a) $A_1 = \{5, 6, 7\}$;
b) $A_2 = \{1, 2, 3, 4\}$;
c) $A_3 = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$;
d) $A_4 = \{8, 9, 10, \ldots\}$;
e) $A_5 = \{8, 9, 10\}$;
f) $A_6 = \{1, 2, 3, 4, 11, 12, 13, \ldots\}$;
g) $A_7 = \{1, 2, 3, 4, 8, 9, 10\}$.

12. Determine the set E, in the case it is not indicated and its parts A, B, C simultaneously satisfy the conditions:

a) $A \cup B = \{1,2,3,4,5,6,7\}$, $A \cap B = \{1,2\}$, $A \setminus B = \{5\}$;

b) $\overline{A} = \{2,5,9,13,18,20\}$, $\overline{B} = \{2,6,18,20\}$, $A \cup B = \{1,5,6,9,13,14\}$;

c) $A \cap B = \{1,3\}$, $\overline{A} = \{5,6,7,9,10\}$, $A \triangle B = \{2,4,5,8,9,10\}$;

d) $A \cup B = \{1,2,3,4,5\}$, $A \setminus B = \{1,4\}$, $A \cap B \not\subseteq \{3,4,5\}$, $E = \{1,2,3,4,5\}$;

e) $A \cup B \cup C = \{1,2,3,4,5\}$, $A \cap B \cap C = \{4\}$, $A \setminus B = \{1,2\}$, $A \setminus C = \{1,3\}$, $5 \notin A \cup B$, $E = \{1,2,3,4,5\}$;

f) $E = \{1,2,3,4\}$, $1 \in A$, $\{2,4\} \cap B = \varnothing$, $3 \in A \cap B \cap C$, $4 \in A \cap C$, $A \cap B \not\subset C$, $B \cup C \not\subset A$, $A \cup B \cup C = E$;

g) $E = \{1,2,3,4,5\}$, $A \cup B = E$, $A \cap B = \{2,3\}$, $\{2,3,4,5\} \cap B \not\subseteq A$, $\{1,4\} \cap A \not\subseteq B$;

h) $E = \{1,2,3\}$, $A \cup B \cup C = E$, $A \cap B \not\subset C$, $A \cap C \not\subset B$, $B \cap C = \{2\}$, $1 \in B \setminus C$;

i) $E = \{1,2,3,4,5\}$, $A \cup B = E$, $A \cap B = \{1,2\}$, $5 \notin A \setminus B$, A has more elements than B;

j) $E = \{1,2,3,4,5,6\}$, $A \cup B \cup C = E$, $A \cap B \cap C = \{5\}$, $A \setminus B = \{1,3,6\}$, $A \setminus C = \{1,2,4\}$;

k) $E = \{1,2,3,4\}$, $A \cap B = \{1,2\}$, $A \setminus B = \{1,2,4\}$, $\{1,2,3\} \not\subset B$, A has more elements than B;

l) $\overline{A} = \{1,2,3,4,5,6\}$, $\overline{B} = \{1,5,6,7\}$, $A \cup B = \{2,3,4,7,8,9,10\}$, $A \cap B = \{8,9,10\}$;

m) $E = \{a,b,c,d,e,f,g,h,i\}$, $A \cap B = \{d,f,i\}$, $A \cup \{c,d,e\} = \{a,c,d,e,f,h,i\}$, $B \cup \{d,h\} = \{b,c,d,e,f,g,h,i\}$;

n) $E = \{1,2,3,\ldots,9\}$, $A \cap B = \{4,6,9\}$, $A \cup \{3,4,5\} = \{1,3,4,5,6,8,9\}$, $B \cup \{4,8\} = \{2,3,\ldots,9\}$.

13. Determine the sets A, B, $A \cup B$, $A \cap B$, $B \setminus A$, $A \setminus B$, $A \Delta B$, $A \cup (B \setminus A)$, , $A \setminus (B \setminus A)$, $A \times B$, $B \times A$, $(A \cup B) \times A$, $B \times (A \setminus B)$, if:

a) $A = \left\{x \in \mathbb{N} \middle| x = \frac{8n-18}{2n-9}, n \in \mathbb{N}\right\}$,

$B = \left\{x \in \mathbf{Z} \middle| x = \frac{9n^2-48n+16}{3n-8}, n \in \mathbb{N}\right\}$;

b) $A = \left\{x \in \mathbf{Q} \middle| x = \frac{2n+1}{2n-3}, n \in \mathbb{N}\right\}$,

$B = \left\{y \in \mathbf{Q} \middle| y = \frac{3k+1}{3k-2}, k \in \mathbb{N}\right\}$;

c) $A = \left\{x \in \mathbb{N} \middle| x = \frac{4n}{n+2}, n \in \mathbb{N}\right\}$,

$B = \left\{y \in \mathbf{Z} \middle| y = \frac{6n+7}{3n+1}, n \in \mathbb{N}\right\}$;

d) $A = \left\{x \in \mathbb{N} \middle| \frac{2x+5}{x+1} \in \mathbb{N}\right\}$,

$B = \left\{x \in \mathbb{N} \middle| y = \frac{2n^2+4n+2}{n^2+1}, n \in \mathbb{N}\right\}$;

e) $A = \left\{x \in \mathbf{Z} \middle| \frac{x+1}{3-x} \geq 0\right\}$, $B = \left\{x \in \mathbf{Z} \middle| \frac{x^2-6x+6}{x-1} \in \mathbf{Z}\right\}$;

f) $A = \left\{x \in \mathbf{Z} \middle| \frac{3-x}{5+x} \geq 1\right\}$, $B = \left\{x \in \mathbb{N} \middle| \frac{3-x}{5+x} \geq 1\right\}$;

g) $A = \left\{x \in \mathbf{Z} \middle| \frac{x+1}{x-3} \in \mathbf{Z}\right\}$, $B = \left\{x \in \mathbb{N} \middle| \frac{x+1}{x-3} \in \mathbf{Z}\right\}$;

h) $A = \left\{x \in \mathbf{Z} \middle| \frac{x^2-5x+6}{x+3} \in \mathbf{Z}\right\}$,

$B = \left\{x \in \mathbb{N} \middle| \frac{x^2-5x+6}{x+3} \in \mathbf{Z}\right\}$;

i) $A = \{x \in \mathbb{N}^* | x = 2n, n = \overline{1,10}\}$,

$B = \{y \in I\!N^* | y = 4m + 6n,\ m = \overline{1,3},\ n \in \{-1, 0\}\}$;
j) $A = \{x \in I\!R |\ |x - 7| + |x + 7| = 14\}$,
$B = \{x \in \mathbf{Z} |\ |x + 3| + |x - 9| = 14\}$;
k) $A = \{n^2 - 5 | n \in I\!N\}$, $B = \{n^2 + 5 | n \in I\!N\}$;
l) $A = \{n^2 - 50 | n \in I\!N\}$, $B = \{n^2 + 50 | n \in I\!N\}$;
m) $A = \{n^2 - 500 | n \in I\!N\}$, $B = \{n^2 + 500 | n \in I\!N\}$;
n) $A = \{x \in I\!R | x - \sqrt{x^2 - 1} = \sqrt{2}\}$, $B = \{x \in I\!R | 8x^2 - 2\sqrt{2}x + 3 = 0\}$.

14. Let

$$M = \left\{ x \in \mathbf{Q} \middle| x = \frac{7n - 4}{n + 3},\ n \in N^* \right\}.$$

a) Determine the sets:
$A = \{x \Leftrightarrow M | x \leq 6\}$, $B = \{x \in M | x < 7\}$, $C = \{x \in M | x \in \mathbf{Z}\}$.

b) How many elements does the set $D = \left\{x \in M \middle| x \leq (699/100)\right\}$ have?

15. Determine the sets $A, B \subseteq E$, if
$A \vartriangle B = \{2, 4, 5, 8, 9, 10\}$, $A \cap B = \{1, 3\}$, $\overline{A} = \{5, 6, 7, 9, 10\}$,
$E = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

16. Determine the set E and its parts A and B, if
$\overline{A} = \{2, 5, 9, 13, 18, 20\}$, $\overline{B} = \{2, 6, 18, 20\}$,
$A \cup B = \{1, 5, 6, 9, 13, 14\}$.

17. Compare the sets A and B, if
a) $A = \{x \in I\!R | \sqrt{x^2 - 25} < x + 1\}$,

$$B = \left\{ x \in I\!R \middle| \begin{cases} x + 1 > 0, \\ x^2 - 25 < (x + 1)^2 \end{cases} \right\};$$

b) $A = \{x \in I\!R | \sqrt{x^2 - 16} \cdot (x^2 - 80) \leq \sqrt{x^2 - 16}\}$,

$$B = \left\{ x \in I\!R \middle| \begin{cases} x^2 - 16 \geq 0, \\ x^2 - 80x \leq 1 \end{cases} \right\};$$

c) $A = \{x \in \mathbb{R} | \sqrt{6 + x - x^2} > 2x - 1\}$,

$$B = \left\{x \in \mathbb{R} \middle| \begin{cases} 6 + x - x^2 \geq 0, \\ 2x - 1 \geq 0, \\ 6 + x - x^2 > (2x - 1)^2, \end{cases} \right\};$$

d) $A = \left\{x \in \mathbb{R} | 2x - 3 - \frac{1}{x - 5} < x - 4 - \frac{1}{x - 5}\right\}$,

$B = \{x \in \mathbb{R} | 2x - 3 < x - 4\}$;

e) $A = \left\{x \in \mathbb{R} \middle| -\frac{5}{2}(x - x^2 - 1)(x + 4) < -\frac{5}{2}(x - x^2 - 1)(3x + 1)\right\}$,

$B = \{x \in \mathbb{R} | x + 4 < 3x + 1\}$;

f) $A = \left\{x \in \mathbb{R} \middle| \frac{2}{\sqrt{x} + 1} < 0\right\}$, $B = \left\{x \in \mathbb{R} \middle| \left(\frac{2}{\sqrt{x} + 1}\right)^2 < 0\right\}$;

g) $A = \{x \in \mathbb{R} | \sqrt{x + 3} \cdot \sqrt{x - 3} < 1/2\}$,

$B = \{x \in \mathbb{R} | 2\sqrt{(x + 3)(x - 3)} < 1\}$.

18. Determine the values of the real parameter m for which the set A has one element, two elements or it is void, if:

a) $A = \{x \in \mathbb{R} | x^2 + mx + 1 = 0\}$;

b) $A = \{x \in \mathbb{R} | mx^2 - 5x + m = 0\}$;

c) $A = \{x \in \mathbb{R} | x^2 - mx + 3 = 0\}$;

d) $A = \{x \in \mathbb{R} | x^2 - 2(m - 2)x + m^2 - 4m + 3 = 0\}$;

e) $A = \{x \in \mathbb{R} | (m + 1)x^2 - (5m + 4)x + 4m + 3 = 0\}$;

f) $A = \{x \in \mathbb{R} | x^2 - mx + 36 = 0\}$;

g) $A = \{x \in \mathbb{R} | (2m - 1)x^2 + 2(1 - m)x + 3m = 0\}$;

h) $A = \{x \in \mathbb{R} | mx^2 - (m + 1)x + m - 1 = 0\}$.

19. Determine the number of elements of the set A:

a) $A = \{x \in \mathbf{Q} | x = (n^2 + 3)/(n^2 + n),\ n = \overline{1, 50}\}$;

b) $A = \left\{x \in \mathbf{Q} \middle| x = \frac{z}{(z+6)(z+5)},\ z \in \mathbf{Z},\ |z| \leq 45\right\}$;
c) $A = \{x \in \mathbf{Z} | (x^2+1)(5-x^2) \geq 0\}$;
d) $A = \{x \in \mathbf{Z} | (x^2-3)(x^2-33)(x^2-103)(x^2-203) < 0\}$;
e) $A = \left\{x \in \mathbf{Z} \middle| x = \frac{z+4}{z+1},\ z \in \mathbf{Z}\right\}$;
f) $A = \left\{x \in I\!N \middle| x = \frac{z+4}{z+1},\ z \in \mathbf{Z}\right\}$.

20. The sets A, B, C are given. Determine $A \cap B \cap C$.
a) $A = \{10x+3 | x \in I\!N\}$, $B = \{12y+7 | y \in I\!N\}$,
$C = \{15z+13 | z \in I\!N\}$;
b) $A = \{15n-700 | n \in I\!N\}$, $B = \{270-10m | m \in I\!N\}$,
$C = \{48k+56 | k \in I\!N\}$.

21. Determine $A \cap B$, if:
a) $A = \{6n+7 | n \in I\!N\}$, $B = \{114-7m | m \in I\!N\}$;
b) $A = \{3p+28 | p \in I\!N\}$, $B = \{107-14q | q \in I\!N\}$;
c) $A = \{3n-2 | n \in I\!N\}$, $B = \{1003-2m | m \in I\!N\}$.

22. Prove the properties of the operations with sets.

23. Determine the equalities *(*A, B, C etc. are arbitrary sets):
a) $A \setminus B = A \setminus (A \cap B) = (A \cup B) \setminus B$;
b) $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$;
c) $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$;
d) $(A \cap B) \setminus C = (A \setminus C) \cap (B \setminus C)$;
e) $(A \setminus B) \cap C = (A \cap C) \setminus (B \cap C) = (A \cap C) \setminus B$;
f) $(A \cup B) \setminus C = (A \setminus C) \cup (B \setminus C)$;
g) $(A \setminus B) \setminus C = A \setminus (B \cup C)$;
h) $A \setminus (B \setminus C) = (A \setminus B) \cup (A \cap C)$;

i) $A \cap (B \vartriangle C) = (A \cap B) \vartriangle (A \cap C)$;
j) $(A \cap B) \vartriangle A = A \setminus B$;
k) $A \cup (\bigcap_{i=1}^{n} B_i) = \bigcap_{i=1}^{n} (A \cup B_i)$;
l) $A \setminus (\bigcap_{i=1}^{n} B_i) = \bigcup_{i=1}^{n} (A \setminus B_i)$;
m) $A \setminus (\bigcup_{i=1}^{n} B_i) = \bigcap_{i=1}^{n} (A \setminus B_i)$.

24. Given the sets $A = \{1, 2, 3\}$, $B = \{2, 3, 4\}$, $C = \{3, 4, 5\}$ and $D = \{4, 5, 6\}$, write the elements of the following sets:
a) $(A \times A) \cap (B \times B)$;
b) $A^2 \times C^2$;
c) $(A \setminus B) \times (C \setminus D)$;
d) $(A \cap B) \times (C \cap B)$;
e) $(A \cup B) \times (B \cup D)$;
f) $(A \times B) \setminus (C \times D)$;
g) $(A \setminus B) \times (C \cap B)$;
h) $(A \setminus C) \times (B \setminus D)$;
i) $(A \setminus (C \setminus D)) \times ((D \setminus B) \cup A)$;
j) $(A \vartriangle B) \times (D \vartriangle B)$.

25. Given the sets A, B and C, solve the equations $(AfB)\Delta X = C$, where $f \in \{\cup, \cap, \setminus \Delta\}$:
a) $A = \{1, 4, 6\}$, $B = \{5, 7, 9\}$, $C = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$;
b) $A = \{4, 5, 6\}$, $B = \{1, 2, 3, 4, 5, 6, 7\}$, $C = \{1, 5, 6, 7\}$.

26. Determine the sets $A \cap B$, $A \cup B$, $\bar{A}$, $\bar{B}$, $A \setminus B$, $B \setminus A$, $A \cup (B \setminus \bar{A})$, $A \cap (\bar{A} \setminus B)$, if:

a) $A = \{x \in \mathbb{R} | (x-3)(2+x)(4-x) > 0\}$, $B = \{x \in \mathbb{R} | x^2 - 7x + 12 \leq 0\}$;
b) $A = \{x \in \mathbb{R} | 4x^2 - 12x + 5 < 0\}$, $B = \{x \in \mathbb{R} | 1/2 < x < 5/2\}$;
c) $A = \{x \in \mathbb{R} | x^2 - 5x + 6 \leq 0\}$, $B = \{x \in \mathbb{R} | 1 \leq 2x + 7 \leq 3\}$;
d) $A = \{x \in \mathbb{R} | (x^2 - 4)(x + 1) > 0\}$, $B = \{x \in \mathbb{R} | x^2 - 2x - 3 > 0\}$;
e) $A = \{x \in \mathbb{R} | 2x(x+7) = x^2 + 3x\}$, $B = \left\{x \in \mathbb{R} \middle| \left[\begin{array}{l} 5x^2 = 6x, \\ 5x + 4 = 0 \end{array} \right. \right\}$;
f) $A = \{x \in \mathbb{R} | (x^2 - 4x)(x + 1) < 0\}$, $B = \{x \in \mathbb{R} | x^2 - 2x - 3 < 0\}$;
g) $A = \{x \in \mathbb{R} | 3x(x - 2) - (x + 1)(x - 13) = 0\}$,

$$B = \left\{x \in \mathbb{R} \middle| \left[\begin{array}{l} x^2 + 7 = 0, \\ 13x^2 - 14x + 9 = 0 \end{array} \right. \right\};$$

h) $A = \{x \in \mathbb{R} | 3(x - 9)^2 - 2(x - 9) - 16 = 0\}$,

$$B = \left\{x \in \mathbb{R} \middle| \left[\begin{array}{l} x^2 - 14x + 49 = 0, \\ x - \frac{9}{5}\left(x + \frac{4}{3}\right) = \frac{39}{2} \end{array} \right. \right\};$$

i) $A = \{x \in \mathbb{R} | 4(2x - 3)^2 - 4(2x - 3) + 1 = 0\}$,

$$B = \left\{x \in \mathbb{R} \middle| \left[\begin{array}{l} \sqrt{3}x^2 - x + 2 = 0, \\ 3(4x - 7)(x^2 + 1) = 0 \end{array} \right. \right\};$$

j) $A = \{x \in \mathbb{R} | \, |2x-1| < |4x+1|\}$, $B = \{x \in \mathbb{R} | \, |3x-1| - |2x+3| \geq 0\}$;
k) $A = \{x \in \mathbb{R} | \, |4 - 3x| \geq 2 - x\}$, $B = \{x \in \mathbb{R} | \, |2x - 3| \geq 2x - 3\}$;
l) $A = \{x \in \mathbb{R} | 6x^2 - 2x + 1 \leq 1\}$, $B = \{x \in \mathbb{R} | x^2 + 2|x| - 3 \leq 0\}$;
m) $A = \{x \in \mathbb{R} | \, |x| + |x-1| < 5\}$, $B = \{x \in \mathbb{R} | \, |x+1| + |x-2| > 5\}$;
n) $A = \{x \in \mathbb{R} | \, ||x - 3| + 1| \geq 2\}$, $B = \{x \in \mathbb{R} | \, ||x - 1| + x| < 3\}$.

2. Relations, functions

2.1. Definitions and notations

Relations, types. Relations formation

Let A and B be two non-empty sets, and $A \times B$ their Cartesian product. Any subset $R \subseteq A \times B$ is called a **relation** between the elements of A and the elements of B. When $A = B$, a relation like $R \subseteq A \times A$ is called a **binary relation** on set A.

If there exists a relation like $R \subseteq A \times B$, then for an ordered pair $(a, b) \in A \times B$ we can have $(a, b) \in R$ or $(a, b) \notin R$. In the first case, we write aRb and we read "a is in relation R with b", and in the second case – $a\not{R}b$, that reads "a is not in relation R with b". We hold that $aRb \Leftrightarrow (a, b) \in R$.

By the **definition domain of relation** $R \subseteq$ A X B, we understand the subset $\delta_R \subseteq A$ consisting of only those elements of set A that are in relation R with an element from B, namely

$$\delta_R = \{x \in A | (\exists)\, y \in B,\ (x, y) \in R\}.$$

By **values domain of relation** $R \subseteq A \times B$ we understand the subset $\rho_R \subseteq B$ consisting of only those elements of set B that are in relation R with at least one element of A, namely

$$\rho_R = \{y \in B | (\exists)\, x \in A,\ (x, y) \in R\}.$$

The inverse relation. If we have a relation $R \subseteq A \times B$, then by the **inverse relation** of the relation R we understand the relation $R^{-1} \subseteq B \times A$ as defined by the equivalence

$$(b, a) \in R^{-1} \Leftrightarrow (a, b) \in R,$$

namely

$$R^{-1} = \{(b, a) \in B \times A | (a, b) \in R\}.$$

Example 1. Let us have $A = \{1, 2\}, B = \{4, 5, 6\}$ and also the relations

$\alpha = \{(1,5), (2,4), (2,6)\} \subseteq A \times B,$

$\beta = \{(2,4), (2,5), (2,6)\} \subseteq A \times B$

$\gamma = \{(4,1), (4,2), (5,1), (5,2)\} \subseteq B \times A.$

Determine the definition domain and the values domain of these relations and their respective inverse relations.

Solution.

a) $\delta_\alpha = \{1,2\} = A, \ \rho_\alpha = \{4,5,6\} = B; \ \alpha^{-1} =$
$= \{(5,1), (4,2), (6,2)\}; \ \delta_{\alpha^{-1}} = B, \ \rho_{\alpha^{-1}} = A;$
b) $\delta_\beta = \{2\}, \ \rho_\beta = \{4,5,6\} = B; \quad \beta^{-1} = \{(4,2), (5,2), (6,2)\};$
$\delta_{\beta^{-1}} = B, \ \rho_{\beta^{-1}} = \{2\};$
c) $\delta_\gamma = \{4,5\}, \ \rho_\gamma = \{1,2\} = A; \ \gamma^{-1} = \{(1,4), (2,4), (1,5), (2,5)\};$
$\delta_{\gamma^{-1}} = A, \ \rho_{\gamma^{-1}} = \{(4,5)\}.$

The formation of relations. Let A, B, C be three sets and let us consider the relations $R \subseteq A \times B, S \subseteq B \times C$. The relation $R \circ S \subseteq A \times C$ formed in accordance with the equivalence

$$(a,c) \in R \circ S \Leftrightarrow (\exists) b \in B \ ((a,b) \in R \wedge (b,c) \in S),$$

namely

$$R \circ S = \{(a,c) \in A \times C | (\exists) b \in B((a,b) \in R \wedge (b,c) \in S)\} \subseteq$$
$$\subseteq A \times C.$$

is called the formation of relation R and S.

Example 2. Let us have $A, B, \alpha, \beta, \gamma$, from Example 1.

Determine $\alpha \circ \alpha, \alpha \circ \beta, \alpha \circ \gamma, \beta \circ \gamma, \beta^{-1} \circ \alpha, \beta^{-1} \circ \beta, \gamma \circ \beta^{-1}$, $\gamma^{-1} \circ \beta^{-1}$ and $(\beta \circ \gamma)^{-1}$.

Solution. We point out that the compound of the relations $\alpha \subseteq C \times D$ with $\beta \subseteq E \times E$ exists if and only if $D = E$.

Because $\alpha \subseteq A \times B$, $\beta \subseteq A \times B$, it follows that $\alpha \circ \alpha$ and $\alpha \circ \beta$ doesn't exist.

b) $\alpha \subseteq A \times B$ şi $\gamma \subseteq B \times A \Rightarrow \alpha \circ \gamma \in A \times A.$

We determine $\alpha \circ \gamma$:

$(1,5) \in \alpha$ şi $(5,1) \in \gamma \Rightarrow (1,1) \in \alpha \circ \gamma$,

$(1,5) \in \alpha$ şi $(5,2) \in \gamma \Rightarrow (1,2) \in \alpha \circ \gamma$,

$(2,4) \in \alpha$ and $\{(4,1),(4,2)\} \subseteq \gamma \Rightarrow \{(2,1),(2,2)\} \subseteq \alpha \circ \gamma$;

$(2,6) \in \alpha$, but in γ we don't have a pair with the first component 6. It follows that

$\alpha \circ \gamma = \{(1,1), (1,2), (2,1), (2,2)\}$.

c) $\beta \subseteq A \times B$, $\gamma \subseteq A \times B \Rightarrow \beta \circ \gamma$ exists. By repeating the reasoning from b),

$\beta \circ \gamma = \{(2,1), (2,2)\}$.

d) $\beta^{-1} \subseteq B \times A$ şi $\alpha \subseteq A \times B \Rightarrow \beta^{-1} \circ \alpha \subseteq B \times B$,

$\beta^{-1} \circ \alpha = \{(4,4), (4,6), (5,4), (5,6), (6,4), (6,6)\}$.

$\beta^{-1} \subseteq B \times A$ şi $\beta \subseteq A \times B \Rightarrow \beta^{-1} \circ \beta \subseteq B \times B$,

$\beta^{-1} \circ \beta = \{(4,4), (4,5), (4,6), (5,4), (5,5), (5,6), (6,4), (6,5),$
$(6,6)\} = B^2$.

f) $\beta \subseteq A \times B$ şi $\beta^{-1} \subseteq B \times A \Rightarrow \beta \circ \beta^{-1} \subseteq A \times A$

and $\beta \circ \beta^{-1} = \{(2,2)\}$.

g) $\gamma^{-1} \subseteq A \times B$ şi $\beta^{-1} \subseteq B \times A \Rightarrow \gamma^{-1} \circ \beta^{-1} \subseteq A \times A$

and $\gamma^{-1} \circ \beta^{-1} = \{(1,2),(2,2)\}$.

h) $(\beta \circ \gamma)^{-1} = \{(1,2),(2,2)\} = \gamma^{-1} \circ \beta^{-1}$.

The equality relation. Let A be a set. The relation

$1_A = \{(x,x) | x \in A\} = \Delta \subseteq A \times A$

is called **an equality relation** on A. Namely,

$x 1_A y \Leftrightarrow x = y$.

Of a real use is:

1[st] *Theorem.* Let A, B, C, D be sets and $R \subseteq A \times B$, $S \subseteq B \times C$, $T \subseteq C \times D$ relations. Then,

1) $(R \circ S) \circ T = R \circ (S \circ T)$ (associativity of relations formation)

2) $1_A \circ R = R \circ 1_B = R$;

3) $(R \circ S)^{-1} = S^{-1} \circ R^{-1}$;

4) $(R^{-1})^{-1} = R$.

The equivalence Relations. The binary relation $R \subseteq A^2$ is called:

a) **reflexive**, if xRx whatever $x \in A$ is;

b) **symmetrical**, if $(xRy \Rightarrow yRx), (\forall)x, y \in A$;

c) **transitive**, if $((xRy \wedge yRz) \Rightarrow xRz)$, $(\forall)\ x, y, z \in A$;

d) **anti-symmetrical**, if $((xRy \wedge yRx) \Rightarrow x = y)$, $(\forall)\ x, y, z \in A$;

e) **equivalence relation on** A, if it is reflexive, symmetrical and transitive;

f) **irreflexive**, if $x\not{R}x$ whatever $x \in A$ is.

Let R be an equivalence relation on set A. For each element $x \in A$, the set

$$R_x = \{y \in A | xRy\}$$

is called the **equivalence class** of x modulo R (or in relation to R), and the set

$$A/R = \{R_x | x \in A\}$$

is called a **factor set** (or **quotient set)** of A through R.

The properties of the equivalence classes. Let R be an equivalence relation on set A and $x, y \in A$. Then, the following affirmations have effect:

1) $x \in R_x$;

2) $R_x = R_y \Leftrightarrow xRy \Leftrightarrow y \in R_x$;

3) $R_x \neq R_y \Leftrightarrow R_x \cap R_y = \varnothing$;

4) $\bigcup_{x \in A} R_x = A$.

Partitions on a set. Let A be a non-empty set. A family of subsets $\{A_i \,|\, i \in I\}$ of A is called a partition on A (or of A), if the following conditions are met:

1) $i \in I \Rightarrow A_i \neq \varnothing$;

2) $A_i \neq A_j \Rightarrow A_i \cap A_j = \varnothing$;

3) $\bigcup_{i \in I} A_i = A$.

2nd Theorem. For any equivalence relation R on set A, the factor set $A/R = \{R_x \,|\, x \in A\}$ is a partition of A.

3rd Theorem. For any partition $S = \{A_i \mid i \in I\}$ of A, there exists a sole equivalence relation α_S on A, hence

$$A/\alpha_S = \{A_i | i \in I\}.$$

The relation $\alpha_S \subseteq A$ is formed according to the rule

$$x\alpha_S y \Leftrightarrow (\exists) i \in I \quad (x \in A_i \wedge y \in A_i).$$

It is easily established that α_S is an equivalence relation on A and the required equality.

Example 3. We define on set $\mathbb{Z}$ the binary relation α according to the equivalence $a\alpha b \Leftrightarrow (a - b) \vdots n$, where $n \in \mathbb{N}^*$, n fixed, $(\forall) a, b \in \mathbb{Z}$.

a) Prove that α is an equivalence relation on $\mathbb{Z}$.

b) Determine the structure of the classes of equivalence.

c) Form the factor set $\mathbb{Z}/\alpha$. Application: $n = 5$.

Solution. a) Let $a, b, c \in \mathbb{Z}$. Then:

1) **reflexivity** $a - a = 0 \vdots \Rightarrow n\ a\alpha a;$

2) **symmetry**

$$a\alpha b \Rightarrow (a-b) \vdots n \Rightarrow -(b-a) \vdots n \Rightarrow (b-a) \vdots n \Rightarrow b\alpha a;$$

3) **transitivity**

$$(a\alpha b \wedge b\alpha c) \Rightarrow ((a-b) \vdots n \wedge (b-c) \vdots n) \Rightarrow$$
$$\Rightarrow ((a-b)+(b-c)) \vdots n \Rightarrow (a-c) \vdots n \Rightarrow a\alpha c.$$

From 1) - 3) it follows that α is an equivalence relation on $\mathbb{Z}$.

b) Let $a \in \mathbb{Z}$. Then

$$\alpha_a = \{b \in \mathbf{Z} | a\alpha b\} = \{b \in \mathbf{Z} | (b-a) \vdots n\} =$$
$$= \{b \in \mathbf{Z} | (\exists) t \in \mathbf{Z}:\ b - a = nt\} =$$
$$= \{b \in \mathbf{Z} | (\exists) t \in \mathbf{Z}:\ b = a + nt\} = \{a + nt | t \in \mathbf{Z}\}.$$

In accordance with the theorem of division with remainder for integer numbers a and n, we obtain

$$a = nq + r_a, \quad 0 \leq r_a \leq n - 1.$$

Then

$$a + nt = nq + r_a + nt = r_a + (nt + nq) = r_a + ns,$$

where $s = t + q \in \mathbb{Z}$, and therefore

$$\alpha_a = \{r_a + ns | s \in \mathbf{Z}\},$$

where r_a is the remainder of the division of a through n. But,

$$a = nq + r_a \Leftrightarrow a - r_a = nq \Leftrightarrow (a - r_a) \vdots n \Leftrightarrow a\alpha r_a \Leftrightarrow \alpha_a = \alpha_{r_a}.$$

In other words, the equivalence class of $a \in \mathbb{Z}$ coincides with the class of the remainder of the division of a through n.

c) Because by the division to n only the remainders $0, 1, 2, \ldots, n-1$, can be obtained, from point b) it follows that we have the exact n different classes of equivalence:

$$\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_{n-1}.$$

The following notation is customarily used

$\alpha_i = \hat{i},\ i = \overline{0, n-1}$. Then

$$\mathbf{Z}/\alpha = \{\hat{0}, \hat{1}, \hat{2}, \ldots, \widehat{n-1}\},$$

where $\hat{\imath}$ consists of only those integer numbers that divided by n yield the remainder

$$i,\ i = \overline{0, n-1}.$$

For $n = 5$, we obtain

$$\mathbf{Z}/\alpha = \{\hat{0}, \hat{1}, \hat{2}, \hat{3}, \hat{4}\},$$

with

$$\hat{0} = \{\pm 0, \pm 5, \pm 10, \pm 15, \ldots\} = \{5t | t \in \mathbf{Z}\},$$
$$\hat{1} = \{1 + 5q | q \in \mathbf{Z}\} = \{\ldots, -9, -4, 1, 6, 11, \ldots\},$$
$$\hat{2} = \{2 + 5s | s \in \mathbf{Z}\} = \{\ldots, -8, -3, 2, 7, 12, \ldots\},$$
$$\hat{3} = \{3 + 5u | u \in \mathbf{Z}\} = \{\ldots, -7, -2, 3, 8, 13, \ldots\},$$
$$\hat{4} = \{4 + 5v | v \in \mathbf{Z}\} = \{\ldots, -6, -1, 4, 9, 14, \ldots\}.$$

Definition. The relation α is called a **congruency relation modulo** n on $\mathbb{Z}$, and class $\hat{a} = \alpha_a$ is called a **remainder class modulo** n and its elements are called the representatives of the class.

The usual notation:

$a\alpha b \Leftrightarrow (a - b) \vdots n \Leftrightarrow a \equiv b$ (mode n) (a is congruent with b modulo n), and

$$\mathbf{Z}/\alpha = \mathbf{Z}_n = \{\hat{0}, \hat{1}, \hat{2}, \ldots, \widehat{n-1}\}$$

is the set of all remainder classes modulo n.

Example 4 (geometric). Let π be a plane and L the set of all the straight lines in the plane. We define L as the binary relation β in accordance with

$$d_1 \beta d_2 \Leftrightarrow d_1 \parallel d_2, \ (\forall)\, d_1, d_2 \in L.$$

a) Show that β is an equivalence relation on L.

b) Describe the equivalence classes modulo β.

c) Indicate the factor set.

Solution. a) It is obvious that β is an equivalence relation (each straight line is parallel to itself; if $d_1 \parallel d_2 \Rightarrow d_2 \parallel d_1$ and

$$(d_1 \parallel d_2 \wedge d_2 \parallel d_3) \Rightarrow d_1 \parallel d_3).$$

b) Let $d \in L$. Then the class

$$\beta_d = \{l \in L | l \alpha d\} = \{l \in L | l \| d\}$$

consists only of those straight lines from L that are parallel with the straight line d.

c) $L/\beta = \{\beta_d \,\big|\, d \in L\}$ is an infinite set, because there are an infinity of directions in plane π.

Example 5. The set $A = \{1,2,3,4,5,6,7,8,9\}$ is given and its parts $A_1 = \{1,2\}$, $A_2 = \{3,4,6\}$, $A_3 = \{5,7,8\}$, $A_4 = \{9\}$, $B_1 = \{1,2,4\}$, $B_2 = \{2,5,6\}$, $B_3 = \{3,7,8,9,10\}$.

a) Show that $S = \{A_1, A_2, A_3, A_4\}$ is a partition of A.

b) Determine the equivalence relation α_S on A.

c) Is $T = \{B_1, B_2, B_3\}$ a partition on A?

Solution. a) 1) We notice that

$$A_i \in S \Rightarrow A_i \neq \varnothing, \ i = \overline{1,4};$$

2) $A_1 \cap A_2 = A_1 \cap A_3 = A_1 \cap A_4 = A_2 \cap A_3 = A_2 \cap A_4 = A_3 \cap A_4 = \varnothing$.

3) $A_1 \cup A_2 \cup A_3 \cup A_4 = A$,

meaning the system S of the subsets of the set A defines a partition on A.

b) In accordance to the 3rd Theorem, we have

$$(x, y) \in \alpha_S \Leftrightarrow x\alpha_S y \Leftrightarrow (\exists) i \in \{1,2,3,4\}\ (x \in A_i \wedge y \in A_i).$$

So:

α_S
$= \{(1,1), (1,2), (2,1), (2,2), (3,3), (3,4), (3,6), (4,3), (4,4), (4,6), (6,3), (6,4), (6,6), (5,5), (5,7), (5,8), (8,5), (8,8), (8,7), (7,5), (7,7), (7,8), (9,9)\}$.

c) 1) $B_i \in T \Rightarrow B_i \neq \varnothing;\ i = \overline{1,3}$;

2) $B_1 \cup B_2 \cup B_3 = A$;

3) $B_1 \cap B_2 = \{2\} \neq \varnothing$,

which proves that system T does not define a partition on A.

Order relations. A binary relation R on the set A is called an **order relation on** A, if it is reflexive, anti-symmetrical and transitive.

If R is an order relation on A, then R^{-1} is also an order relation on A (verify!). Usually, the relation R is denoted by "$\leq$" and the relation R^{-1} by "$\geq$", hence

$$x \leq y \Leftrightarrow y \geq x.$$

With this notation, the conditions that "$\leq$" is an order relation on the set A are written:

- reflexivity $x \in A \Rightarrow x \leq x$;
- asymmetry $(x \leq y \wedge y \leq x) \Rightarrow x = y$;
- transitivity $(x \leq y \wedge y \leq z) \Rightarrow x \leq z$.

Example 6. On the set $\mathbb{N}$ we define the binary relation γ in accordance with

$$a\gamma b \Leftrightarrow (\exists) k \in \mathbb{N} (a = b \cdot k).$$

Show that γ is an order relation on $\mathbb{N}$.

Solution. We verify the condition from the definition of order relations.

1) **Reflexivity**

$$a = a \cdot 1 \Rightarrow a\gamma a,\ (\forall) a \in \mathbb{N}.$$

2) **Asymmetry**. Let $a, b \in \mathbb{N}$ with $a\gamma b$ and $b\gamma a$. It follows that there are the natural numbers $c, d \in \mathbb{N}$ with $a = b.c$ and $b = a.d$. Then

$$a = b \cdot c = (a \cdot d) \cdot c = a \cdot (d \cdot c) \Rightarrow d \cdot c = 1 \Rightarrow d = c = 1,$$

which implies that

$$a = b \cdot c = b \cdot 1 = b.$$

3) **Transitivity**. Let $a, b, c \in \mathbb{N}$ with $a\gamma b$ and $b\gamma c$. Then there exists $u, v \in \mathbb{N}$ with $a = bu$ and $b = cv$ hence,

$$a = bu = (cv)u = c(vu) \Rightarrow a\gamma c.$$

Because $u, v \in \mathbb{N}$, the implication is true

$$(a\gamma b \wedge b\gamma c) \Rightarrow a\gamma c,$$

which proves that γ is an order relation on the set $\mathbb{N}$.

Remark. The order relation γ is called a divisibility relation on $\mathbb{N}$ and it is written as $a \vdots b$, namely

$$a\gamma b \Rightarrow a \vdots b \Leftrightarrow (\exists)\, k \in \mathbb{N}(a = b \cdot k) \Leftrightarrow b|a$$

($b \,|\, a$ reads "b divides a", and $a \vdots b$ "a is divisible through b").

Functional relations

Let A and B be two sets. A relation $R \subseteq A \times B$ is called an **application** or a functional relation, if the following conditions are met:

1)$\forall\, x \in A (\exists) y \in B$, so as xRy;

2) $(xRy \wedge xRy_1) \Rightarrow y = y_1$.

An application (or function) is a triplet $f = (A, B, R)$, where A and B are two sets and $R \subseteq A \times B$ is a functional relation.

If $R \subseteq A \times B$ is an application, then for each $x \in A$ there exists, according to the conditions 1) and 2) stated above, a single element $y \in B$, so that xRy; we write this element y with $f(x)$. Thus,

$$f(x) = y \Leftrightarrow xRy.$$

The element $f(x) \in B$ is called the **image of** element $x \in A$ through application f, set A is called **the definition domain** of f noted by $D(f) = A$, and set B is called the **codomain** of f and we usually say that f is an application defined on A with values in B. The functional relation R is also called the **graphic** of the application (function) f, denoted, later on, by G_f. To show that f is an application defined on A with values in B, we write $f: A \longrightarrow B$ or $A \xrightarrow{f} B$, and, instead of describing what the graphic of R (or $f's$) is, we indicate for each $x \in A$ its image $f(x)$. Then,

$$y = f(x) \Leftrightarrow xRy \Leftrightarrow xRf(x) \Leftrightarrow (x, f(x)) \in R = G_f,$$

i.e.

$$G_f = \{x, f(x))|x \in A\} \subseteq A \times B.$$

The equality of applications. Two applications $f = (A, B, R)$ and $g = (C, D, S)$ are called **equal** if and only if they have the same domain $A = C$, the same codomain $B = D$ and the same graphic $R = S$. If $f, g: A \longrightarrow B$, the equality $f = g$ is equivalent to $f(x) = g(x), (\forall)x \in A$, that is to say

$$f = g \Leftrightarrow (\forall)x \in A\ (f(x) = g(x)).$$

The identical application. Let A be a set. The triplet $(A, A, 1_A)$ is, obviously, an application, denoted by the same symbol 1_A (or ε) and it is called the **identical application** of set A.

We have

$$1_A(x) = y \Leftrightarrow (x, y) \in 1_A \Leftrightarrow x = y.$$

Consequently,

$1_A: A \longrightarrow A$ and $1_A(x) = x$ for $(\forall)x \in A$.

By $F(A, B)$ we will note the set of functions defined on A with values in B. For $B = A$, we will use the notation $F(A)$ instead of $F(A, A)$.

In the case of a finite set $A = \{a_1, a_2, \ldots, a_n\}$, a function $\varphi: A \longrightarrow A$ is sometimes given by means of a table as is:

x	a_1	a_2	$\ldots$	a_n
$\varphi(x)$	$\varphi(a_1)$	$\varphi(a_2)$	$\ldots$	$\varphi(a_n)$

where we write in the first line the elements $a_1, a_2, \dots, a_n$ of set A, and in the second line their corresponding images going through φ, namely $\varphi(a_1), \varphi(a_2), \dots, (a_n)$.

When $A = \{1,2,\dots,n\}$, in order to determine the application, we will use $\varphi: A \longrightarrow A$ and the notation

$$\varphi = \begin{pmatrix} 1 & 2 & 3 & \dots & n-1 & n \\ \varphi(1) & \varphi(2) & \varphi(3) & \dots & \varphi(n-1) & \varphi(n) \end{pmatrix},$$

more frequently used to write the bijective applications of the set A in itself. For example, if $A = \{1,2\}$, then the elements of $F(A)$ are:

$$\varepsilon = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \quad \alpha = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, \quad \gamma = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}.$$

If $f: A \longrightarrow B$, $X \subseteq A$, $Y \subseteq B$, then we introduce the notations:

$$f(X) = \{b \in B | (\exists) x \in X: \ f(x) = b\} = \{f(x) | x \in X\} \subseteq B$$

- the **image of subset** X through the application f.

Particularly, $\varphi(A) = Im\varphi$, the **image of application** φ;

$$f^{-1}(Y) = \{a \in A | (\exists) y \in Y: \ f(a) = y\} = \{a \in A | f(a) \in Y\} \subseteq A$$

is the **preimage of subset** Y through the application f.

Particularly, for $y \in B$, instead of writing $f^{-1}(\{y\})$, we simply note $f^{-1}(y)$, that is

$$f^{-1}(y) = \{a \in A | f(a) = y\}$$

- the **set of all preimages** of y through the application f, and

$$f^{-1}(B) = \{a \in A | \varphi(a) \in B\}$$

- the **complete preimage of set** B through the application f.

The formation of applications. We consider the applications $f = (A, B, R)$ and $g = (B, C, S)$, so as the codomain of f to coincide with the domain of g. We form the triplet $g \circ f = (A, C, R \circ S)$.

Then $g \circ f$ is also an application, named the compound of the application g with the application f, and the operation "$\circ$" is called **the formation of applications**. We have

$$(g \circ f)(x) = z \Leftrightarrow (x, z) \in R \circ S \Leftrightarrow$$
$$\Leftrightarrow (\exists) y \in B((x, y) \in R \wedge (y, z) \in S) \Leftrightarrow$$

$\Leftrightarrow (\exists)\, y \in B\colon xRy \wedge ySz \Leftrightarrow$
$\Leftrightarrow (\exists)\, y \in B\colon (f(x) = y \wedge g(y) = z) \Leftrightarrow$
$\Leftrightarrow g(f(x)) = z,$

namely

$$(g \circ f)(x) = g(f(x)),\ (\forall)\, x \in A.$$

4^{th} *Theorem.* Given the applications

$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$. It follows that *a)* $(h \circ g) \circ f = h \circ (g \circ f)$ *(associativity of the formation of applications)*; *b)* $f \circ 1_A = 1_B \circ f = f$.

Example 7. We consider the relations $R \subseteq \mathbb{R} \times \mathbb{R}$ and $S \subseteq [0,+\infty) \times [0,+\infty)$, $T \subseteq \mathbb{R} \times \mathbb{R}$, defined as it follows: $xRy \Leftrightarrow x^2 = y$, $xSy \Leftrightarrow x = y^2$, $xTy \Leftrightarrow y = x + 1$.

A. Determine which of the relations $R, R^{-1}, S, S^{-1}, T, T^{-1}$ are functional relations.

B. Find the functions determined at point A.

C. Calculate $f \circ g, g \circ f, f \circ h, h \circ f, h \circ h^{-1}, h^{-1} \circ h$ (f, g, h are the functions from point B.)

D. Calculate $(f \circ h)(-3), (h^{-1} \circ h)(1/2), (h \circ f)(1/3)$.

Solution. We simultaneously solve A and B.

a) We examine the relation R.

$x \in \mathbb{R} \Rightarrow x^2 \in \mathbb{R} \Rightarrow y \in \mathbb{R}$, i.e.

1)$\forall\, x \in \mathbb{R} (\exists) y = x^2 \in \mathbb{R}$, so that xRy;

2) $(xRy \wedge xRy_1) \Rightarrow (y = x^2 \wedge y_1 = x^2) \Rightarrow y = y_1$,

which means that R is a functional relation. We write the application determined by R through f, $f = (\mathbb{R}, \mathbb{R}, R)$.

b) We examine the relation R^{-1}.

$$yR^{-1}x \Leftrightarrow xRy \Leftrightarrow y = x^2,$$

which means that if $x \in \mathbb{R} = \{a \in \mathbb{R} \,|\, a < 0\}$, then there is no $x \in \mathbb{R}$, so that $yR^{-1}x$, which proves that R^{-1} is not a functional relation.

c) For the relation S, we have: S and S^{-1} are functional relations. We denote by $g = ([0,+\infty), [0,+\infty), S)$,and $g^{-1} = ([0,+\infty), [0,+\infty), S^{-1})$, the functions (applications) defined by S and S^{-1}, respectively.

d) We examine the relation T:

1) $x \in \mathbb{R} \Rightarrow x+1 \in \mathbb{R}$ and so

$(\forall)x \in \mathbb{R}\,(\exists)\,y = x+1 \in \mathbb{R}$,

so that xTy;

2) $(xTy \wedge xTy_1) \Rightarrow (y = x+1 \wedge y_1 = x+1) \Rightarrow y = y_1$,
which proves that T is a functional relation.

e) For the relation T^{-1}, we obtain:

$yT^{-1}x \Leftrightarrow xTy \Leftrightarrow y = x+1 \Rightarrow x = y-1$.

1)$(\forall)x \in \mathbb{R}$ $(\exists)x = y-1 \in \mathbb{R}$, so that $yT^{-1}x$;

2) $(yT^{-1}x \wedge yT^{-1}x_1) \Rightarrow (xTy \wedge x_1Ty) \Rightarrow (y = x+1 \wedge y = x_1+1) \Rightarrow x_1+1 = x+1 \Rightarrow x_1 = x$,

meaning that T^{-1} is also a functional relation. We denote by $h = (\mathbb{R}, \mathbb{R}, T)$ and $h^{-1} = (\mathbb{R}, \mathbb{R}, T^{-1})$.

C. From A and B, it follows that:

$f(x) = x^2, x \in \mathbb{R}$, $g(x) = \sqrt{x}, x \in [0,+\infty)$, $h(x) = x+1$,
$x \in \mathbb{R}$, $h^{-1}(x) = x-1, x \in \mathbb{R}$.

a) Then $f \circ g$, $g \circ f$ and $g \circ h$ don't exist, because the domains and codomains do not coincide:

$f = (\mathbb{R}, \mathbb{R}, R)$, $g = ([0,+\infty), [0,+\infty), S)$, $h = (\mathbb{R}, \mathbb{R}, T)$).

b) We calculate $f \circ h, h \circ f, h \circ h^{-1}$ and $h^{-1} \circ h$.

$(f \circ h)(x) = f(h(x)) = f(x+1) = (x+1)^2$;

$(h \circ f)(x) = h(f(x)) = h(x^2) = x^2+1$;

$(h \circ h^{-1})(x) = h(h^{-1}(x)) = h(x-1) = (x-1)+1 = x = 1_{\mathbb{R}}(x)$;

$(h^{-1} \circ h)(x) = h^{-1}(h(x)) = h^{-1}(x+1) = (x+1)-1 = x = 1_{\mathbb{R}}(x)$.

So $f \circ h \neq h \circ f$ and $h \circ h^{-1} = h^{-1} \circ h = 1_{\mathbb{R}}$.

D. We calculate:

$(f \circ h)(-3) = (x+1)^2/_{x=-3} = 4$, $\quad (h^{-1} \circ h)(1/3) =$

$= 1/2$, $(h \circ f)(1/3) = (x^2 + 1)/_{x=1/3} = 1/9 + 1 = 10/9$.

Injective, surjective and bijective applications. An application $f: A \longrightarrow B$ is called:

1) **injective**, if

$f(a_1) = f(a_2) \Rightarrow a_1 = a_2$, $(\forall)\, a_1, a_2 \in A$

(equivalent: $a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2)$);

2) **surjective**, if

$(\forall)\, b \in B\, (\exists)\, a \in A$: $f(a) = b$

(any element from B has at least one preimage in A);

3) **bijective**, if f is injective and surjective.

Remark. To prove that a function $f: A \longrightarrow B$ is a surjection, the equation $f(x) = b$ must be solvable in A for any $b \in B$.

Of a real use are:

5th Theorem. Given the applications $A \xrightarrow{f} B \xrightarrow{g} C$, we have:

a) if f and g are injective, then $g \circ f$ is injective;

b) if f and g are surjective, then $g \circ f$ is surjective;

c) if f and g are bijective, then $g \circ f$ is bijective;

d) if $g \circ f$ is injective, then f is injective;

e) if $g \circ f$ is surjective, then g is surjective.

6th Theorem. The application $f: A \longrightarrow B$ with the graphic $G_f = R$ is a bijective application, if and only if the inverted relation R^{-1} is a functional relation (f^{-1} is an application).

This theorem follows immediately from

$(y, x) \in R^{-1} \Leftrightarrow yR^{-1}x \Leftrightarrow f^{-1}(y) = x \Leftrightarrow xRy \Leftrightarrow y = f(x)$.

Inverted application. Let $f: A \longrightarrow B$ be a bijective application with the graphic $G_f = R$. From the 6th Theorem, it follows that the triplet $f^{-1} = (B, A, R^{-1})$ is an application (function). This function is called the inverse of function f.

We have

$f^{-1}: B \longrightarrow A$, and for $y \in R$,

$f^{-1}(y) = x \Leftrightarrow yR^{-1}x \Leftrightarrow xRy \Leftrightarrow f(x) = y$, i.e.

$$(y,x) \in R^{-1} = G_{f^{-1}} \Leftrightarrow (x,y) \in R = G_f.$$

7th Theorem. Application $f: A \longrightarrow B$ is bijective, if and only if there exists an application $g: B \longrightarrow A$ with $g \circ f = 1_A$ and $f \circ g = 1_B$. In this case, we have $g = f^{-1}$.

Real functions

The function $f: A \longrightarrow B$ is called a **function of real variable**, if $A = D(f) \subseteq \mathbb{R}$. The function of real variable $f: A \longrightarrow B$ is called a **real function**, if $B \subseteq \mathbb{R}$. In other words, the function $f: A \longrightarrow B$ is called **a real function**, if $A \subseteq \mathbb{R}$ and $B \subseteq \mathbb{R}$. The graphic of the real function $f: A \longrightarrow B$ is the subset G_f of $\mathbb{R}^2$, formed by all the pairs $(x,y) \in \mathbb{R}^2$ with $x \in A$ and $y = f(x)$, i.e.:

$$G_f = \{(x, f(x)) | x \in A\}.$$

If the function f is invertible, then

$$(y,x) \in G_{f^{-1}} \Leftrightarrow (x,y) \in G_f.$$

Traditionally, instead of $f^{-1}(y) = x$ we write $y = f^{-1}(x)$. Then the graphic of the inverted function f^{-1} is symmetrical to the graphic function f in connection to the bisector $y = x$.

Example 8. For the function

$$f: [0,\infty) \longrightarrow [0,\infty),\ y = f(x) = x^2,$$

the inverted function is

$$f^{-1}: [0,\infty) \longrightarrow [0,\infty),\ y = f^{-1}(x) = \sqrt{x}.$$

The graphic of function f is the branch of the parabola $y = x^2$ comprised in the quadrant I, and the graphic of function f^{-1} is the branch of parabola $x = y^2$ comprised in the quadrant I (see below Fig. 2.1).

Algebra operations with real functions

Let $f, g: D \longrightarrow \mathbb{R}$ be two real functions defined on the same set. We consider the functions:

$s = f + g: D \longrightarrow \mathbb{R}$,

defined through $s(x) = f(x) + g(x), (\forall)x \in D$;

$s = f + g$ – **sum-function.**

$p = f \cdot g: D \longrightarrow \mathbb{R}$,

defined through $p(x) = f(x) \cdot g(x)$, $(\forall)x \in D$;

$p = f . g$ – **product-function.**

$d = f - g: D \longrightarrow \mathbb{R}$,

defined through $(x) = f(x) - g(x), (\forall)x \in D$;

$d = f - g$ – **difference-function.**

$q = \frac{f}{g}: D_1 \longrightarrow \mathbb{R}$,

defined through $q(x) = \frac{f(x)}{g(x)}, (\forall)x \in D_1$

where $D_1 = \{x \in D \,|\, g(x) \neq O\}$;

$q = \frac{f}{g}$ – **quotient-function.**

$|f|: D \longrightarrow \mathbb{R}$,

defined through $|f|\ (x) = |f\ (x) = \begin{Bmatrix} f(x), if\ f(x) \geq 0, \\ -f(x), if\ f(x) < 0, \end{Bmatrix}$

for $(\forall)x \in D$;

$|f|$ - **module-function.**

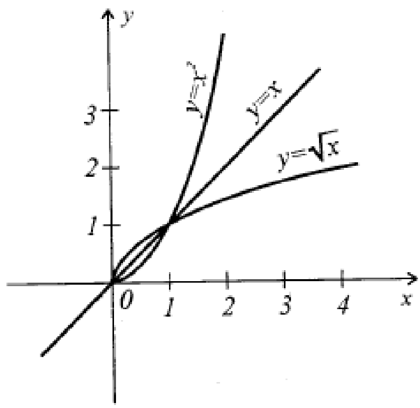


Figure 2.1.

Example 9. Let $f, g: [0, +\infty) \longrightarrow \mathbb{R}$ with $f(x) = x^2$ and $g(x) = \sqrt{x}$. Determine $f \pm g, f \cdot g$ and f/g.

Solution. As f, g share the same definition domain, the functions $f \pm g, f \cdot g$ and f/g make sense and

$$s(x) = (f+g)(x) = f(x) + g(x) = x^2 + \sqrt{x},\ (\forall) x \in [0, +\infty);$$
$$d(x) = (f-g)(x) = f(x) - g(x) = x^2 - \sqrt{x},\ (\forall) x \in [0, +\infty);$$
$$p(x) = f(x) \cdot g(x) = x^2 \cdot \sqrt{x},\ (\forall) x \in [0, +\infty);$$
$$q(x) = f(x)/g(x) = x^2/\sqrt{x} = x\sqrt{x},\ (\forall) x \in D_1 = (0, +\infty).$$

Example 10. Let $f: \mathbb{R} \longrightarrow [0, +\infty), f(x) = x^2$and $g: [0, +\infty) \longrightarrow \mathbb{R}, g(x) = \sqrt{x}$. Determine a) $g \circ f$ and b) $f \circ g$.

Solution. a) $g \circ f = \varphi$ makes sense and we obtain the function $\varphi: \mathbb{R} \longrightarrow \mathbb{R}$ with

$$\varphi(x) = (g \circ f)(x) = g(f(x)) = g(x^2) = \sqrt{x^2} = |x|, (\forall) x \in \mathbb{R}.$$

b) $\psi = f \circ g: [0, +\infty) \longrightarrow [0, +\infty)$, also makes sense, with

$$\psi(x) = (f \circ g)(x) = f(g(x)) = f(\sqrt{x}) = (\sqrt{x})^2 = x, (\forall) x \in [0, \infty).$$

2.2. Solved exercises

1. Let $A = \{2,4,6,8\}$ and $B = \{1,3,5,7\}$, $\alpha \subseteq A \times B, \alpha = \{(x,y) | x \geq 6 \ or \ y \leq 1\}$:

a) What is the graphic of the relation α?

b) Make the schema of the relation α.

Solution.

a) Because $x \in A$, and $y \in B$, it follows that

$$x \geq 6 \Leftrightarrow x \in \{6, 8\},\ y \leq 1 \Leftrightarrow y = 1.$$

So:

$$\alpha = \{(6,1), (6,3), (6,5), (6,7), (8,1), (8,3), (8,5), (8,7), (2,1), (4,1)\}$$

b) See Fig. 2.2 below.

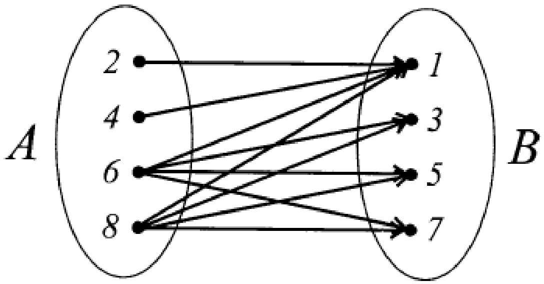


Figure 2.2.

2. Let $A = \{1,2,3,4\}$ and $B = \{1,3,5,7,8\}$. Write the graphic of the relation $\alpha = \{(x,y) \mid 3x + 4y = 10\} \subseteq A \times B$.

Solution. We observe that $3x + 4y = 10 \Leftrightarrow 3x = 2(5 - 2y) \Rightarrow x$ is even and $(5 - 2y) \vdots 3$. Considering that $x \in A$, and $y \in B$, we obtain: $x \in (2,4)$, and $y \in (1,7)$. The equality $3x + 4y = 10$ is met only by $x = 2$ and $y = 1$. So $G_\alpha = \{(2,1)\}$.

3. Let $A = \{1,3,4,5\}$ and $B = \{1,2,5,6\}$. Write the relation α using letters $x \in A$, and $y \in B$, if the graphic of the relation α is known.

$G_\alpha = \{(1,1), (1,2), (1,5), (1,6), (3,6), (5,5)\}$.

Solution. $(x,y) \in G_\alpha \Leftrightarrow y \vdots x$.

4. On the natural numbers set $\mathbb{N}$ we consider the relations $\alpha, \beta, \gamma, \omega \subseteq \mathbb{N}^2$, defined as it follows: $(x, y \in \mathbb{N})$: $\alpha = \{(3,5), (5,3), (3,3), (5,5)\}$; $x\beta y \Leftrightarrow x \leq y$; $x\gamma y \Leftrightarrow y - x = 12$; $x\omega y \Leftrightarrow x = 3y$.

a) Determine $\delta_\alpha, \rho_\alpha, \delta_\beta, \rho_\beta, \delta_\gamma, \rho_\gamma, \delta_\omega, \rho_\omega$.

b) What properties do each of the relations $\alpha, \beta, \gamma, \omega$ have?

c) Determine the relations $\alpha^{-1}, \beta^{-1}, \gamma^{-1}$ and ω^{-1}.

d) Determine the relations $\beta \circ \gamma$, $\gamma \circ \beta$, $\gamma^{-1} \circ \beta^{-1}$, $(\beta \circ \gamma)^{-1}$, $\gamma \circ \omega$ and $\omega^{-1} \circ \omega$.

Solution:

a) 1) $\delta_\alpha = \{3,5\} = \rho_\alpha$.

2) $\delta_\beta=\{x \in \mathbb{N} | (\exists)\, y \in \mathbb{N}\colon x\beta y\}=\{x \in \mathbb{N} | (\exists)\, y \in \mathbb{N}\colon x \le y\}=\mathbb{N}$

(for any natural numbers, there are natural numbers that exceed it).

$\rho_\beta = \{y \in \mathbb{N} | (\exists)\, x \in \mathbb{N}\colon x\beta y\} = \{y \in \mathbb{N}\, (\exists)\, x \in \mathbb{N}\colon x \le y\} = \mathbb{N}$

(for any natural numbers there is at least one natural number that exceeds it).

3) $\delta_\gamma=\{x \in \mathbb{N} | (\exists)\, y \in \mathbb{N}\colon x\gamma y\}=\{x \in \mathbb{N} | (\exists)\, y \in \mathbb{N}\colon y - x=12\}=$
$= \{x \in \mathbb{N} | (\exists)\, y \in \mathbb{N}\colon x = y - 12\} = \mathbb{N}$

(for example, equation $2 = x + 12$ doesn't have solutions in $\mathbb{N}$).

4) $\delta_\omega = \{x \in \mathbb{N} | (\exists)\, y \in \mathbb{N}\colon x\omega y\} = \{x \in \mathbb{N} | (\exists)\, y \in \mathbb{N}\colon x =$
$= 3y\} = \{x \in \mathbb{N} | x \vdots 3\} = \{0,3,6,9,\ldots\} = \{3k | k \in \mathbb{N}\}$.
$\rho_\omega=\{y \in \mathbb{N} | (\exists)\, x \in \mathbb{N}\colon x=3y\}=\{y \in \mathbb{N} | (\exists)\, x \in \mathbb{N}\colon y=x : 3\}=\mathbb{N}$

(for any $n \in \mathbb{N}$, we have $n = 3n/n$).

b) 1) α is symmetrical, transitive, but is not reflexive. For example $(2,2) \in \alpha$; because $(3,3) \in \alpha$:, it follows that α isn't antireflexive either.

2) β is - reflexive($x \le x, (\forall)\, x \in \mathbb{N}$),
- anti-symmetrical ($(x\beta y \wedge y\beta x) \Rightarrow$
$(x \le y \wedge y \le x) \Rightarrow x = y$),
- transitive ($(x \le y \wedge y \le z) \Rightarrow x \le z$).

So β is an order relation on set $\mathbb{N}$.

3) γ is antireflexive ($x \le x, (\forall)\, x \in \mathbb{N}$), anti-symmetrical $\big((x\gamma y \wedge y\gamma x) \Rightarrow (y - x = 12 \text{ and } x - y = 12) \Rightarrow x - y = y - x \Rightarrow x = y\big)$, but it is not symmetrical ($x\gamma y \Rightarrow y - x = 12 \Rightarrow x - y = -12 \Rightarrow x \not\gamma y$), and it is not transitive ($(x\gamma y \wedge y\gamma z) \Rightarrow (y - x = 12 \wedge z - y = 12) \Rightarrow z - x = 24 \Rightarrow x \not\gamma z$).

4) ω is not antireflexive ($x \ne 3x, (\forall)\, x \in \mathbb{N}^*, 0 = 3.0$), it is anti-symmetrical ($(x\omega y \wedge y\omega x) \Rightarrow (x = 3y \wedge y = 3x) \Rightarrow x = 9x \Rightarrow x = 0 = y$), and for $x \ne 0 \ne y$, the equalities $x = 3y$ and $y = 3x$ cannot take place), it is not reflexive (for example, $1 \ne 3.1 \Rightarrow 1 \not\omega 1$), it is not

transitive $((x\omega y \wedge y\omega z) \Rightarrow (x = 3y \wedge y = 3z) \Rightarrow x = 9z \neq 3z$, generally $\Rightarrow x \not\omega z)$.

c) 1) $\alpha^{-1} = \{(5,3), (3,5), (3,3), (5,5)\} = \alpha$.

2) $\beta^{-1} = \{(y,x) \in \mathbb{N}^2 | (x,y) \in \beta\} = \{(y,x) \in \mathbb{N}^2 | x\beta y\} =$
$= \{(y,x) \in \mathbb{N}^2 | x \leq y\} = \{(y,x) \in \mathbb{N} | y \geq x\}$.

So

$$y\beta^{-1}x \Leftrightarrow y \geq x. \qquad (1)$$

3) $(y,x) \in \gamma^{-1} \Leftrightarrow (x,y) \in \gamma \Leftrightarrow x\gamma y \Leftrightarrow y - x = 12 \Leftrightarrow x = y - 12$,

i.e.

$$y\gamma^{-1}x \Leftrightarrow x = y - 12. \qquad (2)$$

4) $(y,x) \in \omega^{-1} \Leftrightarrow (x,y) \in \omega \Leftrightarrow x = 3y \Leftrightarrow y = x/3$,

i.e.

$$y\omega^{-1}x \Leftrightarrow y = x/3. \qquad (3)$$

d) 1) $(x,y) \in \beta \circ \gamma \Leftrightarrow (\exists) z \in \mathbb{N}\colon (x,z) \in \beta \wedge (z,y) \in \gamma \Leftrightarrow$
$\Leftrightarrow (\exists) z \in \mathbb{N}\colon x \leq z \wedge y - z = 12 \Leftrightarrow (\exists) z \in \mathbb{N}\colon x \leq z \wedge z =$
$= y - 12 \Leftrightarrow x \leq y - 12 \Leftrightarrow x + 12 \leq y$,

i.e.

$$\beta \circ \gamma = \{(x,y) \in \mathbb{N}^2 | x + 12 \leq y\}. \qquad (4)$$

2) $(x,y) \in \gamma \circ \beta \Leftrightarrow (\exists) z \in \mathbb{N}\colon (x,z) \in \gamma \wedge (z,y) \in \beta \Leftrightarrow (\exists) z \in$
$\in \mathbb{N}\colon z - x = 12 \wedge z \leq y \Leftrightarrow (\exists) z \in \mathbb{N}\colon z = x + 12 \wedge z \leq y \Leftrightarrow x + 12 \leq y$.

which implies

$$\gamma \circ \beta = \{(x,y) \in \mathbb{N}^2 | x + 12 \leq y\}.$$

3) $(u,v) \in \gamma^{-1} \circ \beta^{-1} \Leftrightarrow (\exists) w \in \mathbb{N}\colon (u,v) \in \gamma^{-1} \wedge (w,v) \in \beta^{-1} \Leftrightarrow$
$\overset{(1),(2)}{\Longleftrightarrow} (\exists) w \in \mathbb{N}\colon w = u - 12 \wedge w \geq v \Leftrightarrow u - 12 \geq v \Leftrightarrow u \geq v + 12$,

i.e.

$$\gamma^{-1} \circ \beta^{-1} = \{(u,v) \in \mathbb{N}^2 | u \geq v + 12\}. \qquad (5)$$

4) $(s,t) \in (\beta \circ \gamma)^{-1} \Leftrightarrow (t,s) \in \beta \circ \gamma \overset{(4)}{\Leftrightarrow} t + 12 \leq s \overset{(5)}{\Leftrightarrow} (s,t) \in \gamma^{-1} \circ \beta^{-1}$.

In other words,

$$(\beta \circ \gamma)^{-1} = \gamma^{-1} \circ \beta^{-1}.$$

5) $(x,y) \in \gamma \circ \omega \Leftrightarrow (\exists) z \in \mathbb{N}\colon (x,z) \in \gamma \wedge (z,y) \in \omega \Leftrightarrow (\exists) z \in$
$\in \mathbb{N}\colon z - x = 12 \wedge z = 3y \Leftrightarrow (\exists) z \in \mathbb{N}\colon z = x + 12 \wedge z = 3y \Leftrightarrow$

$\Leftrightarrow x + 12 = 3y$, which implies

$\gamma \circ \omega = \{(x, y) \in \mathbb{N}^2 | x + 12 = 3y\}$.

6) $(u, v) \in \omega^{-1} \circ \omega \Leftrightarrow (\exists)\, w \in \mathbb{N}\colon (u, w) \in \omega^{-1} \wedge (w, v) \in \omega \Leftrightarrow$

$\overset{(3)}{\Leftrightarrow} (\exists)\, w \in \mathbb{N}\colon u = w/3 \wedge w = 3v \Leftrightarrow u = v \Leftrightarrow (u, v) \in 1_{\mathbb{N}}$,

which implies

$\omega^{-1} \circ \omega = 1_{\mathbb{N}}$.

5. We consider the binary relation defined on $\mathbb{R}$ as it follows:

$x\alpha y \Leftrightarrow (x = y \vee x + y = 2)$.

a) Prove that α is an equivalence relation.

b) Determine the factor set $\mathbb{R}/\alpha$.

Solution.

1) Because $x = x$, $(\forall)\, x \in \mathbb{R}$, we have $x\alpha x$, $(\forall)\, x \in \mathbb{R}$.

2) $a\alpha b \Rightarrow (a = b \vee a + b = 2) \Rightarrow (b = a \vee b + a = 2) \Rightarrow b\alpha a$.

3) Let $a\alpha b$ and $b\alpha c$, $a, b, c \in \mathbb{R}$. Then

$(a\alpha b \wedge b\alpha c) \Rightarrow ((a = b \vee a + b = 2) \wedge (b = c \vee b + c = 2) \Rightarrow$
$\Rightarrow ((a = b \wedge b = c) \vee (a = b \wedge b + c = 2) \vee (a + b = 2 \wedge b = c) \vee (a + b =$
$= 2 \wedge b + c = 2)) \Rightarrow (a = c \vee a + c = 2) \Rightarrow a\alpha c$.

In other words, the binary relation α is reflexive, symmetrical and transitive, which proves that α is an equivalence relation on $\mathbb{R}$.

b) Let $\alpha \in \mathbb{R}$. We determine the class $\alpha_a$of equivalence of a in connection to α.

$\alpha_a = \{x \in \mathbb{R} | x\alpha a\} = \{x \in \mathbb{R} | x = a \vee x + a = 2\} =$
$= \{x \in \mathbb{R} | x = a \vee x = 2 - a\} = \{a, 2 - a\}$.

Then the set factor is

$\mathbb{R}/\alpha = \{\alpha_x | x \in \mathbb{R}\} = \{\{x, 2 - x\} | x \in \mathbb{R}\}$.

We observe that

$|\alpha_a| = 1 \Leftrightarrow a = 1$; $|\alpha_a| = 2 \Leftrightarrow a \neq 1$; $(a \neq b \Rightarrow (\alpha_a = \alpha_b \Leftrightarrow a + b = 2))$.

In other words, for $a \neq 1$, we have $a \neq 2 - a$ and therefore $\alpha_a = \alpha_{2-a}$, and the factor can also be written

$\mathbb{R}/\alpha = \{\alpha_a | a \in \mathbb{R},\, a \geq 1\} = \{\{a, 2 - a\} | a \geq 1\}$.

6. Let $A = \{1,2,3,4,5\}$ and $B = \{a,b,c,d,e\}$. Which of the diagrams in Fig. 2.3 represents a function defined on A with values in B and which doesn't?

Solution. The diagrams a), c), e) reveal the functions defined on A with values in B. Diagram b) does not represent a function defined on A with values in B, because the image of 2 is not indicated. Diagram d) also does not represent a function defined on A with values in B, because 1 has two corresponding elements in B, namely a, d.

7. Let $A = \{1,2,3\}$ and $B = \{a,b\}$. Write all the functions defined on A with values in B, indicating the respective diagram.

Solution. There are eight functions defined on A with values in B. Their diagrams are represented in Fig. 2.4.

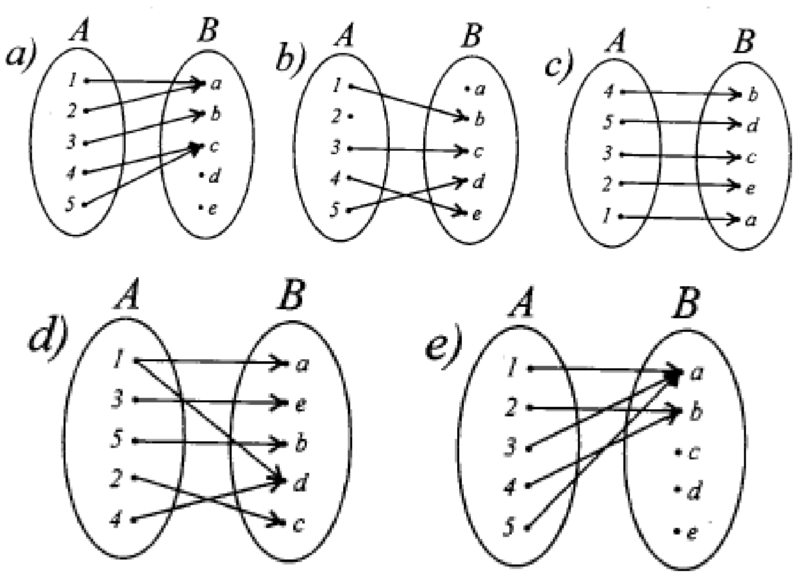


Figure 2.3.

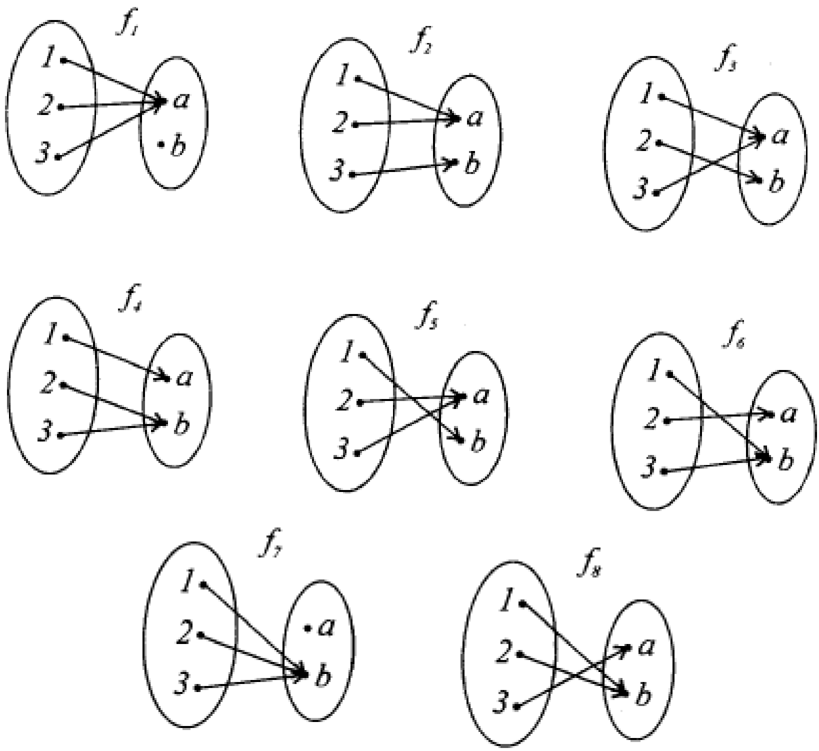


Figure 2.4.

8. Let $A = \{1, 2, 3, 4\}$, $B = \{a, b, c\}$, $C = \{x, y, z, t\}$ and $D = \{1, 2\}$.

a) Draw the corresponding diagrams for two surjective functions defined on A with values in$B.$

b) Draw the diagram of the injective functions defined on D with values in $B.$

c) Draw the diagram of a function denied on B with values in C, that is not injective.

d) Draw the corresponding diagrams for two bijective functions defined on A with values in C.

Solution. a) The diagrams of two surjective and two non-surjective functions defined on A with values in B are represented in Fig. 2.5 a) and b).

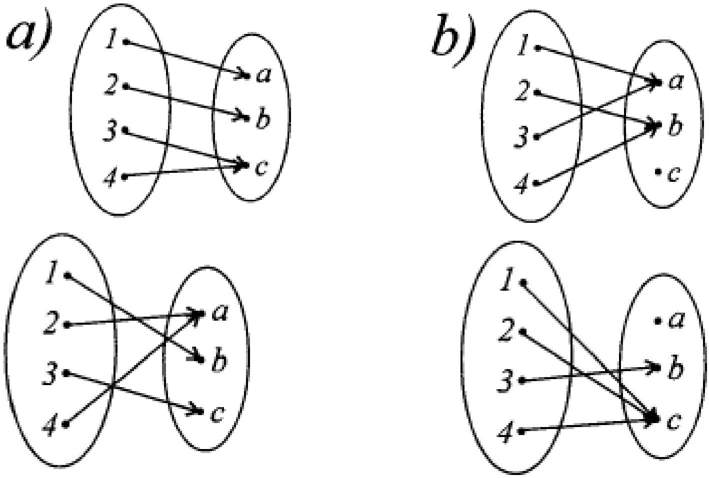
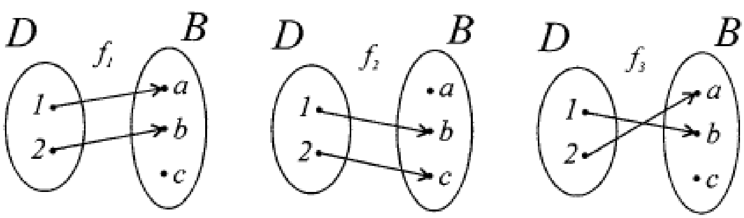


Figure 2.5.

b) The diagram of the injective functions defined on D with values in B are represented in Fig. 2.6.

c) The diagram of a non-injective function defined on B with values in C is represented in Fig. 2.7.

d) The diagrams of the two bijective functions defined on A with values in C are represented in Fig. 2.8.



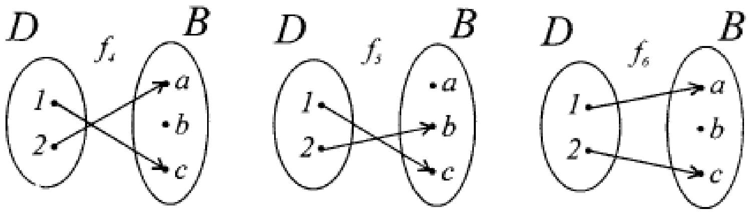


Figure 2.5.

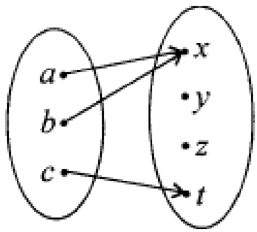


Figure 2.6.

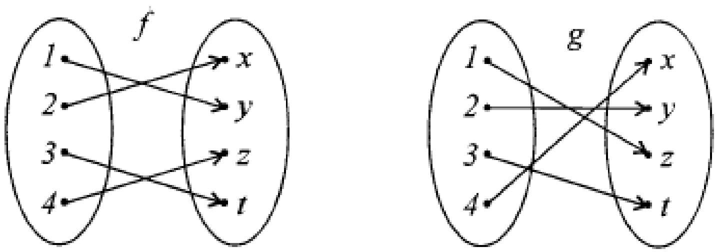


Figure 2.7.

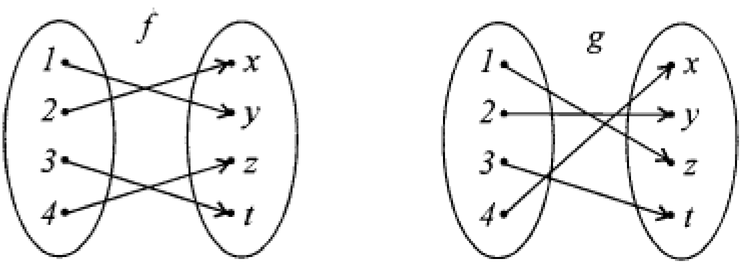


Figure 2.8.

9. Let $A = \{0,1,5,6\}$, $B = \{0,3,8,15\}$ and the function $f: A \longrightarrow B$ as defined by the equality $f(x) = x^2 - 4x + 3, (\forall)\, x \in A.$

a) Is the function f surjective?

b) Is f injective?

c) Is f bijective?

Solution. a) We calculate $f(A) = \{f(0), f(1), f(5), f(6)\} = \{3,0,8,15\} = B.$ So, f is surjective.

b) The function f is also injective, because $f(0) = 3$, $f(1) = 0$, $f(5) = 8$ and $f(6) = 15$, i.e. $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

c) The function f is bijective.

10. Using the graphic of the function $f: A \longrightarrow B$, $f: \mathbb{R} \longrightarrow \mathbb{R}$, show that the function f is injective, surjective or bijective.

a) $f(x) = \begin{cases} x - 2, if\ x \in (-\infty, 2), \\ 3x - 6, if\ x \in [2, +\infty). \end{cases}$

b) $f\colon [1;5] \longrightarrow [-1;3]$, $f(x) = |x^2 - 6x + 8|$.

c) $f: [-1, +\infty] \longrightarrow \mathbb{R}, f(x) = \begin{cases} -3x, if\ -1 < x \leq 0, \\ -x, if\ 0 < x \leq 1, \\ -0.5(x+1), if\ x > 1. \end{cases}$

d) $f\colon I\!R \longrightarrow [0, +\infty)$, $f(x) = \max(x+1, 1-x)$.

Solution. If any parallel to the axis of the abscises cuts the graphic of the function in one point, at most (namely, it cuts it in one point or not at all), then the function is injective. If there is a parallel to the abscises' axis that intersects the graphic of the function in two or more points, the function in not injective. If $E(f)$ is the values set of the function f and any parallel to the abscises' axis, drawn through the ordinates axis that are contained in $E(f)$, cuts the graphic of the function f in at least one point, the function is surjective. It follows that a function f is bijective if any parallel to the abscises' axis, drawn through the points of $E(f)$, intersects the graphic of f in one point.

a) The graphic of the function f is represented in Fig. 2.9.

We have $E(f) = (-\infty, +\infty)$ and any parallel $y = m, m \in \mathbb{R}$ to the abscises' axis intersects the graphic of the function f in one point. Consequently, f is a bijectve function.

$$f(x) = \begin{cases} x^2 - 6x + 8, if\ x^2 - 6x + 8 \geq 0, \\ -x^2 + 6x - 8, if\ x^2 - 6x + 8 < 0 \end{cases} =$$

$$= \begin{cases} x^2 - 6x + 8, \text{if}\ x \in (-\infty, 2] \cup [4, +\infty), \\ -x^2 + 6x - 8, \text{if}\ x \in (2,4). \end{cases}$$

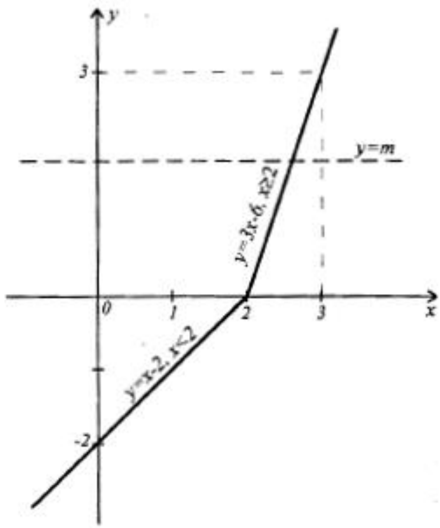


Figure 2.9.

The graphic of the function f is represented in Fig. 2.10.

We have $E(f) = [0; 3] \subset [-1; 3]$. Any parallel $y = m, m \in (0,1)$ intersects the graphic of the function f in four points; the parallel $y = 1$ intersects the graphic in three points; any parallel $y = m, m \in (1; 3]$ intersects the graphic of the function f in two points. So the function f is neither surjective, nor injective.

c) The graphic of the function is represented in Fig. 2.11. We have

$$D(f) = [-1, +\infty), \ E(f) = (-\infty, 3].$$

Any parallel $y = m$ to the abscises' axis cuts the graphic of the function $f(x)$ in one point, at most ($y = -3, y = 2, y = 4$) and that is why $f(x)$ is injective. The equation $f(x) = r \in \mathbb{R}$ has a solution only for $r \leq 3$, hence $f(x)$ is not a surjective function.

d) We explain the function f:

$$f(x) = \max(x+1, 1-x) = \begin{cases} x+1, if\ 1-x \leq x+1, \\ 1-x, if\ x+1 < 1-x \end{cases} =$$
$$= \begin{cases} x+1, if\ x \geq 0, \\ 1-x, if\ x < 0. \end{cases}$$

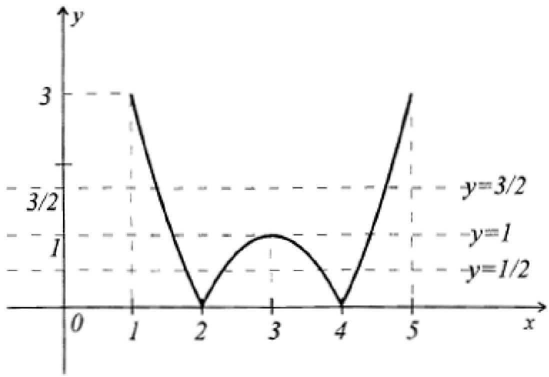


Figure 2.10.

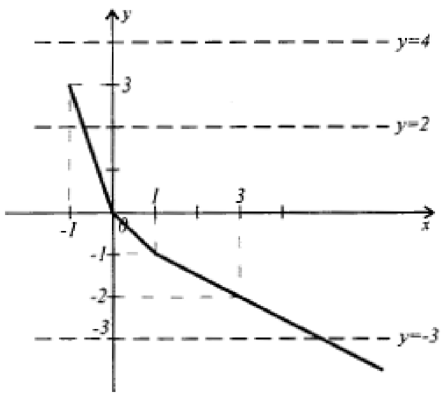


Figure 2.11.

The graphic of the function f is represented in Fig. 2.12.

We have $D(f) = \mathbb{R}$, $E(f) = [1, +\infty)$. Any parallel $y = m$, $m \in (1, +\infty)$ intersects the graphic in two points, therefore f is not injective. Straight line $y = 1/2 \in [0, +\infty)$ doesn't intersect the graphic of the function f, so f is not surjective either.

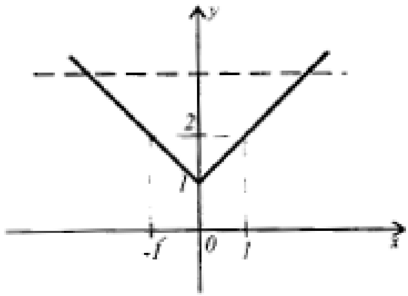


Figure 2.12.

11. Determine which of the following relations are applications, and which ones are injective, surjective, bijective?

a) $\varphi = \{(x, y) \in |N^2| x - y = 3\}$;

b) $f = \{(x, y) \in [-1; 0] \times [-1; 1] | x^2 + y^2 = 1\}$;

c) $g = \{(x, y) \in [0, +\infty) \times (-\infty, +\infty) | y = x^2\}$;

d) $\psi = \{(x, y) \in [0, +\infty)^2 | y = x^2\}$.

For the bijective function, indicate its inverse.

Solution.

a) $(x, y) \in \varphi \Leftrightarrow x - y = 3 \Leftrightarrow x = y + 3$.

Because

$D(\varphi) = \delta_\varphi = \{x \in I\!N \mid (\exists) y \in I\!N : x = y + 3\} \neq I\!N$

the equation $2 = y + 3$ doesn't have solutions in $\mathbb{N}$, it follows that φ is not a functional relation.

b) $(x, y) \in f \Leftrightarrow x^2 + y^2 = 1$.

Then, because $D(f) = \delta_f = [-1; 0]$, and

$$\left.\begin{array}{l}\left(-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2=1,\\ \left(-\frac{1}{2}\right)^2+\left(-\frac{\sqrt{3}}{2}\right)^2=1,\end{array}\right\} \Rightarrow \left\{\begin{array}{l}\left(-\frac{1}{2},\frac{\sqrt{3}}{2}\right)\in f,\\ \left(-\frac{1}{2},-\frac{\sqrt{3}}{2}\right)\in f\end{array}\right.$$

with $\sqrt{3}/2 \neq -\sqrt{3}/2$, and f is not a functional relation.

c) We have: $(x,y) \in g \Leftrightarrow xgy \Leftrightarrow y = g(x) = x^2$. Then:

1) $x \in [0,+\infty) \Rightarrow y = x^2 \in \mathbb{R} \Rightarrow (x,x^2) \in g$;

2) $(xgy_1 \wedge xgy_2) \Rightarrow y_1 = x^2 = y_2$;

3) $g(x_1) = g(x_2) \Rightarrow x_1^2 = x_2^2 \Rightarrow |x_1| = |x_2| \Rightarrow x_1 = x_2$, because $x_1, x_2 \in (0,+\infty)$;

4) The equation $g(x) = r \in \mathbb{R}$ has solutions in $[0,+\infty)$, if and only if $r \geq 0$, which proves that g is not a surjection (number -2, for example, doesn't have a preimage in $[0,+\infty)$.

d) We have:

$$(x,y) \in \psi \Leftrightarrow x\psi y \Leftrightarrow y = \psi(x) = x^2$$

Repeating the reasoning from c), we obtain that ψ is an injection. Furthermore, the equation $\psi(x) = r \in [0,+\infty)$ has the solution $x = \sqrt{r} \in [0,+\infty)$, and that is why ψ is a surjective application. Then ψ is a bijective application and, in accordance with the 6th Theorem, the relation ψ^{-1} is also an application (function).

In accordance with the same 6th Theorem, we have

$$\psi^{-1}(x) = \sqrt{x},\ x \in [0,+\infty).$$

12. Let $\{A = 1,2,3,4,5,6,7,8,9\}$ and $\varphi \in F(A)$ given; using the following table:

x	1	2	3	4	5	6	7	8	9
$\varphi(x)$	2	2	1	3	4	5	3	2	1

determine:

a) $\varphi(\{2,3,5\})$; $\varphi(\{1,3,7,9\})$; $\mathcal{I}m\varphi$.

b) $\varphi^{-1}(\{1,2,3,4,5\})$; $\varphi^{-1}(\{2,3\})$; $\varphi^{-1}(\{7,8,9\})$.

c) Calculate $\varphi^{-1}(1)$; $\varphi^{-1}(4)$; $\varphi^{-1}(7)$.

Solution.

a) $\varphi(\{2,3,5\}) = \{\varphi(2), \varphi(3), \varphi(5)\} = \{2,1,4\}$;

$\varphi(\{1,3,7,9\}) = \{\varphi(1), \varphi(3), \varphi(7), \varphi(9)\} = \{2,1,3,1\} = \{1,2,3\}$;

$\mathcal{I}m\varphi = \varphi(A) = \{\varphi(1), \varphi(2), \varphi(3), \varphi(4), \varphi(5), \varphi(6), \varphi(7), \varphi(8), \varphi(9)\} =$
$= \{1,2,3,4,5\}$.

b) $\varphi^{-1}(\{1,2,3,4,5\}) = \{y \in A | \varphi(y) \in \{1,2,3,4,5\}\} = A$;

$\varphi^{-1}(\{2,3\}) = \{a \in A | \varphi(a) \in \{2,3\}\} = \{1,2,4,7,8\}$;

$\varphi^{-1}(\{7,8,9\}) = \{b \in A | \varphi(b) \in \{7,8,9\}\} = \varnothing$.

c) $\varphi^{-1}(1) = \{a \in A | \varphi(a) = 1\} = \{3,9\}$;
$\varphi^{-1}(4) = \{b \in A | \varphi(b) = 4\} = \{5\}$;
$\varphi^{-1}(7) = \{c \in A | \varphi(c) = 7\} = \varnothing$.

13. Let $A = \{1,2,3\}$ and $B = \{a,b,c\}$. We examine the relations

$\alpha = \{(1,a),(2,b),(3,a),(3,c)\}, \beta = \{(1,b),(2,a),(3,c)\}$,
$\gamma = \{(2,a),(3,c),(1,c)\}$.

a) Determine $\delta_\alpha, \delta_\beta, \delta_\gamma$, and $\rho_\alpha, \rho_\beta, \rho_\gamma$.

b) Which of the relations α, β *și* γ are applications? Indicate the application type.

c) Determine the relations α^{-1}, β^{-1} and γ^{-1}. Which one is a function?

d) Determine the relations $\alpha \circ \beta^{-1}, \beta \circ \alpha^{-1}, \alpha \circ \gamma^{-1}, \gamma \circ \alpha^{-1}, \beta \circ \gamma^{-1}$ and $\gamma \circ \beta^{-1}$. Which ones are applications?

Solution.

a) $\delta_\alpha = \{1,2,3\} = A = \delta_\beta = \delta_\gamma$; $\rho_\alpha = \{a,b,c\} = B = \rho_\beta$;
$\rho_\gamma = \{a,c\}$.

b) α is not an application, because element 3 has two images a and c. β is a bijective application; γ is an application, nor injective (the elements $3 \neq 1$ have the same image *c)*, nor surjective *(b* doesn't have a preimage).

c) $\alpha^{-1} = \{(a,1),(b,2),(a,3),(c,3)\}$, $\beta^{-1} = \{(b,1),(a,2),(c,3)\}$, $\gamma^{-1} = \{(a,2),(c,3),(c,1)\}$.

We have:

$\delta_{\alpha^{-1}} = B = \delta_{\beta^{-1}}$, $\delta_{\gamma^{-1}} = \{a,c\} \neq B$, $\rho_{\alpha^{-1}} = A = \rho_{\beta^{-1}} = \rho_{\gamma^{-1}}$.

α^{-1} is not an application, because the element a has two images 1 and 3; β^{-1} is a bijective application; $\gamma^{-1}$is not an application, because $\delta_{\gamma^{-1}} \neq B$ (the element b doesn't have an image);

d) $\alpha \circ \beta^{-1} = \{(1,2),(2,1),(3,2),(3,3)\}$;

$\beta \circ \alpha^{-1} = \{(1,2),(2,1),(2,3),(3,3)\}$;

$\alpha \circ \gamma^{-1} = \{(1,2),(3,2),(3,3),(3,1)\}$;

$\gamma \circ \alpha^{-1} = \{(2,1),(3,3),(1,3),(2,3)\}$;

$\beta \circ \gamma^{-1} = \{(2,2),(3,3),(3,1)\}$;

$\gamma \circ \beta^{-1} = \{(2,2),(3,3),(1,3)\}$.

Only the relation $\gamma \circ \beta^{-1}$is an application, neither injective, nor surjective.

14. Given the functions: $f, g\colon \mathbb{R} \longrightarrow \mathbb{R}$,

$$f(x) = \begin{cases} 3 - x, & if\ x \leq 2, \\ x - 1, & if\ x > 2, \end{cases} \quad g(x) = \max(x-2, 3-x),$$

show that $f = g$.

Solution. The functions f and g are defined on $\mathbb{R}$ and have values in $\mathbb{R}$. Let's show that $f(x) = g(x), (\forall)x \in \mathbb{R}$.

We explain the function g:

$$g(x) = \begin{cases} 3 - x, if\ x - 1 \leq 3 - x, \\ x - 1, if\ 3 - x < x - 1, \end{cases} = \begin{cases} 3 - x, if\ x \leq 2, \\ x - 1, if\ x > 2, \end{cases} = f(x)$$

no matter what $x \in \mathbb{R}$ is. So $f = g$.

15. The functions $f, g: \mathbb{R} \longrightarrow \mathbb{R}$,

$$f(x) = \begin{cases} x+2, \text{if } x \le 2, \\ \frac{x+10}{3}, \text{if } x > 2, \end{cases} \quad g(x) = \begin{cases} x-2, \text{if } x < 3, \\ 2x-5, \text{if } x \ge 3 \end{cases}$$ are given.

a) Show that f and g are bijective functions.

b) Represent graphically the functions f and f^{-1} in the same coordinate register.

c) Determine the functions: $s = f + g, d = f - g, p = f.g, q = f/g$.

d) Is function d bijective?

Solution. a), b) The graphic of the function f is represented in Fig. 2.13 with a continuous line, and the graphic of f^{-1} is represented with a dotted line.

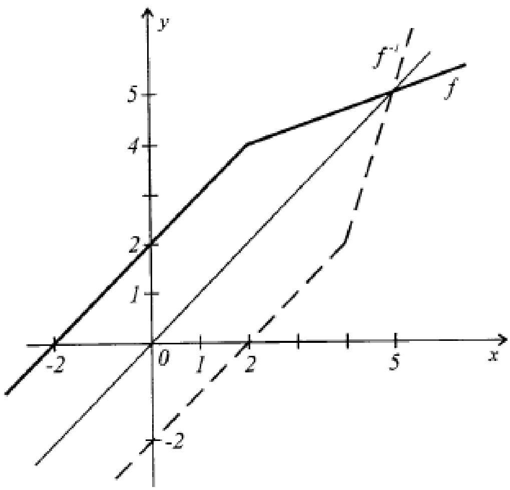


Figure 2.13.

We have:

$g^{-1}, f^{-1}: \mathbb{R} \longrightarrow \mathbb{R}$,

$$f^{-1}(x) = \begin{cases} x-2, & \text{if } x \le 4, \\ 3x-10, & \text{if } x > 4, \end{cases} \quad g(x) = \begin{cases} x+2, & \text{if } x < 1, \\ \frac{1}{2}(x+5), & \text{if } x \ge 1. \end{cases}$$

c) We draw the following table:

x	$(-\infty, 2]$	$(2,3)$	$[3, +\infty)$
f	$x+2$	$(x+10)/3$	$(x+10)/3$
g	$x-2$	$x-2$	$2x-5$
$f+g$	$2x$	$4(x+1)/3$	$(7x-5)/3$
$f-g$	4	$(16-2x)/3$	$(25-5x)/3$
$f \cdot g$	x^2-4	$(x^2+8x-20)/3$	$(2x^2+15x-50)/3$
f/g	$\frac{x+2}{x-2}, x \neq 2$	$\frac{x+10}{3(x-2)}$	$\frac{x+10}{3(2x-5)}$

We have

$$s(x) = \begin{cases} 2x, if\ x \leq 2, \\ \frac{4}{3}(x+1), if\ 2 < x < 3, \\ \frac{1}{3}(7x-5), if\ x \geq 3, \end{cases}$$

$$d(x) = \begin{cases} \frac{4}{2}, if\ x \leq 2, \\ \frac{2}{3}(8-x), if\ 2 < x < 3, \\ \frac{5}{3}(5-x), if\ x \geq 3, \end{cases}$$

$$p(x) = \begin{cases} x^2-4, if\ x \leq 2, \\ \frac{1}{3}(x^2+8x-20), if\ 2 < x < 3, \\ \frac{1}{3}(2x^2+15x-50), if\ x \geq 3, \end{cases}$$

$$q(x) = \begin{cases} \frac{x+2}{x-2}, if\ x \leq 2, \\ \frac{x+10}{3(x-2)}, if\ 2 < x < 3, \\ \frac{x+10}{3(2x-5)}, if\ x \geq 3. \end{cases}$$

d) We have $d(x) = 4$, $(\forall)x \in (-\infty, 2]$, therefore d is not injective, in other words, d is not bijective. This proves that the difference (sum) of two bijective functions isn't necessarily a bijective function.

16. Let's consider the function $f: \mathbb{R} \setminus \{-d/c\} \longrightarrow \mathbb{R} \setminus \{a/c\}$, $f(x) = (ax+b)/(cx+d)$, $ad - bc \neq 0$, $c \neq 0$.

a) Show that function f is irreversible.

b) Determine f^{-1}.

c) In what case do we have $f = f^{-1}$?

Solution:

a) 1) $f(x_1) = f(x_2) \Leftrightarrow \dfrac{ax_1+b}{cx_1+d} = \dfrac{ax_2+b}{cx_2+d} \Leftrightarrow$

$\Leftrightarrow acx_1x_2 + adx_1 + bcx_2 + bd = acx_1x_2 + bcx_1 + adx_2 + bd \Leftrightarrow$

$\Leftrightarrow adx_1 + bcx_2 = adx_2 + bcx_1 \Leftrightarrow ad(x_1 - x_2) - bc(x_1 - x_2) = 0 \Leftrightarrow$
$\Leftrightarrow (ad - bc)(x_1 - x_2) = 0 \Leftrightarrow x_1 = x_2 \Rightarrow f$ is injective.

2) Let $x \in \mathbb{R} \setminus \{a/c\}$. We examine the equation $f(x) = r$. Then
$f(x) = r \Leftrightarrow (ax+b)/(cx+d) = r \Leftrightarrow ax + b = crx + dr \Leftrightarrow$
$\Leftrightarrow (a - cr)x = dr - b \overset{r \neq \frac{a}{c}}{\Longleftrightarrow} x = (dr - b)/(a - cr)$.

If
$x = (dr-b)/(a-cr) = -d/c \Leftrightarrow cdr - bc = -ad + cdr \Leftrightarrow ad - bc = 0$.

Imposibile. So $(dr - b)/(a - cr) \in \mathbb{R} \setminus \{-d/c\}$, which proves that the equation $f(x) = r$ has solutions in $\mathbb{R} \setminus \{-d/c\}$ for any $\mathbb{R} \setminus \{a/c\}$. This proves that f is a surjective function.

From 1) - 2), it follows that f is a bijective function, hence it is also inversibile.

b) We determine

$f^{-1}: \mathbb{R} \setminus \{a/c\} \longrightarrow \mathbb{R} \setminus \{-d/c\}$:

$f(x) = \dfrac{ax+b}{cx+d} \Leftrightarrow (cf(x) - a)x = b - df(x) \overset{f(x) \neq \frac{a}{c}}{\Longleftrightarrow} x = \dfrac{-df(x)+b}{cf(x)-a} \Rightarrow$

$$\Rightarrow f^{-1}(x) = \frac{-dx+b}{cx-a}.$$

c) In order to have $f = f^{-1}$, it is necessary that the functions f and f^{-1} have the same definition domain and so $d = -a$. This condition is also sufficient for the equality of the functions f and f^{-1}. It is true that, if $d = -a$, the functions f and $f^{-1}$are defined on $\mathbb{R} \setminus \{a/c\}$, take values in $\mathbb{R} \setminus \{a/c\}$, and $f^{-1}(x) = \frac{ax+b}{cx+d} = f(x), (\forall) x \in \mathbb{R} \setminus \left\{\frac{a}{c}\right\}$, namely $f = f^{-1}$.

17. Represent the function $f(x) = \sqrt{5x^2 - 2x + 8}$ as a compound of two functions.

Solution. We put

$u(x) = x^2 - 2x + 8$ şi $v(x) = \sqrt{x}$.

Then

$$f(x) = \sqrt{5x^2 - 2x + 8} = v(5x^2 - 2x + 8) = v(u(x)) = (v \circ u)(x).$$

Answer:

$$f(x) = (v \circ u)(x) \text{ cu } u(x) = 5x^2 - 2x + 8, \quad v(x) = \sqrt{x}.$$

18. Calculate $f \circ g, g \circ f, (f \circ g)(4)$ and $(g \circ f)(4)$, where:

a) $f(x) = \frac{3}{x-1}$ and $g(x) = \sqrt{x}$;

b) $f(x) = \sqrt{x+3}$ and $g(x) = x^2 - 4$;

c) $f(x) = \begin{cases} x^2 + 6x, & if \ \ x \leq -3 \\ -2x - 5, & if \ x \geq -3 \end{cases}$

and $g(x) = \begin{cases} 5x - 2, & if \ x \leq 1, \\ x^2 - 2x + 4, & if \ x > 1. \end{cases}$

Indicate $D(f \circ g)$ and $D(g \circ f)$.

Solution: a) We have

$$(f \circ g)(x) = f(g(x)) = f(\sqrt{x}) = \frac{3}{\sqrt{x} - 1};$$

$$(f \circ g)(4) = \frac{3}{\sqrt{4} - 1} = 3; \ D(f \circ g) = [0; 1) \cup (1, +\infty).$$

$$(g\circ f)(x)=g(f(x))=g\left(\frac{3}{x-1}\right)=\sqrt{\frac{3}{x-1}};\ (g\circ f)(4)=\sqrt{\frac{3}{4-1}}=1;$$
$$D(g\circ f)=(1,+\infty).$$

So $(g\circ f)(4)\neq(f\circ g)(4)$, which proves that the comutative law of function composition, generally, does not take place: $f\circ g\neq g\circ f$.

b) $(f\circ g)(x)=f(g(x))=\sqrt{(x^2-4)+3}=\sqrt{x^2-1}$;
$x\in D(f\circ g)\Leftrightarrow x^2-1\geq 0\Leftrightarrow x\in(-\infty,-1]\cup[1,+\infty)=D(f\circ g)$.
$(g\circ f)(x)=g(f(x))=g(\sqrt{x+3})=(\sqrt{x+3})^2-4=x-1;\ D(g\circ f)=\mathbb{R}$.
$(f\circ g)(4)=\sqrt{4^2-1}=\sqrt{15},\ (g\circ f)(4)=4-1=3$.

c) To determine the functions $f\circ g$ and $g\circ f$, in this particular case we proceed as it follows:

$$(f\circ g)(x)=(f\circ g)\big(g(x)\big)==\begin{cases}g^2(x)+6g(x), & if\ g(x)<-3\\ -2g(x)-5, & if\ g(x)\geq-3\end{cases}=$$

$$=\begin{cases}g^2(x)+6g(x)\ \ if\ g(x)<-3,\\ -2g(x)-5,\ \ if\ g(x)\geq-3 \text{ and } x\leq 1,\\ -2(x^2-2x+4)-5,\ \ if\ (x)\geq-3 \text{ and } x>1\end{cases}=$$

$$\begin{cases}(5x-2)^2+6(5x-2)\ \ if\ 5x-2<-3,\\ -2(5x-2)-5,\ \ if\ 5x-2\geq-3 \text{ and } x\leq 1,\\ -2(x^2-2x+4)-5,\ \ if\ 5x-2\geq-3 \text{ and } x>1\end{cases}=$$

$$=\begin{cases}25x^2+10x-8\ \ if\ x<-1/5,\\ -10x-1,\ \ if\ -1/5\leq\ x\leq 1,\\ 2x^2+4x-13,\ \ if\ \ x>1.\end{cases}$$

$$(f\circ g)(4)=-2\cdot4^2+4\cdot4-13=-29.$$

$$(g\circ f)(x)=g\big(f(x)\big)=\begin{cases}5f(x), & if\ f(x)\leq 1,\\ f^2(x)-sf(x)+4, & if\ f(x)>1.\end{cases}$$

We notice that

$$f(x)\leq 1\Leftrightarrow\left[\begin{array}{l}\begin{cases}x^2+6x\leq 1,\\ x<-3\end{cases}\\ \begin{cases}-2x-5\leq 1,\\ x\geq-3\end{cases}\end{array}\right.\Leftrightarrow\left[\begin{array}{l}\begin{cases}x\in[-3-\sqrt{10},-3+\sqrt{10}],\\ x<-3\end{cases}\\ \begin{cases}-2x\leq 6,\\ x\geq-3\end{cases}\end{array}\right.\Leftrightarrow$$

$$\Leftrightarrow\left[\begin{array}{l}x\in[-3-\sqrt{10};-3),\\ x\geq-3\end{array}\right.\Leftrightarrow x\in[-3-\sqrt{10};-3)\cup[-3,+\infty).$$

So for $f(x) \leq 1$, we obtain

$$(g \circ f)(x) = \begin{cases} 5(x^2 + 6x) - 2, & \text{if } x \in [-3 - \sqrt{10}, -3) \\ 5(-2x - 5) - 2, & \text{if } x \in [-3, +\infty). \end{cases}$$

Similarly,

$$f(x) > 1 \Leftrightarrow \left[\begin{array}{l} \begin{cases} x^2 + 6x > 1, \\ x < -3 \end{cases} \\ \begin{cases} -2x - 5 > 1, \\ x \geq -3 \end{cases} \end{array} \right. \Leftrightarrow \left[\begin{cases} x \in (-\infty, -3 - \sqrt{10}), \\ x \in \varnothing \end{cases} \right.$$

and that is why for $f(x) > 1$, we obtain

$$(g \circ f)(x) = (x^2 + 6x)^2 - 2(x^2 + 6x) + 4 = x^4 + 12x^3 + 34x^2 - 12x + 4$$

for $x \in (-\infty, -3 - \sqrt{10})$.

To sum it up, we have

$$(g \circ f)(x) =$$

$$= \begin{cases} x^4 + 12x^3 + 34x^2 - 12x + 4, & if\ x \in (-\infty, -3 - \sqrt{10}), \\ 5x^2 + 30x - 2, & \text{if } x \in [-3\sqrt{10}, -3), \\ -10x - 7, & \text{if } x \geq -3. \end{cases}$$

$(g \circ f)(4) = -10.4 - 27 = -67.$

19. Solve the following equations:

a) $(g \circ f \circ f)(x) = (f \circ g \circ g)(x)$, if $f(x) = 3x + 1, g(x) = x + 3$;

b) $(f \circ f \circ f)(x) = x$, if $f(x) = \frac{ax+1}{x+a}, a \in \mathbb{R}, x \in \mathbb{R} \setminus \{-a\}$;

c) $(f \circ g)(x) = (g \circ f)(x)$, if $f(x) = 2x^2 - 1, g(x) = 4x^3 - 3x$.

Solution. We determine the functions $(g \circ f \circ f)(x)$ and $(f \circ g \circ g)(x)$:

$$(g \circ f \circ f)(x) = g(f(f(x))) = g(f(3x + 1)) = g(3(3x + 1) + 1) =$$
$$= g(9x + 4) = 9x + 4 + 3 = 9x + 7;$$

$$(f \circ g \circ g)(x) = f(g(g(x))) = f(g(x + 3)) = f((x + 3) + 3) =$$
$$= f(x + 6) = 3(x + 6) + 1 = 3x + 19.$$

The equation becomes

$$9x + 7 = 3x + 19 \Leftrightarrow x = 2.$$

Answer: $x = 2$.

b) We calculate $(f \circ f \circ f)(x)$:

$$(f \circ f \circ f)(x) = f(f(f(x))) = f\left(f\left(\frac{ax+1}{x+a}\right)\right) =$$

$$= f\left(\frac{a \cdot \frac{ax+1}{x+a} + 1}{\frac{ax+1}{x+a} + a}\right) = f\left(\frac{ax^2 + x + 2a}{2ax + 1 + a^2}\right) =$$

$$= \frac{a \cdot \frac{a^2x + x + 2a}{2ax + 1 + a^2} + 1}{\frac{a^2x + x + 2a}{2ax + 1 + a^2} + a} = \frac{a^3x + 3ax + 3a^2 + 1}{3a^2x + x + 3a + a^3} =$$

$$= ((a^3 + 3a)x + (3a^2 + 1))/((3a^2 + 1)x + (a^3 + 3a)).$$

The equation becomes

$$\frac{(a^3 + 3a)x + (3a^2 + 1)}{(3a^2 + 1)x + (a^3 + 3a)} = x \Leftrightarrow (3a^2 + 1)x^2 = 3a^2 + 1 \Leftrightarrow$$

$$\Leftrightarrow x^2 = 1 \Leftrightarrow \left[\begin{array}{l} x = -1, \\ x = 1. \end{array}\right.$$

Answer: $x \in \{-1,1\}$

c) We determine $(f \circ g)$ and $(g \circ f)(x)$:

$$(f \circ g)(x) = f(g(x)) = f(4x^3 - 3x) = 2(4x^3 - 3x)^2 - 1 =$$
$$= 32x^6 - 48x^4 + 18x^2 - 1.$$

$$(g \circ f)(x) = g(f(x)) = g(2x^2 - 1) = 4(2x^2 - 1)^3 - 3(2x^2 - 1) =$$
$$= 4(8x^6 - 12x^4 + 6x^2 - 1) - 6x^2 + 3 = 32x^6 - 48x^4 + 18x^2 - 1$$

The equation becomes

$$32x^6 - 48x^4 + 18x^2 - 1 = 32x^6 - 48x^4 + 18x^2 - 1 \Leftrightarrow 0 = 0,$$

meaning the equality is true for any $x \in \mathbb{R}$.

Answer: $x \in \mathbb{R}$.

20. Let $f, g: \mathbb{R} \longrightarrow \mathbb{R}$ given by $f(x) = x^2 + x + 12$ and $g(x) = x^2 - x + 2$. Show there is no function $\varphi: \mathbb{R} \longrightarrow \mathbb{R}$, so that

$$(\varphi \circ f)(x) + (\varphi \circ g)(x) = (g \circ f)(x), \ (\forall)\ x \in \mathbb{R}. \qquad (A)$$

Solution. The relation (A) can also be written

$$\varphi(x^2 + x + 2) + \varphi(x^2 - x + 2) = (x^2 + x + 2)^2 - (x^2 + x + 2) + 2. \ (A')$$

We suppose there is a function $\varphi: \mathbb{R} \longrightarrow \mathbb{R}$ that satisfies the relation (A'). We put in (A') $x = 1$ and $x = -1$. We obtain:

$\varphi(4) + \varphi(2) = 14, \ \ \varphi(2) + \varphi(4) = 4,$ which implies $\varphi(4) + \varphi(2) \neq \varphi(2) + \varphi(4)$, contradiction with the hypothesis. So there is no function φ with the property from the enunciation.

21. Determine all the values of the parameters $a, b \in \mathbb{R}$ for which $(f \circ g) = (g \circ f)(x)$, $(\forall) \in \mathbb{R}$, where $f(x) = x^2 - x$ and $g(x) = x^2 + ax + b$.

Solution. We determine $f \circ g$ and $g \circ f$:

$$(f \circ g)(x) = f(g(x)) = f(x^2 + ax + b) = (x^2 + ax + b)^2 - (x^2 + ax + b) =$$
$$= x^4 + 2ax^3 + (2b - 1)x^2 + a^2x^2 + b^2 - ax - b + 2abx =$$
$$= x^4 + 2ax^3 + (a^2 + 2b - 1)x^2 + (2ab - a)x + b^2 - b.$$

$$(g \circ f)(x) = g(f(x)) = g(x^2 - x) = (x^2 - x)^2 + a(x^2 - x) + b =$$
$$= x^4 - 2x^3 + (a + 1)x^2 - ax + b.$$

Then

$$(f \circ g)(x) = (g \circ f)(x), \ (\forall)\ x \in R \Leftrightarrow x^4 + 2ax^3 + (a^2 + 2b - 1)x^2 +$$
$$+ (2ab - a)x + b^2 - b = x^4 - 2x^3 + (a + 1)x^2 - ax + b \Leftrightarrow$$
$$\overset{(\forall)x \in R}{\Longleftrightarrow} \begin{cases} 2a = -2, \\ a^2 + 2b - 1 = a + 1, \\ 2ab - a = -a, \\ b = b^2 - b \end{cases} \Leftrightarrow \begin{cases} a = -1, \\ b = 0. \end{cases}$$

Answer: $a = -1, b = 0, g(x) = x^2 - x = f(x)$.

22. Given the functions $f, g, h : \mathbb{R} \longrightarrow \mathbb{R}$,

$f(x) = x^4 + 4x^3 + 3, \ g(x) = x^3 + x + 3$ şi $h(x) = x^3 + 8$.

Show that:

a) f is not injective;

b) g is injective;

c) h is bijective and determine h^{-1}.

Solution. a) Let $f(x_1) = f(x_2)$.

Then

$$x_1^4 + 4x_1^3 + 3 = x_2^4 + 4x_2^3 + 3 \Leftrightarrow (x_1^2 - x_2^2)(x_1^2 + x_2^2) + 4(x_1 - x_2)\times$$
$$\times(x_1^2 + x_1x_2 + x_2^2) = 0 \not\Rightarrow x_1 = x_2.$$

For example, $f(x) = x^3(x+4) + 3$, taking $x_1 = 0$ and $x_2 = -4$, we obtain $f(0) = f(-4) = 3$, $x_1 \neq x_2$.

$$\text{b) } g(x_1) = g(x_2) \Leftrightarrow x_1^3 + x_1 + 3 = x_2^3 + x_2 + 3 \Leftrightarrow$$
$$\Leftrightarrow (x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2 + 1) = 0 \Leftrightarrow x_1 = x_2,$$

because

$$x_1^2 + x_1x_2 + x_2^2 + 1 > 0,\ (\forall)\ x_1, x_2 \in \mathbb{R}.$$

$$\text{c) } h(x_1) = h(x_2) \Leftrightarrow x_1^3 + 8 = x_2^3 + 8 \Leftrightarrow x_1^3 = x_2^3 \Leftrightarrow x_1 = x_2,$$

so h is an injective function.

We prove that h is surjective. Let $r \in \mathbb{R}$. We solve the equation

$$h(x) = r \Leftrightarrow x^3 + 8 = r \Leftrightarrow x^3 = r - 8 \Leftrightarrow x = \sqrt[3]{r-8}$$

(the root of an uneven order exists out of any real number). So h is a surjection, therefore, h is a bijective function.

We determine h^{-1}:

$$(y, x) \in G_{h^{-1}} \Leftrightarrow (x, y) \in G_h \Leftrightarrow h(x) = y = x^3 + 8 \Leftrightarrow x^3 = y - 8 \Leftrightarrow$$
$$\Leftrightarrow x = \sqrt[3]{y-8} \Leftrightarrow h^{-1}(x) = \sqrt[3]{x-8}.$$

23. Let $f: [1, +\infty) \longrightarrow [1, +\infty)$ with

$$f(x) = x^6 - 3x^5 + 6x^4 - 7x^3 + 6x^2 - 3x + 1.$$

a) Prove that f is a bijective function.

b) Determine f^{-1}.

Solution. a) We have $f(x) = (x^2 - x + 1)^3$. We represent this function as a compound of two functions:

u, v: $[1, +\infty) \longrightarrow [1, +\infty)$, where $u(x) = x^3$, $v(x) = x^2 - x + 1$. Then $f(x) = (u \circ v)(x)$, which implies that $f = u \circ v$ is bijective, being the compound of two bijective functions.

b) $f^{-1} = (u \circ v)^{-1} = v^{-1} \circ u^{-1}$. We determine v^{-1} and u^{-1}.
u^{-1}: $[1, +\infty) \longrightarrow [1, +\infty)$, $u^{-1}(x) = \sqrt[3]{x}$.

Because:
$v(x) = x^2 - x + 1 = y \Rightarrow x^2 - x + 1 - y = 0$.
We solve this equation according to $x \in [1, +\infty)$.
The discriminant is:
$D = 1 - 4(1 - y) = 4y - 3$; $x_{1,2} = (1 \mp \sqrt{4y - 3})/2$
$(y \geq 1 \Leftrightarrow 4y - 3 \geq 0!)$.
The equation has only one root in $[1, +\infty)$:
$x = (1 + \sqrt{4y - 3})/2$.

Then

$$v^{-1}(x) = (1 + \sqrt{4x - 3})/2.$$

So,

$$f^{-1}(x) = (v^{-1} \circ u^{-1})(x) = v^{-1}(u^{-1}(x)) = v^{-1}(\sqrt[3]{x}) =$$

$= (1 + \sqrt{4\sqrt[3]{x} - 3})/2$, with f^{-1}: $[1, +\infty) \longrightarrow [1, +\infty)$.

24. Considering the function $f: \mathbb{R} \longrightarrow \mathbb{R}$ with the properties:
1) $f(x_1 + x_2) = f(x_1) + f(x_2)$, $(\forall)\ x_1, x_2 \in \mathbb{R}$;
2) $f(1) = 1$;
3) $f(1/x) = 1/x^2 \cdot f(x)$, $(\forall)\ x \in \mathbb{R}^*$.
a) Determine the function f.
b) Calculate $f(\sqrt{1998})$.

Solution. a) For $x_2 = 0$ from 1), it follows that $f(x_1) = f(x_1) + f(0)$, which implies that $f(0) = 0$. For $x_2 = -x_1$ from 1), we obtain

$$f(0) = f(x_1) + f(-x_1) = 0 \Rightarrow f(-x_1) = -f(x_1) \Rightarrow f(x_2) =$$
$$= -f(-x_2) \Rightarrow f(-x_2) = -f(x_2).$$

Then

$$f(x_1 - x_2) = f(x_1 + (-x_2)) \overset{1)}{=} f(x_1) + f(-x_2) =$$
$$= f(x_1) - f(x_2),\ (\forall)\ x_1, x_2 \in \mathbb{R}. \qquad (1)$$

Let $x \notin \{0, 1\}$. Then

$$f\left(\frac{1}{1-x}\right) \overset{3)}{=} \frac{1}{(1-x)^2} \cdot f(1-x) \overset{(1)}{=} \frac{f(1)-f(x)}{(1-x)^2}. \qquad (2)$$

On the other hand, $\frac{1}{1-x} = \frac{1-x+x}{1-x} = 1 + \frac{x}{1-x}$ implies

$$f\left(\frac{1}{1-x}\right) = f\left(1+\frac{x}{1-x}\right) \overset{1)}{=} f(1) + f\left(\frac{x}{1-x}\right) = 1 + f\left(\frac{x}{1-x}\right) =$$

$$\overset{3)}{=} 1 + \left(\frac{x}{1-x}\right)^2 \cdot f\left(\frac{1-x}{x}\right) = 1 + \left(\frac{x}{1-x}\right)^2 \cdot \left[f\left(\frac{1}{x}-1\right)\right] =$$

$$\overset{1)}{=} 1 + \left(\frac{x}{1-x}\right)^2 \cdot \left(f\left(\frac{1}{x}\right) - f(1)\right) = 1 + \left(\frac{x}{1-x}\right)^2 \cdot \left(\frac{1}{x^2} \cdot f(x) - 1\right) =$$

$$= 1 + \frac{1}{(1-x)^2} \cdot f(x) - \frac{x^2}{(1-x)^2} = \frac{1-2x+f(x)}{(1-x)^2}. \qquad (3)$$

From (2) and (3) it follows that

$$\frac{f(1)-f(x)}{(1-x)^2} = \frac{1-2x+f(x)}{(1-x)^2} \Leftrightarrow f(x) = x.$$

So $f(x) = x, (\forall) \in \mathbb{R}$.

b) $f(\sqrt{1998}) = \sqrt{1998}$.

Answer: a) $f(x) = x$; b) $f(\sqrt{1998}) = \sqrt{1998}$.

25. Using the properties of the characteristic function, prove the equality:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C), \ A, B, C \in P(M).$$

Solution. Using the properties $A = B \Leftrightarrow f_A = f_B$, we will prove the required equality by calculating with the help of f_A, f_B and f_C the characteristic functions of sets $A \cup (B \cap C)$ and $(A \cup B) \cap (A \cup C)$:

$$f_{A\cup(B\cap C)} \overset{6)}{=} f_A + f_{B\cap C} - f_A \cdot f_{B\cap C} \overset{4)}{=} f_A + f_B \cdot f_C - f_A(f_B \cdot f_C) =$$
$$= f_A + f_B \cdot f_C - f_A \cdot f_B \cdot f_C.$$

$$f_{(A\cup B)\cap(A\cup C)} = f_{A\cup B} \cdot f_{A\cup C} \overset{6)}{=} (f_A + f_B - f_A \cdot f_B)(f_A + f_C - f_A \cdot f_C) =$$

$= f_A^2 + f_A \cdot f_C - f_A^2 \cdot f_C + f_B \cdot f_A + f_B \cdot f_C - f_B \cdot f_A \cdot f_C - f_A^2 \cdot f_B -$
$- f_A \cdot f_B \cdot f_C + f_A^2 \cdot f_B \cdot f_C \overset{3)}{=} f_A + f_B \cdot f_C - f_A \cdot f_B \cdot f_C = f_{A \cup (B \cap C)},$
which implies $A \cup (B \cap C)$ and $(A \cup B) \cap (A \cup C)$.

2.3. Suggested exercises

1. Determine the definition domains and the values domains of the following relations:

1) $\alpha = \{(2,4), (3,1), (2,-4), (0,27)\}$;
2) $\beta = \{(100,10), (200,20), (300,30), (400,40)\}$;
3) $\gamma = \{(1,5), (2,7), (3,9), (4,11)\}$;
4) $\delta = \{(1/2, 5)\}$;
5) $\rho = \{(-2,-5), (-2,0), (7,-2), (9,0)\}$;
6) $\omega = \{(-1,2), (-5,-2), (0,-2), (0,9)\}$.

2. Let $A = \{2,4,6,8\}$ and $B = \{1,3,5,7\}$, $\alpha \subseteq A \times B$.
a) Determine the graphic of the relation α.
b) Construct the schema of the relation α:
1) $\alpha = \{(x,y) \mid x < 3 \text{ and } y > 3\}$;
2) $\alpha = \{(x,y) \mid x > 2 \text{ and } y < 5\}$;
3) $\alpha = \{(x,y) \mid x > 6 \text{ or } y > 7\}$;
4) $\alpha = \{(x,y) \mid \max(x,y) \leq 3\}$;
5) $\alpha = \{(x,y) \mid \min(x,y) \leq 2\}$;
6) $\alpha = \{(x,y) \mid \min(x,y) > 6\}$;
7) $\alpha = \{(x,y) \mid \min(x,5) > \max(y,3)\}$;
8) $\alpha = \{(x,y) \mid \max(x,6) > \max(y,5)\}$.

3. Let $A = \{1,2,3,4\}$ and $B = \{1,3,5,7,8\}$. Write the graphic of the relation $\alpha \subseteq A \times B$, if:

1) $\alpha = \{(x,y)|x+y=9\}$;
2) $\alpha = \{(x,y)|2x-y=1\}$;
3) $\alpha = \{(x,y)|x^2-y^2=8\}$;
4) $\alpha = \{(x,y)|x-y \geq 3\}$;
5) $\alpha = \{(x,y)|y \vdots x\}$;
6) $\alpha = \{(x,y)|4x+y=11\}$;
7) $\alpha = \{(x,y)|(x+y) \vdots 3\}$;
8) $\alpha = \{(x,y)|x \geq y\}$.

4. Let $A = \{l, 3, 4, 5\}, B = \{l, 2, 5, 6\}$ and G be the graphic of the relation α. Write the relation α with propositions containing letters x and y, with $x\ E\ A$ and $y \in B$:

1) $G_\alpha = \{(1,5),(4,2),(5,1)\}$;
2) $G_\alpha = \{(1,2),(4,5),(5,6)\}$;
3) $G_\alpha = \{(1,2),(1,5),(1,6),(3,5),(3,6),(4,5),(4,6),(5,6)\}$;
4) $G_\alpha = \{(1,5),(1,6),(3,5),(3,6),(4,5),(4,6),(5,1),(5,2),(5,5),$
$(5,6)\}$;
5) $G_\alpha = \{(3,2),(4,2),(5,2)\}$;
6) $G_\alpha = \{(4,2),(4,6)\}$;
7) $G_\alpha = \{(4,1),(4,2),(4,5),(4,6),(1,6),(3,6),(5,6)\}$;
8) $G_\alpha = \{(1,1),(4,2)\}$.

5. Let $A = \{l, 2.3,4\}$. Analyze the properties of the relation $\alpha \subseteq A^2$(var. 1-6) and $\alpha \subseteq \mathbb{R}^2$ (var. 7-14):

1) $\alpha = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,4)\}$;
2) $\alpha = \{(1,2),(1,3),(2,1),(3,1),(3,4),(4,3)\}$;
3) $\alpha = \{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3)\}$;
4) $\alpha = \{(1,1),(1,2),(2,2),(3,3),(4,4)\}$;
5) $\alpha = \{(1,1),(2,2),(2,3),(3,3),(3,4),(4,4)\}$;

6) $\alpha = \{(1,1),(1,2),(2,1),(2,2),(2,3),(3,2),(3,3),(4,4)\}$;

7) $\alpha = \{(x,y) \in \mathbb{R}^2 \mid x > 1 \text{ and } y > 1\}$;

8) $\alpha = \left\{(x,y) \in \mathbb{R}^2 \,\middle|\, \begin{cases} x > 0, \\ y > 0 \end{cases} \text{or } \begin{cases} x < 0, \\ y < 0 \end{cases}\right\}$;

9) $\alpha = \{(x,y) \in \mathbb{R}^2 \mid x \geq 0 \text{ or } y \geq 0\}$;

10) $\alpha = \{(x,y) \in \mathbb{R}^2 \mid x \geq 1 \text{ or } y \geq 1\}$;

11) $\alpha = \{(x,y) \in \mathbb{R}^2 | x^2 + x = y^2 + y\}$;

12) $\alpha = \{(x,y) \in \mathbb{R}^2 | x^2 - 3x + 2 = y^2 - 3y + 2\}$;

13) $\alpha = \{(x,y) \in \mathbb{R}^2 | x^2 + x = y^2 - y\}$;

14) $\alpha = \{(x,y) \in \mathbb{R}^2 | x^2 = y^2\}$.

6. For each binary relation α defined on set $\mathbb{N}$:

a) determine the definition domain δ_α and the values domain ρ_α;

b) establish the properties (reflexivity, ireflexivity, symmetry, anti-symmetry, transitivity);

c) determine the inverse relation $\alpha^{-1}(x, y \in \mathbb{N})$:

1) $x\alpha y \Leftrightarrow$ c.m.m.d.c. $(x,y) = 1$;

2) $x\alpha y \Leftrightarrow y < 2x$;

3) $x\alpha y \Leftrightarrow |y - x| = 12$;

4) $x\alpha y \Leftrightarrow x = y^2$;

5) $x\alpha y \Leftrightarrow (x - y) \vdots 3$;

6) $x\alpha y \Leftrightarrow x \cdot y = 30$;

7) $x\alpha y \Leftrightarrow y = 2x + 1$;

8) $x\alpha y \Leftrightarrow x < y + 1$;

9) $x\alpha y \Leftrightarrow x < y - 1$;

10) $x\alpha y \Leftrightarrow y = 2x$;

11) $x\alpha y \Leftrightarrow y^2 = x^2$;

12) $x\alpha y \Leftrightarrow x \cdot y = 0$.

7. Given the set A and the binary relation $\subseteq A^2$, prove that α is an equivalence relation and determine the factor set A/α.

1) $A = \{1,2,3\}$, $\alpha = \{(1,1),(1,3),(3,1),(2,2),(3,3)\}$;

2) $A = \{1,2,3,4\}$, $\alpha = \{(1,1),(1,2),(1,3),(2,1),(3,1),(2,2),$
$(3,3),(4,4),(3,2),(2,3)\}$;

3) $A = \{1,2,3,4\}$, $\alpha = \{(1,4),(1,1),(4,1),(1,2),(2,1),(3,3),$

$(2,2),(2,4),(4,2),(4,4)\}$;

4) $A=\{1,2,3\}$, $\alpha=\{(1,1),(2,2),(3,3)\}$;

5) $A=\{1,3,5,6\}$, $\alpha=\{(1,6),(6,1),(1,1),(6,6),(3,3),(5,5)\}$;

6) $A=\{1,2,3,4\}$, $\alpha=\{(1,3),(1,4),(1,1),(3,3),(3,1),(4,1),$
$(4,4),(2,2),(3,4),(4,3)\}$;

7) $A=I\!N^2$, $(a,b)\alpha(c,d)\Leftrightarrow a+d=b+c$;

8) $A=\mathbf{Z}\times\mathbf{Z}^*$, $(a,b)\alpha(c,d)\Leftrightarrow a\cdot d=b\cdot c$;

9) $A=\{1,2,3,4,6,9\}$, $\alpha=\{(1,1),(1,3),(3,1),(2,2),(1,2),$
$(4,4),(3,3),(2,1),(6,6),(9,9)\}$;

10) $A=\{1,2,3,5\}$, $\alpha=\{(1,3),(1,1),(3,1),(1,2),(2,1),(2,2),$
$(3,3),(3,2),(2,3),(5,5)\}$.

8. Given the set $A=\{1,2{,}3{,}4{,}5{,}6{,}7{,}8,9\}$ and the subset system $S=\{A_i\subseteq A, I=\overline{1,n}\}$, prove that S defines a partition on A and build the equivalence relation α_S.

1) $A_1=\{1,2,3,8,9\}$, $A_2=\{4\}$, $A_3=\{5,6,7\}$;

2) $A_1=\{1,2\}$, $A_2=\{3,4\}$, $A_3=\{5,6\}$, $A_4=\{7,8\}$, $A_5=\{9\}$;

3) $A_1=\{1\}$, $A_2=\{2,3,4\}$, $A_3=\{5,6\}$, $A_4=\{7,8,9\}$;

4) $A_1=\{1\}$, $A_2=\{3,4,5\}$, $A_3=\{2,7\}$, $A_4=\{6,9\}$, $A_5=\{8\}$;

5) $A_1=\{1,2\}$, $A_2=\{3,9\}$, $A_3=\{4,8\}$, $A_4=\{5,6,7\}$;

6) $A_1=\{1,2\}$, $A_2=\{3,8,9\}$, $A_3=\{4,5,6\}$, $A_4=\{7\}$;

7) $A_1=\{1,9,7\}$, $A_2=\{2,8,6\}$, $A_3=\{3,4,5\}$;

8) $A_1=\{7,8\}$, $A_2=\{1,9\}$, $A_3=\{2,3,4,5,6\}$;

9) $A_1=\{1,8,9\}$, $A_2=\{2,7\}$, $A_3=\{4\}$, $A_4=\{5\}$, $A_5=\{3,6\}$;

10) $A_1=\{1,3,5,7,9\}$, $A_2=\{2,4,6,8\}$.

9. We define on $\mathbb{R}$ the binary relation α:

$x\alpha y \Leftrightarrow \ln^2 x - \ln x = \ln^2 y - \ln y.$

a) Show that α is an equivalence relation on $\mathbb{R}$.

b) Determine the equivalence classes.

10. We define on $\mathbb{R}$ the binary relation β:

$x\beta y \Leftrightarrow \sin^2 x - 2\sin x = \sin^2 y - 2\sin y.$

a) Show that β is an equivalence relation.

b) Determine the equivalence classes.

11. Let $\subseteq A^2$, where $A = \{-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8\}$ with $x\alpha y \Leftrightarrow x^2 - y^2 = 2(x-y)$.

a) Is α an equivalence relation?

b) If so, determine the equivalence classes.

12. Determine: δ_α, $\rho_\alpha, \alpha^{-1}, \alpha \circ \alpha, \alpha \circ \alpha^{-1}, \alpha^{-1} \circ \alpha$, if:

1) $\alpha = \{(x,y) \in I\!N^2 | y \vdots x\}$;

2) $\alpha = \{(x,y) \in I\!N^2 | x \vdots y\}$;

3) $\alpha = \{(x,y) \in I\!R^2 | x + y \leq 0\}$;

4) $\alpha = \{(x,y) \in I\!R^2 | 2x \geq 3y\}$;

5) $\alpha = \{(x,y) \in [-\pi/2, -\pi/2]^2 | y \geq \sin x\}$.

13. Determine the relations

$\alpha \circ \beta,\ \beta \circ \alpha, \alpha^{-1}, \beta^{-1}, \alpha^{-1} \circ \beta,\ (\beta \circ \alpha)^{-1}$;

1) $\alpha = \{(x,y) \in I\!R^2 | x \geq y\}$, $\beta = \{(x,y) \in I\!R^2 | x \leq y\}$;

2) $\alpha = \{(x,y) \in I\!R^2 | x > y\}$, $\beta = \{(x,y) \in I\!R^2 | x < y\}$;

3) $\alpha = \{(x,y) \in I\!R^2 | x + y < 2\}$, $\beta = \{(x,y) \in I\!R^2 | 2x - y > 0\}$;

4) $\alpha = \{(x,y) \in I\!R^2 | (x-1)^2 + y^2 > 1\}$, $\beta = \{(x,y) \in I\!R^2 | x^2 + y^2 \leq 2\}$;

5) $\alpha = \{(x,y) \in \mathbb{Z}^2 \,|\, x - y \text{ is even}\}$,

$\beta = \{(x,y) \in \mathbb{Z}^2 \,|\, x - y \text{ is uneven}\}$;

6) $\alpha = \{(x,y) \in \mathbb{Z}^2 | |x| = |y|\}$, $\beta = \{(x,y) \in \mathbb{Z}^2 | y = 2^x\}$;
7) $\alpha = \{(x,y) \in \mathbb{N}^2 | x \vdots y\}$, $\beta = \{(x,y) \in \mathbb{N}^2 | y \vdots x\}$;
8) $\alpha = \{(x,y) \in \mathbb{N}^2 | x^y = 1\}$, $\beta = \{(x,y) \in \mathbb{N}^2 | x \cdot y = 1\}$.

14. Let $A = \{1,2,3,4\}$, $B = \{a,b,e,d\}$. Determine the definition domain and the values domain of each of the following relations α, β. Which ones are applications? Determine the application type. Determine the relation $\alpha^{-1} \circ \alpha$, $\alpha \circ \beta^{-1}$, $\beta \circ \alpha^{-1}$, $\beta \circ \beta^{-1}$. Are they applications?

1) $\alpha = \{(1,a),(2,c),(3,c),(4,d)\}$, $\beta = \{(1,d),(2,a),(3,c),(4,b)\}$;
2) $\alpha = \{(1,a),(1,c),(2,b),(3,c),(4,d)\}$,
$\beta = \{(1,a),(2,a),(3,a),(4,a)\}$;
3) $\alpha = \{(2,a),(3,c),(4,d),(1,b),(2,b)\}$,
$\beta = \{(1,a),(1,b),(1,c),(1,d)\}$.

15. We consider the application $\varphi: \mathbb{R} \longrightarrow \mathbb{R}$, $\varphi(x) = \sin x$. Determine: $\varphi(\mathbb{R})$, $\varphi((0,\pi))$, $\varphi^{-1}([-1,0])$, $\varphi^{-1}(1/2)$, $\varphi^{-1}([1,2])$, $\varphi^{-1}((1,2])$.

16. The application $\varphi: A \longrightarrow B$ is given by the table below, where $A = \{1,2,3,4,5,6,7,8,9\}$ and $B = \{a,b,e,d,e,i\}$.

x	1	2	3	4	5	6	7	8	9
$\varphi(x)$	a	d	b	f	b	c	b	d	e

Determine:
$\varphi(A)$, $\varphi(\{2,3,5\})$, $\varphi(\{5,6,7,8\})$, $\varphi(\{1,3,7,9\})$,
$\varphi^{-1}(\{b,f,c\})$, $\varphi^{-1}(\{e,c\})$, $\varphi^{-1}(d)$.

17. We consider the application $\varphi: \mathbb{R} \longrightarrow \mathbb{Z}$, $\varphi(x) = [x]$ ($[x]$ is the integer part of x). Determine
$\varphi(\{2,4,6,7\})$, $\varphi((1,5))$, $\varphi([-2.5;2])$,
$\varphi^{-1}(\{2,4,5\})$ and $\varphi^{-1}(-1)$.

18. The following application is given: $\varphi\colon \mathbb{N} \longrightarrow \mathbb{N}$, $\varphi(x) = x^2$. Determine $\varphi(A)$ and $\varphi^{-1}(A)$, if $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

19. Let A and E be two finite sets, $|A| = m. |B|$. How many surjective applications $\varphi\colon A \longrightarrow B$ are there, if:

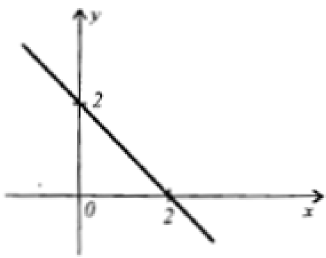
1) $n = 1$; 2) $n = 2$; 3) $m = 4$, $n = 3$; 4) $m = 5$, $n = 3$;
5) $m = 5$, $n = 4$; 6) $m = n = 5$?

20. Given the graphic of the relation α, establish if α is a function. Determine δ_α and ρ_α.

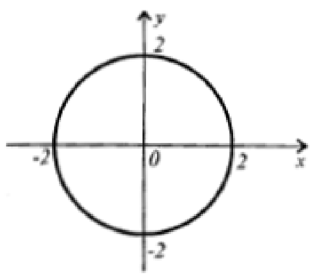
Changing δ_α and ρ_α, make α become an application injective, surjective and bijective.

Draw the graphic for the relation α^{-1}. Is relation α^{-1} a function? What type is it?

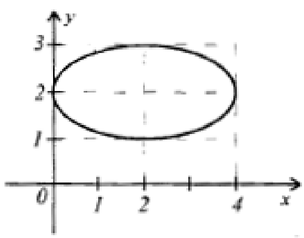
1) $\alpha\colon x + y = 2$



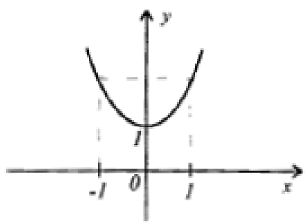
2) $\alpha\colon x^2 + y^2 = 4$



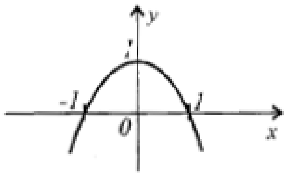
3) $\alpha\colon (x-2)^2/4 + (y-2)^2 = 1$



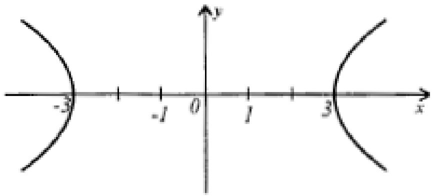
4) $\alpha\colon y = x^2 + 1$



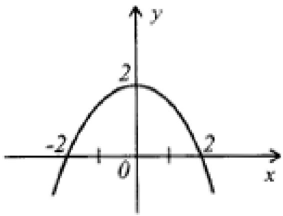
5) $\alpha: y = -x^2 + 1$



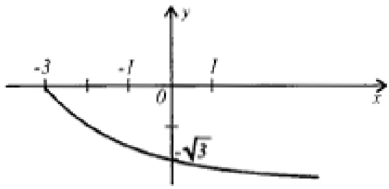
6) $\alpha: x^2 - y^2 = 9$



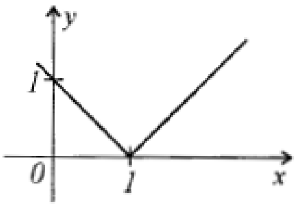
7) $\alpha: y = 4 - x^2$



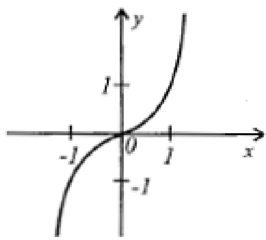
8) $\alpha: y = -\sqrt{x+3}$



9) $\alpha: y = |x - 1|$



10) $\alpha: y = x^3$



21. Let $A = \{1, 2, 3, 4\}$, $B = \{0, 1, 5, 6\}$ and $C = \{7, 8, 9\}$.

a) Draw the diagrams of two injective and two non-injective functions defined on C with values in B.

b) Draw the diagrams of two surjective functions and two non-surjective applications defined on A with values in C.

c) Draw the diagrams of two bijective and two non-bijective functions defined on A with values in B.

22. Let $A=\{1,3,5,6\}$ and $B=\{0,1,2,3,5\}$, $x\in A$, $y\in B$. Which of the relations stated below represents a function defined on A with values in B? And one defined on B with values in A? For functions, indicate their type:

1) α: $x+y=6$; 2) α: $y=x+1$; 3) α: $x=y$;

4) α: $y=x^2$; 5) α: $y=x^3-9x^2+23x-15$;

6) α: $y^5-11y^4+41y^3-61y^2+30y-x+1=0$.

23. Using the graphic of the function $f\colon A\longrightarrow B$, show that f is injective, surjective or bijective. If bijective, determine $f^{-1}$and draw the graphic of f and f^{-1}in the same register of coordinates..

1) f: $(-2;0)\cup[2,+\infty)\longrightarrow[0,+\infty)$, $f(x)=|x|$;

2) f: $\mathbb{R}\longrightarrow[2,+\infty)$, $f(x)=|x|+|x-2|$;

3) f: $[-2,+\infty)\longrightarrow[0,+\infty)$, $f(x)=\begin{cases}-x/2, & -2\le x\le 0,\\ x+1, & x>0;\end{cases}$

4) f: $\mathbb{R}\longrightarrow[0,+\infty)$, $f(x)=\begin{cases}|x^2-1|, & x\le 1,\\ 0, & x>1;\end{cases}$

5) f: $\{0,1,3\}\longrightarrow\{-2,0,4\}$, $f(x)=2x-2$;

6) f: $\{-3,0,2\}\longrightarrow\{1,11/5,3,4\}$, $f(x)=0.2(2x+11)$.

24. Which of the following relations $a\subseteq\mathbb{R}^2$ are functions? Indicate the definition domain of the functions. Establish their type:

1) α: $2y-3x=19$; 2) α: $x\cdot y=9$; 3) α: $3x-7+5y=0$;

4) α: $2x^2+3y-6=0$; 5) α: $y=\sqrt{x-2}$; 6) α: $xy-2y+5x-7=0$;

7) α: $x^2-(y-2)^2=0$; 8) α: $3(4-5x)+4(y+5)=1$;

9) α: $(x-1)^2+(y+3)^2=4$; 10) α: $x^2-y+7x=3$;

11) α: $4x-2y=9x+y$; 12) α: $y^2+xy+1=0$;

13) α: $x^2+y^2=16$; 14) α: $y=x^2-3x+1$; 15) α: $2xy=y^2+5$.

25. Given the relation α with $\beta_\alpha = [-3;\ 5]$ and $\rho_\alpha = [-4;\ 7]$:

a) Does pair $(-4, 5)$ belong to the relation α? Why?

b) Indicate all the ordered pairs $(x, y) \in \alpha$ with $x = O$. Explain.

26. Given the function $f(x)$, calculate its values in the indicated points.

1) $f(x) = -7;\ f(4),\ f(-3),\ f(c),\ \ c \in \mathbb{R};$

2) $f(x) = |x^3 - 2x|;\ f(5),\ f(-2),\ f(-7),\ f(1,4);$

3) $f(x) = x^4 - x^3 - x - 3;\ f(0),\ f(-1),\ f(2+c);$

4) $f(x) = \begin{cases} x^2 - 5, & if\ x > 0, \\ 2x + 3, & if\ x \le 0, \end{cases} f(-2), f(0), f(5);$

5) $f(x) = \begin{cases} x^2 + 1, & if\ x > 0 \\ -4, & if\ x = 0 \\ 1 - 2x, & if\ x < 0 \end{cases} \quad f(5), f(-1), f(1/2);$

6) $f(x) = x^2 - 5x + 2;\ (f(1+c) - f(1))/c,\ \ c \in \mathbb{R}^*.$

27. Determine $D(f)$, if:

1) $f(x) = (2x+3)/(|x-4|);$

2) $f(x) = \sqrt{|2x+1|};$

3) $f(x) = 5/(x^2 + x + 1);$

4) $f(x) = 3 - 2/(5 - x);$

5) $f(x) = \sqrt[3]{6x^2 + 13x - 5};$

6) $f(x) = (4x)/(9 - 4x^2);$

7) $f(x) = (5x)/(\sqrt{4 - 3x});$

8) $f(x) = \pi;$

9) $f(x) = (5x)/(x^2 - 2x - 15).$

28. Given the functions $f(x)$ and $g(x)$, determine the functions $f + g, f - g, f \cdot g$ and $f\ /\ g$, indicating their definition domain.

1) $f(x) = \sqrt{x-5},\ g(x) = \sqrt{x-3};$

2) $f(x) = \dfrac{2}{x-3},\ g(x) = 2x + 1;$

3) $f(x) = x - 5,\ g(x) = x^2 + 1;$

4) $f(x) = x - 3,\ g(x) = 2/x;$

5) $f(x) = x^2 - 4,\ g(x) = 1 - x^2;$

6) $f(x) = 3/x,\ g(x) = 4/x;$

7) $f(x) = x - 1,\ g(x) = x^2 - 5x + 6$; 8) $f(x) = \sqrt{9 - x^2},\ g(x) = x$;
9) $f(x) = 5,\ g(x) = -3$; 10) $f(x) = 1 - x^2,\ g(x) = 4x$.

29. Represent the function $f(x)$ as a compound of some functions:

1) $f(x) = 7(4x - 9)^5 + 4$; 2) $f(x) = (x^2 + 3x)^{\frac{2}{3}} + (x^2 + 3x)^{\frac{1}{3}} - 7$;
3) $f(x) = 1/\sqrt{x^2 - 3}$; 4) $f(x) = 4(x^2 - 3)^6 - 7$;
5) $f(x) = -2(x + 5)^4 + 10$; 6) $f(x) = (2x - 3)^2 - (2x - 3) + 1$;
7) $f(x) = \sqrt{x^2 + x - 2}$; 8) $f(x) = (x - 1)^{\frac{4}{3}} + (x - 1)^{\frac{2}{3}} - 4$;
9) $f(x) = (3x + 5)^{\frac{2}{3}} + 3(3x + 5)^{\frac{1}{3}} + 7$; 10) $f(x) = \dfrac{6}{\sqrt[3]{5 - 3x}}$.

30. Given the functions $f(x)$ and $g(x)$, determine the functions $f \circ g$ and $g \circ f$ and calculate $(f \circ g)(3)$ and $(g \circ f)(3)$ in options 1) – 6) and $(f \circ g)(-1)$ and $(g \circ f)(-1)$ in options 7) – 12).

1) $f(x) = x + 2,\ g(x) = x - 1$; 2) $f(x) = x^2 + 8,\ g(x) = x - 3$;
3) $f(x) = g(x) = x$; 4) $f(x) = x^2 - 1,\ g(x) = x + 1$;
5) $f(x) = 2x^2 + 1,\ g(x) = x^2 - 1$; 6) $f(x) = x^2,\ g(x) = x^3$;
7) $f(x) = x^2 + 2x + 1,\ g(x) = -2x^2 - 1$; 8) $f(x) = 3x^2 + 2,\ g(x) = x - 3$;
9) $f(x) = 2x^4 + 4x^3 + 1,\ g(x) = x^2 + 1$; 10) $f(x) = x - 8,\ g(x) = |x|$;
11) $f(x) = |x + 1| = g(x)$; 12) $f(x) = x - 1,\ g(x) = x + 1$.

31. Given the functions $f(x) = x^2$, $g(x) = 3x$ and $h(x) = x - 1$, calculate:

1) $(f \circ g)(1)$; 2) $(g \circ f)(1)$; 3) $(h \circ f)(3)$;
4) $(f \circ h)(3)$; 5) $(g \circ f)(-2)$; 6) $(f \circ h)(-3)$;
7) $(g \circ h)(-2)$; 8) $(h \circ g)(-2)$; 9) $(f \circ h)(-1/2)$;
10) $(g \circ f)(-1/2)$; 11) $(f \circ h)(\sqrt{2} + 3)$; 12) $(f \circ g)(1 + \sqrt{2})$;

13) $(f \circ g)(c)$; 14) $(g \circ h)(c)$; 15) $(f \circ (g \circ h))(c)$;
16) $((f \circ g) \circ h)(c)$.

32. Determine if in the function pairs f and g one is the inverse of the other:

1) $f(x)=2x+1$, $g(x)=\dfrac{x-1}{2}$; 2) $f(x)=-2x+3$, $g(x)=2x-3$;
3) $f(x)=x+4$, $g(x)=x-4$; 4) $f(x)=x+1$, $g(x)=x-1$;
5) $f(x)=4x-5$, $g(x)=\dfrac{x+5}{4}$; 6) $f(x)=x-\dfrac{1}{2}$, $g(x)=2x+1$;
7) $f(x)=x$, $g(x)=-x$; 8) $f(x)=-2x+3$, $g(x)=-2x-3$.

33. The value of function f is given. Calculate the value of the function f^{-1}, if:

1) $f(3)=4$; 2) $f(1/2)=6$; 3) $f(a)=b$;
4) $f(a+1)=2$; 5) $f(m+n)=p$.

34. Determine f^{-1}, if:

1) $f(x)=(x+2)/2$; 2) $f(x)=(2x+1)/x$;
3) $f(x)=1/\sqrt{x}+2$; 4) $f(x)=(1/x)^2$;
5) $f(x)=(x/(x+4))^2$; 6) $f(x)=\sqrt{x/(x-1)}$;
7) $f(x)=\sqrt{(x-1)/(x+1)}$; 8) $f(x)=\sqrt{\sqrt{x+2}-2}$;
9) $f(x)=((x-3)/(x+1))^2$; 10) $f(x)=(\sqrt{x/(x+4)}-2)^2$.

35. The function: $f: \in \mathbb{R} \longrightarrow \mathbb{R}$ is given:

$$f(x)=\begin{cases} x^2-2x-2, & x \geq 1, \\ 2x-1, & x<1. \end{cases}$$

a) Prove that f is bijective.

b) Determine f^{-1}

c) Calculate $f \circ f^{-1}$ și $f^{-1} \circ f$.

36. Given the functions $f(x)$ and $g(x)$, determine the functions $(f \circ g)(x)$ and $(g \circ f)(x)(f, g: \mathbb{R} \longrightarrow \mathbb{R})$:

1) $f(x) = |x-1| + 2;$ $\qquad g(x) = |x-2| + 1;$

2) $f(x) = \begin{cases} x^2 - 1, & x \leq 0, \\ -5x - 1, & x > 0; \end{cases}$ $\qquad g(x) = \begin{cases} 4x - 2; & x < 0, \\ 3x^2 - 2, & x > 0. \end{cases}$

37. Prove the equality of the functions f and g:

1) $f, g\colon \{-1, 0, 1, 2\} \longrightarrow \mathbb{R}, f(x) = x^4 - 2x^3 - x^2 + 2x + 1;$
$g(x) = x^5 - x^4 - 3x^3 + x^2 + 2x + 1;$

2) $f, g\colon \{-1, 0, 1\} \longrightarrow \mathbb{R}, \quad f(x) = x^3 - x, \quad g(x) = \sin \pi x;$

3) $f, g\colon [1;3] \longrightarrow \mathbb{R}, \qquad f(x) = \max(-t^2 + 4t - 3),\ 1 \leq t \leq x;$
$g(x) = \begin{cases} -x^2 + 4x - 3, 1 \leq x \leq 2, \\ 1 \qquad\qquad 2 < x \leq 3; \end{cases}$

4) $f, g\colon [-1;1] \longrightarrow \mathbb{R}, \quad f(x) = \begin{cases} -x + 1, -1 \leq x \leq 0, \\ x + 1, \quad 0 < x \leq 1; \end{cases}$
$g(x) = \max(-x + 1,\ x + 1);$

5) $f, g\colon \{-1, 0\} \longrightarrow \mathbb{R}, f(x) = 1 + x, \quad g(x) = \sqrt{1 - x^2};$

6) $f, g\colon \{-1, 0\} \longrightarrow \mathbb{R}, f(x) = -1 + \sqrt{4 + 2x - x^2};$
$g(x) = 1 - \sqrt{-2x - x^2};$

7) $f, g\colon \{0, 2\} \longrightarrow \mathbb{R}, \quad f(x) = 2 - x, \quad g(x) = \sqrt{4 - x^2};$

8) $f, g\colon \{0, 2\} \longrightarrow \mathbb{R}, \quad f(x) = \sqrt{4 - x^2};$
$g(x) = 2 - \sqrt{4x - x^2};$

9) $f, g\colon [1, +\infty) \longrightarrow \mathbb{R}, f(x) = \sqrt{x + 2\sqrt{x-1}} + \sqrt{x - 2\sqrt{x-1}};$
$g(x) = \begin{cases} 2, \qquad 1 \leq x \leq 2, \\ 2\sqrt{x-1}, x > 2; \end{cases}$

10) $f, g\colon \{k\pi, 2k\pi \pm \frac{\pi}{3} | k \in \mathbf{Z}\} \longrightarrow \mathbb{R}, \ f(x) = \sin x, \ g(x) = \sin 2x.$

38. Determine the functions $s = f + g, d = f - g, p = f.g$ and $q = f/g$:

1) $f\colon \{1,2,3,4\} \to \{0,1,3,5,6\}$, $f(1)=0$, $f(2)=1$, $f(3)=3$, $f(4)=6$;
$g\colon \{1,2,3,5\} \to \{1,3,4,5\}$, $g(1)=1$, $g(2)=4$, $g(3)=3$, $g(5)=4$;

2) $f,g\colon \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} x+2, x \le 3, \\ -x+3, x > 3; \end{cases}$ $g(x) = \begin{cases} x-2, x \le 0, \\ x+1, x > 0; \end{cases}$

3) $f,g\colon \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} x, & x \le -1, \\ -x, -1 < x < 1, \\ 0, & x \ge 1; \end{cases}$ $g(x) = \begin{cases} x-1, x < 0, \\ x, & x \ge 0; \end{cases}$

4) $f\colon [0,+\infty) \to \mathbb{R}$, $f(x) = \begin{cases} 2x+1, 0 \le x \le 2, \\ 1, & x > 2; \end{cases}$

$g\colon (-\infty,5] \to \mathbb{R}$, $g(x) = \begin{cases} -x, & x < 0, \\ 1, & x = 0, \\ x, 0 < x \le 5. \end{cases}$

5) $f,g\colon \mathbb{R} \to \mathbb{R}$, $f(x) = \max(x+1, x^2)$; $g(x) = \min(-x, x)$.

39. Let $A = \{1,2,3,4\}, B = \{0,1,3,4\}$ and the functions

$f\colon A \longrightarrow B$, $f(1) = 0$, $f(2) = 0$, $f(3) = 1$, $f(4) = 3$;

$g\colon B \longrightarrow A$, $g(0) = 2$, $g(1) = 1$, $g(3) = 4$, $g(4) = 1$.

Can the functions $f \circ g, g \circ f$ be defined? If so, determine these functions. Make their diagrams.

40. a) Show that the function f is bijective.

b) Determine f^{-1}.

c) Graphically represent the functions f and f^{-1} in the same coordinate register.

1) $f\colon \mathbb{R} \longrightarrow \mathbb{R}$, , $f(x) = 6x - 2$;
2) $f\colon [0,+\infty) \longrightarrow [1,+\infty)$, $f(x) = 3x + 1$;
3) $f\colon (-\infty,0) \cup [2;4] \longrightarrow (-\infty,4]$, $f(x) = -x^2 + 4x$;
4) $f\colon \mathbb{R} \longrightarrow (-\infty,3) \cup [4,+\infty)$, $f(x) = \begin{cases} x+3, & x \le 0, \\ \frac{2}{3}x + 4, & x > 0; \end{cases}$
5) $f\colon [0,\pi] \longrightarrow [-1,1]$, $f(x) = \begin{cases} \sin x, & 0 \le x \le \pi/2, \\ \cos x, & \pi/2 < x \le \pi. \end{cases}$

41. Show that the function $f\colon \in \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = x^2 - 6x + 2$ admits irreversible restrictions on:

a) $(-\infty, 3]$; b) $[3, +\infty)$; c) $(-\infty, 0] \cup [3, 6)$.

Determine the inverses of these functions and graphically represent them in the same coordinate register.

42. Using the properties of the characteristic function, prove the following equalities (affirmations):

1) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$;
2) $(A \cup B = B \cap A) \Rightarrow (A = B)$;
3) $\left.\begin{array}{l} A \cap B = A \cap C \\ A \cup B = A \cup C \end{array}\right\} \Rightarrow (B = C)$;
4) $(A \vartriangle B) \vartriangle C = A \vartriangle (B \vartriangle C)$;
5) $A \cap (B \vartriangle C) = (A \cap B) \vartriangle (A \cap C)$;
6) $A \vartriangle B = (A \cup \overline{B}) \cap (A \cup B)$;
7) $A' \vartriangle B' = A \vartriangle B$;
8) $A' \vartriangle B = A \vartriangle B'$;
9) $A \vartriangle B = \varnothing \Leftrightarrow A = B$;
10) $A \vartriangle B = A \cup B \Rightarrow A \cap B = \varnothing$;
11) $A \cap B = A \setminus B \Leftrightarrow A = \varnothing$;
12) $A \cup B = A \setminus B \Leftrightarrow B = \varnothing$;
13) $(A \setminus B) \setminus C = (A \setminus C) \setminus B$;
14) $A \setminus B = B \setminus A \Leftrightarrow A = B$.

3. Elements of combinatorics

3.1. Permutations. Arrangements. Combinations. Newton's Binomial

In solving many practical problems (and beyond), it is necessary to:

1) be able to assess the number of different combinations compound with the elements of a set or a multitude of sets;

2) choose (select) from a set of objects the subsets of elements that possess certain qualities;

3) arrange the elements of a set or a multitude of sets in a particular order etc.

The mathematical domain that deals with these kind of problems and with the corresponding solving methods is called combinatorics. In other words, combinatorics studies certain operations with finite sets. These operations lead to notions of permutations, arrangements and combinations.

Let $M = \{a_1, a_2, \dots, a_n\}$ be a finite set that has n elements. Set M is called **ordered**, if each of its elements associates with a certain number from 1 to n, named the element rank, so that different elements of M associate with different numbers.

Definition 1. All ordered sets that can be formed with n elements of given set M $(n(M) = n)$ are called permutations of n elements.

The number of all permutations of n elements is denoted by the symbol P_n and it is calculated according to the formula:

$$P_n = n!\ (n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n),\ n \in \mathbb{N}. \qquad (1)$$

By definition, it is considered $P_0 = 0! = 1 = 1! = P_1$.

Definition 2. All ordered subsets that contain m elements of set M with n elements are called arrangements of n elements taken m at a time.

The number of all arrangements of n elements taken m at a time is denoted by the symbol A_n^m and it is calculated with the formula

$$A_n^m = \frac{n!}{(n-m)!} = n(n-1)\cdot\ldots\cdot(n-m+1);\ 0 \le m \le n;\ n, m \in N. \quad (2)$$

Definition 3. All subsets that contain elements of set M with n elements are called combinations of n elements taken m at a time.

The number of all combinations of n elements taken m at a time is denoted by the symbol C_n^m and it is calculated with the formula

$$C_n^m = \frac{n!}{m!(n-m)!} = \frac{n(n-1)\cdot\ldots\cdot(n-m+1)}{m!} = \frac{A_n^m}{P_m} = \frac{P_n}{P_m \cdot P_{n-m}}, \quad (3)$$

where $m, n \in \mathbb{N}; 0 \le m \le n$.

Remark. In all the subsets mentioned in definitions 1 - 3, each element of the initial set M appears only once.

Along with the combinations in which each of the different n elements of a set participates only once, we can also consider combinations with repetitions, that is, combinations in which the same element can participate more than once.

Let n groups of elements be given. Each group contains certain elements of the same kind.

Definition $1'$. Permutations of n elements each one containing α_1 elements a_{i_1}, α_2 elements a_{i_2}, α_k elements a_{i_k}, where $\alpha_1 + \alpha_2 + \cdots + \alpha_k$ are called permutations of n elements with repetitions.

The number of all permutations with repetitions is denoted by the symbol $\overline{P}_{\alpha_1,\alpha_2,\ldots,\alpha_k}$ and it is calculated using the formula:

$$\overline{P}_{\alpha_1,\alpha_2,\ldots,\alpha_k} = \frac{(\alpha_1 + \alpha_2 + \ldots + \alpha_k)!}{\alpha_1!\alpha_2!\cdot\ldots\cdot\alpha_k!} = \frac{n!}{\alpha_1!\alpha_2!\cdot\ldots\cdot\alpha_k!}. \quad (4)$$

Definition $2'$. The arrangements of n elements, each one containing m elements, and each one being capable of repeating itself in each arrangement for an arbitrary number of times, but not more than m times, are called arrangements of n elements taken m at a time with repetitions.

The number of all arrangements with repetitions of n elements taken m at a time is denoted by the symbol $\overline{A_n^m}$ and it is calculated with the formula

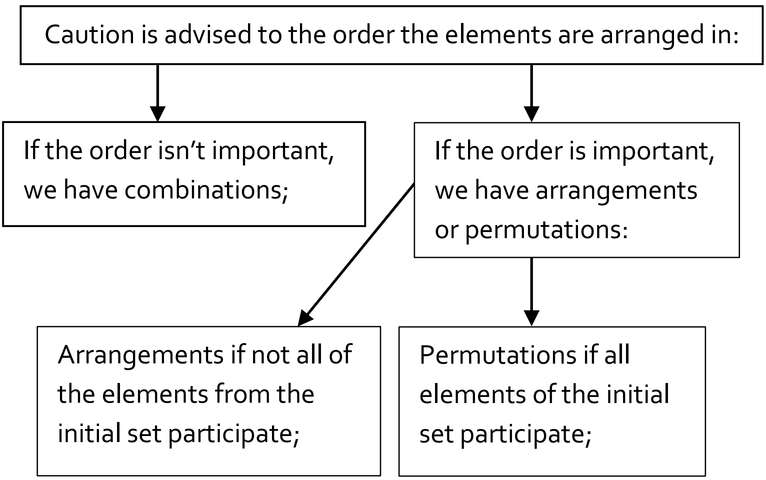
$$\overline{A_n^m} = n^m, \quad n, m \in \mathbb{N}^*. \tag{5}$$

Definition 3'. The combinations of n elements each one containing m elements, and each one being capable of repeating itself more than once, but not more than m times, are called combinations of n elements taken m at a time with repetitions.

The number of all combinations with repetitions is denoted by the symbol $\overline{C_n^m}$ and it is calculated with the formula

$$\overline{C_n^m} = C_{m+n-1}^m = \frac{(n+m-1)!}{m! \cdot (n-1)!}; \quad n, m \in \mathbb{N}^*. \tag{6}$$

In the process of solving combinatorics problems, it is important to firstly establish the type (the form) of combination. One of the rules used to establish the type of combination could be the following table



It is often useful to use the following two rules:

The sum rule. If the object A can be chosen in m ways and the object B in n ways, then the choice "either *A* or *B*" can be made in $m + n$ ways.

The multiplication rule. If the object A can be chosen in m ways and after each such choice the object B can be chosen in n ways, then the choice "*A* and *B*" in this order can be made in $m \,.\, n$ ways.

We will also mention some properties of combinations, namely:

I. $C_n^m = C_n^{n-m}$.

II. $C_n^{m-1} + C_n^m = C_{n+1}^m$.

III. $C_n^k = C_{n-1}^{k-1} + C_{n-2}^{k-1} + C_{n-3}^{k-1} + \ldots + C_{k-1}^{k-1}$

$(C_n^{n-k} = C_{n-1}^{n-k} + C_{n-2}^{n-k-1} + C_{n-3}^{n-k-2} + \ldots + C_{k-1}^0)$.

IV. $C_n^0 + C_n^1 + C_n^2 + \ldots + C_n^n = 2^n$.

The formula

$$(x+a)^n = C_n^0 \cdot x^n + C_n^1 \cdot x^{n-1} \cdot a + C_n^2 \cdot x^{n-2} \cdot a^2 + \ldots + C_n^{n-1} \cdot x \cdot a^{n-1} + C_n^n \cdot a^n \quad (7)$$

is called Newton's Binomial formula *($n \in \mathbb{N}^*$).*

The coefficients $C_n^0, C_n^1, C_n^2, \ldots, C_n^n$ from Newton's Binomial formula are called binomial coefficients; they possess the following qualities:

V. The binomial coefficients from development (7), equally set apart from the extreme terms of the development, are equal among themselves.

VI a. The sum of all binomial coefficients equals 2^n.

VI b. The sum of the binomial coefficients that are on even places equals the sum of the binomial coefficients that are on uneven places.

VII. If n is an even number (i.e. $n = 2k$), then the binomial coefficient of the middle term in the development (namely C_n^k) is the biggest. If n is uneven (i.e. $n = 2k + 1$), then the binomial coefficients of the two middle terms are equals (namely $C_n^k = C_n^{k+1}$) and are the biggest.

VIII. Term $C_n^k x^{n-k} a^k$, namely the $(k+1)$ term in equality (7), is called term of rank $k+1$ (general term of development) and it is denoted by T_{k+1}. So,

$$T_{k+1} = C_n^k \cdot x^{n-k} \cdot a^k, \quad k = 0, 1, 2, \ldots, n. \tag{8}$$

IX. The coefficient of the term of rank $k+1$ in the development of Newton's binomial is equal to the product of the coefficient term of rank k multiplied with the exponent of x in this term and divided by k, namely

$$C_n^k = \frac{n-k+1}{k} \cdot C_n^{k-1}. \tag{9}$$

X. $$C_{n+1}^{k+1} = \frac{n+1}{k+1} \cdot C_n^k. \tag{10}$$

3.2. Solved problems

1. In how many ways can four books be arranged on a shelf?

Solution. As order is important and because all the elements of the given set participate, we are dealing with permutations.

So

$$P_4 = 4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24.$$

Answer: 24.

2. A passenger train has ten wagons. In how many ways can the wagons be arranged to form the train?

Solution. As in the first problem, we have permutations of 10 elements of a set that has 10 elements. Then the number of ways the train can be arranged is

$$P_{10} = 10! = 3\,628\,800.$$

Answer: 3 628 800.

3. In how many ways can 7 students be placed in 7 desks so that all the desks are occupied?

Solution.

$$P_7 = 7! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 = 5\,040.$$

Answer: 5 040.

4. How many phone numbers of six digits can be dialed:

1) the digit participates in the phone number only once;

2) the digit participates more than once?

(The telephone number can also start with 0.)

Solution. We have 10 digits on the whole: $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$. As the phone number can also start with 0, we have:

1) arrangements of 10 digits taken 6 at a time, i.e.

$$A_{10}^{6} = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 = 151\,200;$$

2) because the digit can repeat in the number, we have arrangements with repetitions, i.e.

$$A_{10}^{6} = 10^6 = 1\,000\,000.$$

Answer: 1) $151\ 200$; 2) $1\ 000\ 000$.

5. The volleyball team is made out of 6 athletes. How many volleyball teams can a coach make having 10 athletes at his disposal?

Solution. As the coach is only interested in the team's composition it is sufficient to determine the number of combinations of 10 elements taken 6 at a time, i.e.

$$C_{10}^{6} = C_{10}^{4} = \frac{10!}{4! \cdot 6!} = \frac{10 \cdot 9 \cdot 8 \cdot 7}{1 \cdot 2 \cdot 3 \cdot 4} = 210.$$

Answer: 210.

6. How many 5 digit numbers can be formed with digits 0 and 1?

Solution. As the digits are repeating and their order is important, we consequently have arrangements with repetitions. Here, $m = 5$, $n = 2$.

From digits 0 and 1 we can form $\overline{A_2^5}$ numbers of five digits.

But we have to take into consideration that the number cannot begin with the digit zero. So, from the number $\overline{A_2^5}$ we have to begin with zero. Numbers of this type are $\overline{A_2^4}$. Therefore, the number we seek is

$$\overline{A_2^5} - \overline{A_2^4} = 2^5 - 2^4 = 16.$$

Answer: 16.

7. How many three digit numbers can be formed with the digits 1, 2, 3, 4, 5, if the digits can be repeated?

Solution. As the digits repeat, we obviously have arrangements with 5 digits taken three times. Therefore, there can be formed $\overline{A_5^3} = 5^3$ numbers of three digits.

Answer: 125.

8. Using 10 roses and 8 Bedding Dahlias, bunches are made that contain 2 roses and 3 Bedding Dahlias. How many bunches of this kind can be formed?

Solution. Two roses (out of the 10 we have) can be chosen in C_{10}^2 ways, and three Bedding Dahlias (out of 8) can be taken in C_8^3 ways. Applying the rule of multiplication, we have: the total number of bunches that can be formed is $C_{10}^2 . C_8^3 = 1\,890$.

Answer: 1890.

9. 12 young ladies and 15 young gentlemen participate at a dancing soirée. In how many ways can four dancing pairs be chosen?

Solution. The 12 young ladies can be distributed in four person groups in C_{12}^4 ways, and the 15 young gentlemen - in C_{15}^4 ways.

Because in each group of young ladies (or young gentlemen) order is an issue, each of these groups can be ordered in P_4 ways.

As a result (we apply the multiplication rule), we will have $C_{12}^4 . P_4 . C_{15}^4 = A_{12}^4 . C_{15}^4 = C_{12}^4 . A_{15}^4 = 16216200.$

Answer: 16216200.

10. To make a cosmic flight to Mars, it is necessary to constitute the crew of the spacecraft in the following distribution: the spacecraft captain, the first deputy of the captain, the second deputy of the captain, two mechanical engineers, and a doctor. The command triplet can be selected out of the 25 pilots that are flight-ready, two mechanical engineers out of the 20 specialists who master the construction of the spacecraft, and the doctor – out of the 8 available doctors. In how many ways can the crew be assembled?

Solution. Choosing the captain and his deputies, it is important to establish which of the pilots would best deal with random steering commands. So, the distribution of the tasks between the triplet's members is equally important. So, the commanding triplet can be formed in A_{25}^3 ways.

The tasks of the two engineers are more or less the same. They can accomplish these tasks consecutively. So the pair of engineers can be formed in C_{20}^2 ways. Regarding the doctor – the situation is the same, namely the doctor can be chosen in C_8^1 ways.

Using the multiplication rule, we have: a pair of engineers can be assigned in C_{20}^2 ways to each commanding triplet. We will have, in total, $A_{25}^3 . C_{20}^2$ quintets. To each quintet a doctor can be associated in C_8^1 ways.

As a result, the spacecraft's crew can be assembled in $A_{25}^3 . C_{20}^2 . C_8^1$ ways, or

$$A_{25}^3 \cdot C_{20}^2 \cdot C_8^1 = 20\,976\,000.$$

Answer: 20 976 000.

11. In how many different ways can 5 cakes, of the same type or different, be chosen in a cake shop where there are 11 types of different cakes?

Solution. The five cakes can all be of the same type or four of the same type and one of a different type, or three of the same type and two of a different type, etc., or all can be of different types.

The number of the possible sets comprised of five cakes out of the existing 11 types is equal to the number of the combinations with repetitions of 11 elements taken five at a time, i.e.

$$\overline{C_{11}^5} = \frac{(11+5-1)!}{5!(11-1)!} = \frac{15!}{5!10!} = 3\,003.$$

Answer: 3003.

12. In a group of 10 sportsmen there are two rowers, three swimmers and the others are athletes. A 6 person team must be formed for the upcoming competitions in such a manner that the team will comprise at least one representative from the three nominated sport types. In how many ways can such a team be assembled?

Solution. a) In the team there can be one rower, one swimmer and four athletes. The rower can be chosen in C_2^1 ways, the swimmer in C_3^1 ways, and the athlete in C_5^4 ways. Using the multiplication rule, we have $C_2^1 . C_3^1 . C_5^4$ ways.

b) In the team there can be one rower, two swimmers and three athletes. According to the aforementioned reasoning, the numbers of the teams of this type will be $C_2^1 . C_3^1 . C_5^4$.

c) In the team there can be one rower, three swimmers and two athletes. The number of teams in this case will be $C_2^1 . C_3^3 . C_5^2$.

d) In the team there can be two rowers, one swimmer and three athletes. We will have $C_2^2 . C_3^1 . C_5^3$ of these teams.

e) In the team there can be two rowers, two swimmers and two athletes. The number of teams will be $C_2^2 . C_3^2 . C_5^2$.

f) In the team there can be two rowers, three swimmers and one athlete. The number of teams will be $C_2^2 . C_3^3 . C_5^1$.

Using the addition rule, the total number of teams that can be assembled is:

$$C_2^1 \cdot C_3^1 \cdot C_5^4 + C_2^1 \cdot C_3^2 \cdot C_5^3 + C_2^1 \cdot C_3^3 \cdot C_5^2 + C_2^2 \cdot C_3^1 \cdot C_5^3 + C_2^2 \cdot C_3^2 \cdot C_5^2 + C_2^2 \cdot C_3^3 \cdot C_5^1 = \\ = 175.$$

Answer: 175.

13. Given $k = 15$ capital letters, $m = 10$ vowels and $n = 11$ consonants (in total $k + m + n = 36$ letters) determine: How many different words can be formed using these letters, if in each word the first letter has to be a capital letter, among the other different letters there have to be $\mu = 4$ different vowels (out of the $m = 10$ given) and $v = 6$ different consonants (out of the $n = 11$ given).

Solution. We choose a capital letter. This choice can be made in k ways. Then from m vowels we choose μ letters. This can be done in C_m^μ ways. Finally, we choose v consonants, which can be made in C_n^v ways. Using the multiplication rule, we can choose the required letters to form the word in $k. C_m^\mu . C_n^v$ ways.

After placing the capital letter at the beginning, with the other $\mu + v$ letters we can form $(\mu + v)!$ permutations. Each such permutation yields a new word. So, in total, a number of $k. C_m^\mu . C_n^v (\mu + v)!$ different words can be formed, namely $15. C_{10}^4 C_{11}^6 . 10!$

Answer: $15. C_{10}^4 C_{11}^6 . 10!$

14. In a grocery store there are three types of candy. The candy is wrapped in three different boxes, each brand in its box. In how many ways can a set of five boxes be ordered?

Solution (see Problem 11).

$$\overline{C_3^5} = \frac{(3+5-1)!}{5! \cdot (3-1)!} = \frac{7!}{5! \cdot 2!} = 21.$$

Answer: 21.

15. In order to form the 10 person guard of honor, officers of the following troops are invited: infantry, aviation, frontier guards, artillery; navy officers and rocket officers.

In how many ways can the guard of honor be assembled?

Solution. We have 6 categories of officers. Repeating the reasoning from problem 10, we have to calculate combinations with repetitions of 6 elements taken 10 times each, as follows

$$\overline{C_6^{10}} = \frac{(6+10-1)!}{(6-1)! \cdot 10!} = \frac{15!}{5! \cdot 10!} = \frac{15 \cdot 14 \cdot 13 \cdot 12 \cdot 11}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} = 3\,003.$$

Answer: 3 003.

16. On a shelf there are $m + n$ different books. Among them m have a blue cover, and n a yellow cover. The books are permutated in every way possible. How many positions do the books have, if:

a) the books with the blue cover occupy the first m places;

b) the books in the yellow covers sit beside them?

Solution. a) The books with blue covers can be placed in the first m places in $P_m = m!$ ways. With each such allocation, the books with yellow covers can be placed in $P_n = n!$ ways. Using the multiplication rule, we have in total $m! \cdot n!$ positions in which the blue covers books occupy the first m places.

b) Let the books in blue covers sit beside. It follows that right beside them on the shelf there can be either n books with yellow covers or $n-1$, or $n-2$, ... , or no book with yellow covers. We can thus place the books with blue covers so that they follow each other in $n + 1$ ways. In each of these positions, the books with yellow covers can be permutated in any way, also the books with the blue covers can be permutated in any way. As a result, we will have $m!\,.\,n!\,.\,(n + 1)$ different positions for the books.

Answer: a) $m! \cdot n!$; b) $m! \cdot n! \cdot (n + 1)$.

17. Determine the fourth term of the development of Newton's binomial:

$(2x\sqrt{x} - \sqrt[3]{x})^8.$

Solution. According to the formula (9), the rank 4 term has the form

$$T_4 = C_8^3(2x\sqrt{x})^{8-3} \cdot (-x^{1/3})^3 = -C_8^3 \cdot 2^5 \cdot x^{15/2} \cdot x =$$

$$= -\frac{8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3} \cdot 2^5 \cdot x^{17/2} = -256 \cdot 7 \cdot x^{17/2} = -1792 \cdot x^{17/2}.$$

Answer: $-1792 \cdot x^{17/2}$.

18. Determine the biggest coefficient in the development of the binomial

$[(1+x)(1/x-1)]^m.$

Solution.

$$\left[(1+x)\left(\frac{1}{x}-1\right)\right]^m = \left(\frac{(1+x)(1-x)}{x}\right)^m = \frac{(1-x^2)^m}{x^m} =$$

$$= x^{-m} \cdot \sum_{k=0}^{m} C_m^k (-1)^k \cdot x^{2k}.$$

If m is an even number, i.e. $m = 2s$, $s \in \mathbb{N}^*$, then the development of the binomial contains $2s + 1$ terms, and according to the VII property, the coefficient C_{2s}^s is the biggest.

If m is an uneven number, i.e. $m = 2s + 1$, $s \in \mathbb{N}$, according to the same VII property, the development of the binomial contains two terms have the biggest coefficients $C_{2s+1}^s, C_{2s+1}^{s+1}$.

Answer: C_{2s}^s, if m is an even number; $C_{2s+1}^s, C_{2s+1}^{s+1}$ if m is an uneven number.

19. Determine which term do not contain x in the development of the binomial:

$[(1+x)(1+1/x)]^n.$

Solution.

$$\left[(1+x)\left(1+\frac{1}{x}\right)\right]^n = \frac{(1+x)^{2n}}{x^n}.$$

The term of rank $k + 1$ in the development of this binomial has the form

$$T_{k+1} = \frac{1}{x^n}C_{2n}^k x^k = C_{2n}^k \cdot x^{k-n}.$$

This term does not contain x only if $k - n = 0 \Leftrightarrow k = n$. So the term that doesn't contain x is T_{n+1}.

Answer: T_{n+1}.

20. In the development of the binomial

$(a\sqrt[5]{a/3} - b/\sqrt[7]{a^3})^n$ determine the terms that contains a to the power of three, if the sum of the binomial coefficients that occupy uneven places in the development of the binomial is equal to 2 048.

Solution. We will first determine the exponent n. Based on the VI^{th} property, the sum of the binomial coefficients is 2^n. Because the sum of the binomial coefficients that occupy uneven places in the development of the binomial is equal to 2048, and based on the $VI\ b$ property, it is equal to the sum of the coefficients that occupy even places in the respective development, we have

$$2048 = 2^{n-1} \Leftrightarrow 2^{11} = 2^{n-1} \Leftrightarrow n = 12.$$

Therefore, the degree of the binomial is 12. The term of rank $k + 1$ takes the form

$$T_{k+1} = C_{12}^k(a\sqrt[5]{a/3})^{12-k} \cdot (-1)^k \cdot (b/\sqrt[7]{a^3})^k =$$
$$= C_{12}^k \cdot (-1)^k/(3^{(12-k)15}) \cdot (a^{6/5})^{12-k} \cdot a^{-3k/7} \cdot b^k.$$

Considering the requirements of the problem, we have

$$a^{\frac{6(12-k)}{5} - \frac{3k}{7}} = a^3 \Leftrightarrow \frac{72-6k}{5} - \frac{3k}{7} = 3 \Leftrightarrow$$

$$\Leftrightarrow 24 \cdot 7 - 2 \cdot 7k - 5k = 35 \Leftrightarrow 19k = 133 \Leftrightarrow k = 7,$$

and

$$T_8 = C_{12}^7 \cdot a^3 \cdot 3^{-1} \cdot (-1)^7 \cdot b^7 = -264a^3b^7.$$

Answer: $-264a^3b^7$.

21. For what value of n the binomial coefficients of the second, third and fourth term from the development of the binomial $(x + y)^n$ forms an arithmetic progression?

Solution. Based on the formula (8), we have

$$T_2 = C_n^1 x^{n-1} y, \quad T_3 = C_n^2 x^{n-2} y^2, \quad T_4 = C_n^3 x^{n-3} y^3,$$

and from the conditions of the problem the following relations issues

$$C_n^1 + C_n^3 = 2C_n^2 \Leftrightarrow n + \frac{n(n-1)(n-2)}{6} = 2 \cdot \frac{n(n-1)}{2} \Leftrightarrow$$

$$\Leftrightarrow n(6+(n-1)(n-2)-6(n-1)) = 0 \overset{n\in N^*}{\Longleftrightarrow} n^2-9n+14 = 0 \Leftrightarrow \left[\begin{array}{l} n = 2, \\ n = 7. \end{array}\right.$$

The conditions of the problem are verified by the value $n = 7$.

Answer: $n = 7$.

22. Prove that the difference of the coefficients x^{k+1} and x^k in the development of the binomial $(1 + x)^{n+1}$ is equal to the difference of the coefficients of x^{k+1} and x^{k-1}I in the development of the binomial $(1 + x)^n$.

Solution. The coefficients of x^{k+1} and x^k in the development of the binomial $(1 + x)^{n+1}$ are $C_{n+1}^{k+1}$and C_{n+1}^k respectively. We assess the difference

$$C_{n+1}^{k+1} - C_{n+1}^k \overset{(9)}{=} \frac{(n+1)-k}{k+1} C_{n+1}^k - C_{n+1}^k = \left(\frac{n+1-k}{k+1} - 1\right) \cdot C_{n+1}^k =$$

$$= \frac{n+1-k-k-1}{k+1} \cdot \frac{(n+1)!}{k!(n+1-k)!} = \frac{(n-2k)\cdot(n+1)!}{(k+1)!\cdot(n+1-k)!}. \quad (*)$$

In the development of the binomial $(1 + x)^n$, the coefficients of x^{k+1} and x^{k-1} are C_n^{k+1} and C_n^{k-1}, respectively. We assess the difference

$$C_n^{k+1} - C_n^{k-1} \overset{(9)}{=} \frac{n-k}{k+1} C_n^k - C_n^{k-1} \overset{(9)}{=} \frac{n-k}{k+1} \cdot \frac{n-k+1}{k} \cdot C_n^{k-1} - C_n^{k-1} =$$

$$= \left(\frac{(n-k)(n-k+1)}{(k+1)k} - 1\right) \cdot C_n^{k-1} =$$
$$= \frac{(n-k)^2 + (n-k) - k^2 - k}{(k+1)k} \cdot \frac{n!}{(k-1)!(n-k+1)!} =$$
$$= \frac{(n+1)(n-2k) \cdot n!}{(k+1)!(n-k+1)!} = \frac{(n-2k) \cdot (n+1)!}{(k+1)! \cdot (n-k+1)!}. \qquad (**)$$

As the members from the right in $(*)$ and $(**)$ are equal, the equality of the members from the left results, i.e.

$$C_{n+1}^{k+1} - C_{n+1}^{k} = C_n^{k+1} - C_n^{k-1},$$

which had to be proven.

23. Comparing the coefficients of x in both members of the equality

$$(1+x)^m \cdot (1+x)^n = (1+x)^{m+n},$$

prove that

$$C_n^k C_m^0 + C_n^{k-1} C_m^1 + \ldots + C_n^0 C_m^k = C_{m+n}^k. \qquad (A)$$

Solution.

$$(1+x)^m \cdot (1+x)^n = (C_m^0 + C_m^1 x + C_m^2 x^2 + \ldots + C_m^k x^k + \ldots + C_m^{m-1} x^{m-1} + x^m) \cdot (C_n^0 + C_n^1 x + C_n^2 x^2 + \ldots + C_n^k x^k + \ldots + C_n^{n-1} x^{n-1} + x^n).$$

In the right member of this equality, the coefficient of x^k is

$$C_m^0 \cdot C_n^k + C_m^1 \cdot C_n^{k-1} + C_m^2 \cdot C_n^{k-2} + \ldots + C_n^1 \cdot C_m^{k-1} + C_n^0 \cdot C_m^k,$$

and in the development of the binomial $(1+x)^{n+m}$, the term of rank $k + 1$ has the form

$$T_{k+1} = C_{m+n}^k \cdot x^k.$$

As the polynomials $(1+x)^m . (1+x)^n$ and $(1+x)^{n+m}$ are equal and have the same degree, the equality of the coefficients beside these powers of x result and this also concludes the demonstration of the equality (A).

24. Prove the equality

$$\frac{C_n^0}{n+1}+\frac{C_n^1}{n}+\frac{C_n^2}{n-1}+\ldots+\frac{C_n^n}{1}=\frac{2}{n+1}\left(2^n-\frac{1}{2}\right).$$

Solution. Let:

$$\frac{C_n^0}{n+1}+\frac{C_n^1}{n}+\frac{C_n^2}{n-1}+\ldots+\frac{C_n^n}{1}=A\overset{V}{\Leftrightarrow}\frac{C_n^n}{n+1}+\frac{C_n^{n-1}}{n}+$$
$$+\frac{C_n^{n-2}}{n-1}+\ldots+\frac{C_n^0}{1}=A.$$

We multiply both members of the last equality with $(n+1)$. We obtain

$$\frac{n+1}{n+1}C_n^n+\frac{n+1}{n}C_n^{n-1}+\frac{n+1}{n-1}C_n^{n-2}+$$
$$+\ldots+\frac{n+1}{1}C_n^0=A(n+1)\overset{(10)}{\Leftrightarrow}$$
$$\Leftrightarrow\ C_{n+1}^1+C_{n+1}^2+C_{n+1}^3+$$
$$+\ldots+C_{n+1}^{n+1}=A(n+1)\Leftrightarrow$$
$$\Leftrightarrow C_{n+1}^0+C_{n+1}^1+C_{n+1}^2+\ldots+C_{n+1}^{n+1}=$$
$$C_{n+1}^0+A(n+1)\overset{IV}{\Longleftrightarrow}$$
$$\Leftrightarrow 2^{n+1}=C_{n+1}^0+A(n+1)\Leftrightarrow 2^{n+1}-C_{n+1}^0=A(n+1)\Leftrightarrow$$
$$\Leftrightarrow A=\frac{2^{n+1}-1}{n+1}\Leftrightarrow A=\frac{2}{n+1}\cdot\left(2^n-\frac{1}{2}\right).$$

Returning to the initial expression, we have

$$\frac{C_n^0}{n+1}+\frac{C_n^1}{n}+\frac{C_n^2}{n-1}+\ldots+\frac{C_n^n}{1}=\frac{2}{n+1}\left(2^n-\frac{1}{x}\right),$$

that had to be proven.

25. Prove that

$$nC_n^0-(n-1)C_n^1+(n-2)C_n^2-(n-3)C_n^3+$$
$$+\ldots+(-1)^{n-1}C_n^{n-1}=0.$$

Solution. We write the development of the binomial $(x-1)^n$:

$$(x-1)^n = C_n^0 x^n - C_n^1 x^{n-1} + C_n^2 x^{n-2} - C_n^3 x^{n-3} +$$
$$+ \ldots + (-1)^{n-1} C_n^{n-1} x + (-1) C_n^n. \qquad (A)$$

We derive both members of the equality (A) according to x. We obtain

$$n(x-1)^{n-1} = nC_n^0 x^{n-1} - (n-1)C_n^1 x^{n-2} + (n-2)C_n^2 x^{n-3} -$$
$$-(n-3)C_n^3 x^{n-4} + \ldots + (-1)^{n-1} C_n^{n-1}. \qquad (**)$$

We place in $(**)$ $x = 1$. Then

$$0 = nC_n^0 - (n-1)C_n^1 + (n-2)C_n^2 - (n-3)C_n^3 +$$
$$\ldots + (-1)^{n-1} C_n^{n-1},$$

that had to be proven.

26. Prove that the equality

$$1 - 10C_{2n}^1 + 10^2 C_{2n}^2 - 10^3 C_{2n}^3 + \ldots - 10^{2n-1} C_{2n}^1 + 10^{2n}$$

is true.

Solution. We notice that the expression

$$1 - 10C_{2n}^1 + 10^2 C_{2n}^2 - 10^3 C_{2n}^3 + \ldots - 10^{2n-1} C_{2n}^1 + 10^{2n}$$

represents the development of the binomial $(1-10)^{2n} = 9^{2n} = (81)^n$, that had to be proven.

27. Bring the expression $P_1 + 2P_2 + \cdots + nP_n$ to a simpler form.

Solution. We will make the necessary transformations by applying the method of mathematical induction. Let

$$P_1 + 2P_2 + \ldots + nP_n = A_n. \qquad (*)$$

For:

$n = 1$, we have $P_1 = A_1 \Leftrightarrow A_1 = 1$;

$n = 2$, we have $P_1 + 2p_2 = A_2 \Leftrightarrow 1 + 2.2! = A_2 \Leftrightarrow 5 = A_2 \Leftrightarrow 3! - 1 = A_2 \Leftrightarrow P_3 - 1 = A_2$

$n = 3$, we have $P_1 + 2p_2 + 3P_2 = A_3 \Leftrightarrow A_2 + 3P_3 = A_3 \Leftrightarrow$
$\Leftrightarrow 3! - 1 + 3 \cdot 3! = A_3 \Leftrightarrow 3!(1+3) - 1 = A_3 \Leftrightarrow 4! - 1 =$

$= A_3 \Leftrightarrow P_4 - 1 = A_3.$

We assume that for $n = k$ the equality $(*)$ takes the form

$$P_1 + 2P_2 + \ldots + kP_k = (k+1)! - 1. \qquad (**)$$

We calculate the value of the expression $(*)$ for $n = k + 1$. We have

$$P_1 + 2P_2 + 3P_3 + \ldots + kP_k + (k+1)P_{k+1} \overset{(**)}{=} (k+1)! - 1 + (k+1)P_{k+1} = (k+1)! - 1 + (k+1)(k+1)! = (k+1)!(1+k+1) - 1 = (k+2)! - 1 = P_{k+2} - 1.$$

Based on the principle of mathematical induction, we reach the conclusion that

$$P_1 + 2P_2 + \ldots + nP_n = (n+1)! - 1 = P_{n+1} - 1.$$

Answer:

$$P_1 + 2P_2 + \ldots + nP_n = P_{n+1} - 1.$$

28. Prove that indifferent of what $m, n \in \mathbb{N}$, are $m!\,.\,n!$ divides $(m + n)!$

Solution. According to the definition 3, $\frac{(m+n)!}{m!\,.\,n!} = C_{m+n}^{n}$ is the number of the subsets that have n elements of a set with $(m + n)$ elements, namely C_{m+n}^{n} is a natural number. Consequently, $\frac{(m+n)!}{m!\,.\,n!}$ is an integer number, or this proves that $m!\,.\,n!$ divides $(m + n)!$

29. Deduce the equality

$$(n-k)C_n^{k+1} - (k+1)C_n^k = (n-2k-1)C_{n+1}^{k+1}.$$

Solution. We use property$X$. We have:

$$C_n^{k+1} = (n-k)/(k+1)C_{n+1}^{k+1}, \quad C_n^k = (k+1)/(n+1)C_{n+1}^{k+1}.$$

As a result,

$$(n - k)C_n^{k+1} - (k + 1)C_n^k =$$

$$= \frac{(n-k)^2}{n+1}C_{n+1}^{k+1} - \frac{(k+1)^2}{n+1}C_{n+1}^{k+1} =$$

$$= \frac{(n-k)^2-(k+1)^2}{n+1}\cdot C_{n+1}^{k+1} =$$
$$= \frac{(n-k-k-1)(n-k+k+1)}{n+1}C_{n+1}^{k+1} =$$
$$= \frac{(n-2k-1)(n+1)}{n+1}\cdot C_{n+1}^{k+1} = (n-2k-1)\cdot C_{n+1}^{k+1}$$

that had to be proven.

30. Calculate the sum

$$S_n = \frac{3}{1!+2!+3!} + \frac{4}{2!+3!+4!} + \ldots + \frac{n+2}{n!+(n+1)!+(n+2)!}.$$

Solution. We notice that term a_n of this sum can be transformed as it follows:

$$a_n = \frac{n+2}{n!+(n+1)!+(n+2)!} = \frac{n+2}{n!(1+n+1+(n+1)(n+2))} =$$
$$= \frac{n+2}{n!(n+2)^2} = \frac{1}{n!(n+2)} = \frac{n+1}{n!(n+1)(n+2)} =$$
$$= \frac{n+1}{(n+2)!} = \frac{(n+2)-1}{(n+2)!} = \frac{1}{(n+1)!} - \frac{1}{(n+2)!}.$$

Then the sum S_n takes the form

$$S_n = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{3!} - \frac{1}{4!} + \ldots - \frac{1}{(k-1)!} + \frac{1}{k!} + \ldots -$$
$$- \frac{1}{(n+1)!} + \frac{1}{(n+1)!} - \frac{1}{(n+2)!} = \frac{1}{2} - \frac{1}{(n+2)!}.$$

Answer:

$S_n = 1/2 - 1/(n+2)!$

31. Solve the equation

$C_x^1 + 6C_x^2 + 6C_x^3 = 9x^2 - 14x.$

Solution.

$C_x^1 + 6C_x^2 + 6C_x^3 = 9x^2 - 14x \Leftrightarrow$

$$\Leftrightarrow x + 6 \cdot \frac{x(x-1)}{1 \cdot 2} + 6 \cdot \frac{x(x-1)(x-2)}{1 \cdot 2 \cdot 3} = 9x^2 - 14x \Leftrightarrow$$
$$\Leftrightarrow x + 3x(x-1) + x(x-1)(x-2) = 9x^2 - 14x \overset{x>0}{\Longleftrightarrow}$$
$$\overset{x>0}{\Longleftrightarrow} 1 + 3x - 3 + x^2 - 3x + 2 - 9x + 14 = 0 \Leftrightarrow$$
$$\Leftrightarrow x^2 - 9x + 14 = 0 \Leftrightarrow \left[\begin{array}{l} x = 2, \\ x = 7. \end{array}\right.$$

Because C_x^3 has meaning only for $x \geq 3$, it follows that the solution of the initial equation is $x = 7$.

Answer: $x = 7$.

32. Solve the equation

$$C_{x+1}^{x-2} + 2C_{x-1}^3 = 7(x-1).$$

Solution.

$$C_{x+1}^{x-2} + 2C_{x-1}^3 = 7(x-1) \Leftrightarrow$$
$$\Leftrightarrow \frac{(x+1)!}{(x-2)!((x+1)-(x-2))!} + 2 \cdot \frac{(x-1)!}{(x-1-3)!} = 7(x-1) \Leftrightarrow$$
$$\Leftrightarrow \frac{(x-2)!(x-1)x(x+1)}{(x-2)! \cdot 3!} + 2 \cdot \frac{(x-4)!(x-3)(x-2)(x-1)}{(x-4)! \cdot 3!} =$$
$$= 7(x-1) \overset{x\geq 4}{\Longleftrightarrow} (x-1)x(x+1) + 2(x-3)(x-2)(x-1) -$$
$$-42(x-1) = 0 \Leftrightarrow x^2 - 3x - 10 =$$
$$= 0 \Rightarrow \left[\begin{array}{l} x = -2, \\ x = 5 \end{array}\right. \Rightarrow x = 5.$$

Answer: $x = 5$.

33. Solve the equation

$$(A_{x+1}^{y+1} \cdot P_{x-y}) / P_{x-1} = 72. \qquad (A)$$

Solution. As: $A_{x+1}^{y+1} = (x+1)!/(x-y)!$, $P_{x-y} = (x-y)!$, $P_{x-1} = (x-1)!$,

We have:

$$(A) \Leftrightarrow \frac{(x+1)!}{(x-y)!}\cdot\frac{(x-y)!}{(x-1)!} = 72 \underset{0\le y\le x}{\overset{x\ge 1}{\Leftrightarrow}} x(x+1) =$$

$$= 72 \Leftrightarrow \left[\begin{array}{l} x = 8, \\ x = -9 \end{array}\right. \overset{x\in N*}{\Longrightarrow} x = 8.$$

As $y \in \mathbb{N}$ and $y \le x$, we have $y \in \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$.

Answer: $x = 8$; $y \in \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$.

34. Determine the values of x that verify the equality

$(x+2)! = -15(x-1)! + 5[x! + (x+1)!]$.

Solution.

$$(x+2)! = -15(x-1)! + 5[x! + (x+1)!] \Leftrightarrow$$

$$\Leftrightarrow (x-1)!x(x+1)(x+2) =$$

$$= -15(x-1)! + 5[(x-1)!x + (x-1)!x(x+1)] \Leftrightarrow$$

$$\Leftrightarrow (x-1)!x(x+1)(x+2) =$$

$$= (x-1)![-15 + 5x + 5x(x+1)] \overset{x\ge 1}{\Longleftrightarrow}$$

$$\Leftrightarrow x(x^2 + 3x + 2) =$$

$$= -15 + 5x^2 + 10x \Leftrightarrow x^3 - 2x^2 - 8x + 15 = 0 \Leftrightarrow$$

$$\Leftrightarrow \left[\begin{array}{l} x = 3, \\ x^2 + x - 5 = 0 \end{array}\right. \Rightarrow x = 3\,,$$

because the solutions of the equation $x^2 + x - 5 = 0$ are irational numbers.

Answer: $x = 3$.

35. Solve the equation

$A_{x+1}^{n+1} \cdot (x-n)! = 90(x-1)!$

Solution.

$$A_{x+1}^{n+1} \cdot (x-n)! = 90(x-1)! \Leftrightarrow \frac{(x+1)!}{(x-n)!}\cdot(x-n)! =$$

$$= 90(x-1)! \underset{0\le n\le x}{\overset{x\ge 1}{\Leftrightarrow}} (x-1)!x(x+1) =$$

$$= 90(x-1)! \Leftrightarrow x^2 + x - 90 = 0 \Leftrightarrow$$

$$\Leftrightarrow \left[\begin{array}{l} x = 9, \\ x = -10 \end{array} \right. \Rightarrow x = 9$$

Then $n \in \{0, 1,2,3,4,5,6, 7,8\}$.

Answer: $x = 9, n \in \{0, 1,2,3,4,5,6, 7,8\}$.

36. Solve the system of equations

$$\begin{cases} A_y^x : P_{x-1} + C_y^{y-x} = 126, \\ P_{x+1} = 720. \end{cases} \qquad (B)$$

Solution. From the conditions of the problem, we have $x, y \in \mathbb{N}$ with $x \geq 1$ and $\leq$. Based on the formulas $(1) - (3)$, we have

$$(B) \Leftrightarrow \begin{cases} \dfrac{y!}{(y-x)!} \cdot \dfrac{1}{(x-1)!} + \dfrac{y!}{(y-x)!x!} = 126, \\ (x+1)! = 720 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \dfrac{y!(x+1)}{(y-x)! \cdot x!} = 126, \\ (x+1)! = 6! \end{cases} \Leftrightarrow \begin{cases} \dfrac{y! \cdot 6}{(y-x)! \cdot 5!} = 126, \\ x+1 = 6 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} (y-4)(y-3)(y-2)(y-1)y = 5! \cdot 21, \\ x = 5 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} y^5 - 10y^4 + 35y^3 - 50y^2 + 24y - 2520 = 0, \\ x = 5. \end{cases} \qquad (*)$$

The divisors of the free term in $(*)$ are the numbers

$\pm 1;\ \pm 2;\ \pm 3;\ \pm 4;\ \pm 5;\ \pm 6;\ \pm 7;\ \pm 8;\ \pm 9;\ \pm 10;\ \ldots.$

We use Homer's scheme and the Bezout theorem to select the numbers that are solutions of the equation $(*)$. As $y \in \mathbb{N}$, we will verify only the natural numbers. It verifies that numbers $\{1, 2, 3, 4, 5, 6\}$ are not solutions of the equation $(*)$.

We verify $y = 7$.

	1	−10	35	−50	24	−2520
7	1	−3	14	48	360	0

So

$$(*) \Leftrightarrow (y-7)(y^4 - 3y^3 + 14y^2 + 48y + 360) = 0 \Leftrightarrow y = 7,$$

because for $y > 7$, the expression $y^4 - 3y^3 + 14y^2 + 48y + 360 > 0$.

Hence, the solution of the system (B) is the pair $(5,7)$.
Answer: $\{(5,7)\}$.

37. Find x and y, if
$$C_x^{y-1}\colon (C_{x-2}^{y}+C_{x-2}^{y-2}+2C_{x-2}^{y-1})\colon C_x^{y+1}=3:5:5. \qquad (C)$$
Solution. We will first bring to a simpler form the expression
$$C_{x-2}^{y}+C_{x-2}^{y-2}+2C_{x-2}^{y-1}=(C_{x-2}^{y}+C_{x-2}^{y-1})+(C_{x-2}^{y-1}+C_{x-2}^{y-2})\overset{II}{=}$$
$$=C_{x-1}^{y}+C_{x-1}^{y-1}\overset{II}{=}C_x^y.$$
As a result, the system (C) takes the form
$$C_x^{y-1}:C_x^y:C_x^{y+1}=3:5:5\Leftrightarrow\begin{cases}C_x^{y-1}:C_x^y=3:5,\\ C_x^y:C_x^{y+1}=5:5,\end{cases}\overset{(3)}{\Leftrightarrow}$$
$$\Leftrightarrow\begin{cases}\dfrac{x!}{(y-1)!(x-y+1)!}:\dfrac{x!}{y!(x-y)!}=\dfrac{3}{5},\\ \dfrac{x!}{y!(x-y)!}:\dfrac{x!}{(x-y-1)!(y+1)!}=1\end{cases}\Leftrightarrow$$
$$\Leftrightarrow\begin{cases}\dfrac{y\cancel{!}\,\cancel{(x-y)!}}{\cancel{(y-1)!}\,(x-y+1)\cancel{!}}=\dfrac{3}{5},\\ \dfrac{\cancel{(x-y-1)!}\,(y+1)\cancel{!}}{\cancel{y!}\cdot(x-y)\cancel{!}}=1\end{cases}\Leftrightarrow$$
$$\Leftrightarrow\begin{cases}5y=3x-3y+3,\\ y+1=x-y\end{cases}\Leftrightarrow\begin{cases}8y=6y+3+3,\\ x=2y+1\end{cases}\Leftrightarrow\begin{cases}y=3,\\ x=7.\end{cases}$$
Answer: $\{(7,3)\}$.

38. Determine the values of x, so that
$$(x(x+1)!)/(2\cdot x!)\le 2x+9.$$
Solution. From the enunciation it follows that $x\in\mathbb{N}$. Then
$$\frac{x(x+1)!}{2\cdot x!}\le 2x+9\Leftrightarrow\frac{x\cdot x!(x+1)}{2\cdot x!}\le$$
$$\le 2x+9\Leftrightarrow x^2+x\le 4x+18\Leftrightarrow x^2-3x-18\le 0\Leftrightarrow$$
$$\Leftrightarrow(x+3)(x-6)\le 0\overset{x\in N}{\Longleftrightarrow}x-6\le 0\Leftrightarrow 0\le x\le 6$$

and $x \in \mathbb{N}$. So $x \in \{0, 1, 2, 3, 4, 5, 6\}$.

Answer: $x \in \{0, 1, 2, 3, 4, 5, 6\}$.

39. Determine the values of x that verify the inequation

$$x \cdot C_{x-1}^{x-3} - 7 \cdot C_{x-2}^{x-3} \leq 8(x-2). \qquad (*)$$

Solution. $(*)$ has meaning for $x \in \mathbb{N}$ and $x \geq 3$. Because

$$C_{x-1}^{x-3} = C_{x-1}^{2} = \frac{(x-1)(x-2)}{1 \cdot 2}; \; C_{x-2}^{x-3} = C_{x-2}^{1} = x-2,$$

it follows that

$$(*) \Leftrightarrow \frac{x(x-1)(x-2)}{2} - 7(x-2) \leq 8(x-2) \Leftrightarrow$$

$$\Leftrightarrow (x-2)[x(x-1)-30] \leq 0 \Leftrightarrow (x-2)(x^2-x-30) \leq 0 \Leftrightarrow$$

$$\Leftrightarrow (x-2)(x+5)(x-6) \leq 0 \overset{x \in N}{\Longleftrightarrow}$$

$$\Leftrightarrow (x-2)(x-6) \leq 0 \Leftrightarrow 2 \leq x \leq 6 \Rightarrow \left[\begin{array}{l} x=3, \\ x=4, \\ x=5, \\ x=6. \end{array}\right.$$

Answer: $x \in \{3, 4, 5, 6\}$.

40. Solve the system of inequations:

$$\begin{cases} C_{x+1}^{x-2} - C_{x+1}^{x-1} \leq 100, \\ C_{x+5}^{4} - \dfrac{143 \cdot P_{x+5}}{96 \cdot P_{x+3}} < 0. \end{cases} \qquad (D)$$

Solution.

$$(D) \Leftrightarrow \begin{cases} \dfrac{(x+1)!}{(x-2)! \cdot 3!} - \dfrac{(x+1)!}{(x-1)! \cdot 2!} \leq 100, \\ \dfrac{(x+5)!}{4! \cdot (x+1)!} - \dfrac{143 \cdot (x+5)!}{96 \cdot (x+3)!} < 0 \end{cases} \Leftrightarrow$$

$$\overset{x\geq 2}{\Leftrightarrow}\begin{cases}\dfrac{(x-1)x(x+1)}{3!}-\dfrac{x(x+1)}{2!}\leq 100,\\ \dfrac{(x+2)(x+3)(x+4)(x+5)}{4!}-\dfrac{143(x+4)(x+5)}{96}<0\end{cases}\Leftrightarrow$$

$$\underset{x\in N}{\overset{x\geq 2}{\Leftrightarrow}}\begin{cases}x^3-3x^2-4x-600\leq 0,\\ x\in\{2,3\}\end{cases}\Leftrightarrow\left[\begin{array}{l}\begin{cases}x^3-3x^2-4x-600\leq 0,\\ x=2,\end{cases}\\ \begin{cases}x^3-3x^2-4x-600\leq 0,\\ x=3\end{cases}\end{array}\right.\Leftrightarrow$$

$$\Leftrightarrow\left[\begin{array}{l}\begin{cases}8-12-8-600\leq 0,\\ x=2,\end{cases}\\ \begin{cases}27-27-12-600\leq 0,\\ x=3\end{cases}\end{array}\right.\Rightarrow\left[\begin{array}{l}x=2,\\ x=3.\end{array}\right.$$

Answer: $x\in\{2,3\}$.

3.3. Suggested exercises

1. A commission is formed of one president, his assistant and five other persons. In how many ways can the members of the commission distribute the functions among themselves?

2. In how many ways can three persons be chosen from a group of 20 persons to accomplish an assignment?

3. In a vase there are 10 red daffodils and 4 pink ones. In how many ways can three flowers be chosen from the vase?

4. The padlock can be unlocked only if a three digit number is correctly introduced out of five possible digits. The number is guessed, randomly picking 3 digits. The last guess proved to be the only successful one. How many tries have proceeded success?

5. On a shelf there are 30 volumes. In how many ways can the books be arranged, so that volumes 1 and 2 do not sit beside each other on the shelf?

6. Four sharpshooters have to hit eight targets (two targets each). In how many ways can the targets be distributed among the sharpshooters?

7. How many four digit numbers, made out of digits 0, 1, 2, 3, 4, 5, contain digit 3, if: a) the digits do not repeat in the number; b) the digits can repeat themselves?

8. In the piano sections there are 10 participating persons, in the reciter section, 15 persons, in the canto section, 12 persons, and in the photography section - 20 persons. In how many ways can there a team be assembled that contains 4 reciters, 3 pianists, 5 singers and a photographer?

9. Seven apples and three oranges have to be placed in two bags such that every bag contains at least one orange and the number of fruits in the bags is the same. In how many ways can this distribution be made?

10. The matriculation number of a trailer is composed of two letters and four digits. How many matriculation numbers can there be formed using 30 letters and 10 digits?

11. On a triangle side there are taken n points, on the second side there are taken m points, and on the third side, k points. Moreover, none of the considered points is the triangle top. How many triangles with tops in these points are there?

12. Five gentlemen and five ladies have to be sited around a table so that no two ladies and no two gentlemen sit beside each other. In how many ways can this be done?

13. Two different mathematic test papers have to be distributed to 12 students. In how many ways can the students be arranged in two rows so that the students that sit side by side have different tests and those that sit behind one another have the same test?

14. Seven different objects have to be distributed to three persons. In how many ways can this distribution be made, if one or two persons can receive no object?

15. How many six digit numbers can be formed with the digits 1, 2, 3, 4, 5, 6, 7, in such a way that the digits do not repeat, and that the digits at the beginning and at the end of the number are even?

16. How many different four digit numbers can be formed with the digits 1, 2, 3, 4, 5, 6, 7, 8, if in each one the digit one appears only once, and the other digits can appear several times?

17. To award the winners of the Mathematics Olympiad, three copies of a book, two copies of another book and one copy of a third book have been provided. How many prizes can be awarded if 20 persons have attended the Olympiad, and no one will receive two books simultaneously? Same problem, if no one will receive two copies of the same book but can receive two or three different books?

18. The letters of the Morse alphabet is comprised of symbols (dots and dashes). How many letters can be drawn, if it is required that each letter contains no more than five symbols?

19. To find their lost friend, some tourists have formed two equal groups. Only four persons among them know the surroundings. In how many ways can the tourists divide so that each group has two persons that know the surroundings and considering they are 16 persons in total?

20. Each of the 10 radio operators located at point A is trying to contact each of the 20 radio operators located at point B. How many different options for making contact are there?

Prove the equalities:

21. $C_n^{m+1} + C_n^{m-1} + 2C_n^m = C_{n+2}^{m+1}$;

22. $C_n^m + C_{n-1}^m + \ldots + C_{n-10}^m = C_{n+1}^{m+1} - C_{n-10}^{m+1}$;

23. $C_n^1 + 2C_n^2 + 3C_n^3 + \ldots + nC_n^n = n \cdot 2^{n-1}$;

24. $C_n^0 + 2C_n^1 + 3C_n^2 + \ldots + (n+1)C_n^n = (n+1) \cdot 2^{n-1}$;

25. $C_n^0 - \frac{1}{2}C_n^1 + \frac{1}{3}C_n^2 - \ldots + (-1)^n \cdot \frac{C_n^n}{n+1} = \frac{1}{n+1}$;

26. $\frac{C_n^1}{C_n^0} + \frac{2C_n^2}{C_n^1} + \frac{3C_n^3}{C_n^2} + \ldots + \frac{nC_n^n}{C_n^{n-1}} = \frac{n(n+1)}{2}$;

27. $C_n^0 + 2C_n^1 + 2^2C_n^2 + \ldots + 2^nC_n^n = 3^n$;

28. $2C_n^0 + \frac{2^2C_n^1}{2} + \frac{2^3C_n^2}{3} + \ldots + \frac{2^{n+1}C_n^n}{n+1} = \frac{3^{n-1} - 1}{n+1}$;

29. $\frac{C_n^k}{C_n^k + C_n^{k+1}} + \frac{C_n^{k+5}}{C_n^{k+2} + C_n^{k+3}} = \frac{2C_n^{k+1}}{C_n^{k+1} + C_n^{k+2}}$

(where $n, k \in \mathbb{N}, n > k + 3$);

30. $\frac{2^n}{n!} + \frac{2^{n-1}}{1!(n-1)!} + \frac{2^{n-2}}{2!(n-2)!} + \ldots + \frac{2^0}{n!} = \frac{3^n}{n!}$;

31. $\sum_{k=0}^{n} C_{n+k}^n \cdot 1/2^k = 2^n$;

32. $\sum\limits_{p=1}^{n}(pC_n^p)^2 = n \cdot C_{2n-1}^{n-1}$;

33. $\sum\limits_{k=0}^{m} C_p^k \cdot C_q^{m-k} = C_{p+q}^{m}$;

34. $\sum\limits_{k=0}^{m}(-1)^k C_m^k = (-1)^n C_{n-1}^{m}$;

35. $\sum\limits_{k=m}^{n} C_k^m \cdot C_n^k = C_n^m \cdot 2^{n-m}$

Solve the equations and the systems of equations**:**

36. $A_x^2 : C_x^{x-1} = 48$;

37. $C_{x+1}^{x-2} + 2C_{x-1}^{3} = 7(x-1)$;

38. $A_x^4 : (A_{x+1}^3 - C_x^{x-4}) = 24/23$;

39. $A_x^3 + C_x^{x-2} = 14x$;

40. $A_x^3 - 2A_x^4 = 3A_x^2$;

41. $A_x^5 : C_{x-1}^{x-5} = 336$;

42. $A_x^{x-3} = xP_{x-2}$;

43. $P_{x+2} : (A_{x-1}^{x-4} \cdot P_3) = 210$;

44. $A_{x+1}^{x-1} + 2P_{x-1} = (30/7) \cdot P_x$;

45. $C_x^{x-1} + C_x^{x-2} + C_x^{x-3} + \ldots + C_x^{x-8} + C_x^{x-9} + C_x^{x-10} = 1023$;

46. $P_{x+3} : (A_x^5 \cdot P_{x-5}) = 720$;

47. $C_{2x}^{x+1} : C_{2x+1}^{x-1} = 2/3$;

48. $A_{x-1}^2 - C_x^1 = 79$;

49. $3C_{x+1}^2 - 2A_x^2 = x$;

50. $C_{x+1}^2 : C_x^3 = 4/5$;

51. $12C_x^1 + C_{x+4}^2 = 162$;

52. $A_{x+1}^3 + C_{x+1}^{x-2} = 14(x+1)$;

53. $P_{x+6} : (A_{x+4}^{n+4} \cdot P_{x-n}) = 240$;

54. $C_{x+1}^{x-4} = 7/15 \cdot A_{x+1}^3$;

55. $C_{x+1}^3 : C_x^4 = 6:5$;

56. $C_{x+1}^2 \cdot A_x^2 - 4x^3 = (A_{2x}^1)^2$;

57. $3C_{x+1}^2 + P_2 \cdot x = 4A_x^2$;

58. $A_{x+3}^2 = C_{x+2}^3 + 20$;

59. $C_x^3 + C_x^4 = 11C_{x+1}^2$;

60. $11C_x^3 = 24C_{x+1}^2$;

61. $(A_{x+1}^8 + A_x^7) : A_{x-1}^6 = 99$;

62. $A_{x+1}^{n+1} \cdot (x-n)! = 90(x-1)!$;

63. $C_x^5 = C_x^3$;

64. $C_x^3 = 2C_x^{x-2}$;

65. $A_x^6 - 24xC_x^4 = 11A_x^4$;

66. $A_{x-2}^2 + C_x^{x-2} = 101$;

67. $\dfrac{1}{P_{x-1}} - \dfrac{1}{P_x} = \dfrac{(x-1)^3}{P_{x+1}}$;

68. $12C_{x-3}^{x+1} = 55A_{x+1}^2$;

69. $C_{5x-x^2+5}^{x} = C_{11}^{8}$;

70. $A_x^5 = 18A_{x-2}^4$;

71. $(A_x^{10} - A_x^8) : A_x^8 = 109$;

72. $(n+2)! : (A_n^k \cdot (n-k)!) = 132$;

73. $\begin{cases} A_x^y + 3C_x^y = 90, \\ A_x^y - 2C_x^y = 40; \end{cases}$

74. $C_{x+1}^y : C_x^{y+1} : C_x^{y-1} = 6:5:2$;

75. $(A_{x-1}^y + yA_{x-1}^{y-1}) : A_x^{y-1} : C_x^{y-1} = 10:2:1$;

76. $A_x^{y-1} : A_{x-1}^y : (C_{x-2}^y + C_{x-2}^{y-1}) = 21:60:10$;

77. $C_x^{y-1} : C_x^y : C_x^{y+1} = 2:3:4$;

78. $\begin{cases} xA_{x-1}^{y-1} \cdot P_{x-y} = 15P_{x-1}, \\ 9C_{x+1}^y = 16C_x^{y+1}; \end{cases}$

79. $\begin{cases} A_{2x}^{3y} - 8A_{2x}^{3y-1} = 0, \\ 9C_{2x}^{3y} - 8C_{2x}^{3y-1} = 0; \end{cases}$

80. $\begin{cases} A_x^y = 7A_x^{y-1}, \\ 6C_x^y = 5C_x^{y+1}; \end{cases}$

81. $\begin{cases} xC_{n-2}^{k-1} + \dfrac{n-1}{k-1}y = \dfrac{k}{n-1}, \\ xC_{n-2}^{k-2} - \dfrac{n-1}{k}y = \dfrac{k-1}{n-1}; \end{cases}$

82. $\begin{cases} 7C_x^1 \cdot y^{x-1} \cdot z = A_8^4, \\ C_x^2 \cdot y^{x-2} \cdot z^2 = A_{10}^3, \\ 11C_x^3 \cdot y^{x-3} \cdot z^3 = A_{13}^4. \end{cases}$

Solve the inequations and the systems of inequations:

83. $\dfrac{(x-1)!}{(x-3)!} < 72$;

84. $\dfrac{(n+2)!}{(n+1)(n+2)} < 1000$;

85. $x(x-3)! < 108(x-4)!$;

86. $C_x^5 < C_x^6$;

87. $C_x^5 > C_x^7$;

88. $C_{20}^{x-1} < C_{20}^{x}$;

89. $C_{16}^{x-2} > C_{16}^{x}$;

90. $C_x^5 < C_x^3$;

91. $C_{13}^{x} < C_{13}^{x+2}$;

92. $C_{18}^{x-2} > C_{18}^{x}$;

93. $C_x^6 < C_x^4$;

94. $5C_x^3 < C_{x+2}^4$;

95. $C_{x+1}^{x-1} > 3/2$;

96. $2C_x^5 > 11C_{x-2}^3$;

97. $C_x^{x-1} \le C_x^{x-3}$;

98. $C_{2x}^{2x-8} \ge C_{2x}^{2x-12}$;

99. $xC_{x-1}^{x-2} - 7C_{x-2}^{x-3} \le 8(x-2)$;

100. $C_{x+1}^{x-2} - C_{x+1}^{x-1} \le 100$;

101. $A_{x+1}^4 : C_{x-1}^{x-3} > 14P_3$;

102. $C_{x-1}^4 - C_{x-1}^3 - \dfrac{5}{4}A_{x-2}^4 < 0$;

103. $C^4_{x+5} - \frac{143P_{x+5}}{96P_{x+3}} < 0;$ **104.** $\frac{A^3_{x+2}}{P_{x+2}} - \frac{143}{4P_{x-1}} < 0;$

105. $\begin{cases} C^7_{2x} > C^5_{2x}, \\ C^x_{13} < C^{x+2}_{13}, \end{cases}$ **106.** $\begin{cases} C^{x-1}_{x+1} < 21, \\ 5C^3_x < C^4_{x+2}, \\ C^{x-3}_{x-1} : A^4_{x+1} < 1:14P_3. \end{cases}$

107. Determine the fifth term from the development of the binomial $(2x\sqrt{x} - \sqrt[3]{x})^8$.

108. Determine the middle term from the development of the binomial $(2x + y/2)^8$.

109. Determine the value of exponent m from the development of the binomial $(1 + a)^m$, if the coefficient of the 5th term is equal to the coefficient of the 9th term.

110. Determine A^2_n, if the 5th term from the development of the binomial $(\sqrt[3]{x} + 1/x)^n$ doesn't depend on x.

111. In the development of the binomial $(\sqrt{1+x} - \sqrt{1-x})^n$ the coefficient of the third term is equal to 28. Determine the middle term from the development of the binomial.

112. Determine the smallest value of $m's$ exponent from the development of the binomial $(1 + a)^m$, if the relation of the coefficients of two neighboring arbitrary is equal to $7:15$.

113. Determine the term from the development of the binomial $(\sqrt{x} + 1/\sqrt[3]{x})^{16}$ that contains x^3.

114. Determine the term from the development of the binomial $(\sqrt[3]{a}+\sqrt{a^{-1}})^{15}$ that doesn't depend on a.

115. Determine the term from the development of the binomial $((a\sqrt[3]{a})/6+1/\sqrt[15]{a^{28}})^n$ that doesn't contain a, if the sum of the coefficients of the first three terms from the development is equal to 79.

116. Determine the term from the development of the binomial $(1/\sqrt[3]{a^2}+\sqrt[4]{a^3})^{17}$ that doesn't contain a.

117. Determine the free term from the development of the binomial $(\sqrt{\sqrt{x}}+2/\sqrt[3]{x})^{1989}$.

118. Determine the third term from the development of the binomial $(z^2+1/z\cdot\sqrt[3]{z})^m$, if the sum of the binomial coefficients is 2048.

119. Determine the term from the development of the binomial $(1/\sqrt[3]{b^2}-\sqrt[4]{b}/\sqrt[8]{a^3})^n$ that contains b^6, if the relation of the binomial coefficients of the fourth and second term is equal to 187.

120. Determine the term from the development of the binomial $(x\sqrt[4]{x}-1/\sqrt[8]{x^5})^n$ that doesn't contain x, if the sum of the coefficients of the second term from the beginning of the development and the third term from the end of the development is equal to 78.

121. The relation of the coefficient of the third term to the coefficient of the 5th term in the development of the binomial $(x^{-3/2} - \sqrt[3]{x})^n$ is equal to $2/7$. . Determine the term from the development of the binomial that contains $x^{-5/2}$.

122. Determine x, y and z, if it is known that the 2nd, 3rd and 4th terms from the development of the binomial $(x + y)^z$ are equal to $240, 720, 1080$, respectively.

123. For what value of the exponent n, the coefficients of the 2nd, the 3rd and the 4th term from the development of the binomial $(x + y)^n$ form an arithmetic progression?

124. Determine the terms from the development of the binomial $(\sqrt[5]{3} + \sqrt[7]{2})^{24}$ that do not contain irrationalities.

125. How many rational terms does the following development of the binomial contain $(\sqrt{2} + \sqrt[4]{3})^{100}$?

126. Determine the ranks of three consecutive terms from the development of the binomial $(a + b)^{23}$ the coefficients of which form an arithmetic progression.

127. Determine the term from the development of the binomial $(\sqrt{x} + \sqrt[4]{x^{-3}})^n$ that contains $x^{6.5}$, if the 9th term has the biggest coefficient.

128. The 3rd term from the development of the binomial $(sx + \frac{1}{x^2})^m$ doesn't contain x. For which value of x is this term equal to the 2nd term from the development of the binomial $(1 + x^3)^{30}$?

129. For what positive values of x the biggest term from the development of the binomial $(5+3x)^{10}$ is the 4th term?

130. In the development of the binomial $(\sqrt{x}+\frac{1}{2\sqrt[4]{x}})^n$ the first three terms form an arithmetic progression. Determine all rational terms from the development of this binomial.

131. Determine the values of x for which the difference between the 4th and the 6th term from the development of the binomial $(\frac{\sqrt{2x}}{\sqrt[16]{8}}+\frac{\sqrt[16]{32}}{\sqrt{2x}})^m$ is equal to 56, if it is known that the exponent m of the binomial is smaller by 20 than the binomial coefficient of the 3rd term from the development of this binomial.

132. Given that n is the biggest natural number only if $log_{l/3}\, n\ +\ llog_{n/3}\, n\ >\ 0$, determine the term that contains b^2 from the development of the binomial $(\sqrt{a}-\sqrt[3]{b})^n$.

133. Determine x for which the sum of the 3rd and al 5th term from the development of the binomial $(\sqrt{2x}+\sqrt{2^{1-x}})^n$ is equal to 135, knowing that the sum of the last three binomial coefficients is 22.

134. Determine x, knowing that the 6th term from the development of the binomial $(a+b)^n$, where $a=\sqrt{2^{\lg(10-3^x)}}$, $b=\sqrt[5]{2^{(x-2)lg3}}$ is 21, and the coefficients of the binomial of the terms ranked $2, 3$ and 4 are respectively the 1st, the 3rd and the 5th term of an arithmetic progression.

135. Are there any terms independent of x in the development of the binomial $\left(\sqrt{\sqrt[3]{x^2}}+2/\sqrt[3]{x}\right)^{1988}$? Write these terms.

136. How many terms from the development of the binomial $(\sqrt[5]{3}+\sqrt[3]{7})^{36}$ are integer terms?

137. In the development of the binomial $\left(\sqrt{y}+\frac{1}{2\sqrt[4]{y}}\right)^n$ the first three coefficients form an arithmetic progression. Determine all terms from the development of the binomial that contain the powers of y with a natural exponent.

138. Determine x, if the 3rd term from the development of the binomial $(x+x^{\lg x})^5$ is equal to 10^6.

139. In the development of $(1+x-x^2)^{25}$ find the term corresponding to the exponent of x that is 3 times as big as the sum of all the development's coefficients sum.

140. Determine the rank of the biggest term from the development of the binomial $(p+q)^n$ according to the descending powers of p, assuming that $p > 0$; $q > 0$; $p + q = 1$. In which conditions:

a) the biggest term will be the first?

b) the biggest term will be the last?

c) the development will contain two consecutive equal terms that are bigger than all the other terms of the development?

Answers

Note. For chapters 1 and 2 only the less „bulky" answers have been provided and those that are relatively more complicated (in our own opinion).

1. Sets. Operations with sets

3. a) $A = \{5, 7\}$; $B = \{-7, 2\}$; $C = \{1/7\}$.
4. $A = \{2, 4, 6, 8, 10, 12, 14, 16, 18\}$; $B = \{2, 4, 6, 8, 12\}$.

5. a) $A = \{10, 22, 24, \ldots\}$; b) $26, 28 \in A$, $33 \notin A$
(because A consists of even natural numbers).
6. $A = \{x \in N^* | x = 1 + 3n,\ n = \overline{1,3}\}$; $B = \{x \in N^* | x = 3 \cdot 2^{n-1},\ n = \overline{1,3}\}$; $C = \{y \in N^* | y = n^2,\ n = \overline{1,5}\}$;
$D = \{z \in N^* | z = n^3,\ n = \overline{1,5}\}$.
7. $n(A) = 50$; $n(B) = 9$;
a)if $ad - bc = 0 \Rightarrow C = \left\{\frac{b}{d}\right\}$; b) if $ad - bc \neq 0 \Rightarrow C\ has\ p\ elements$
8. $A_1 = \{-1\}$; $A_2 = \{1, 2, 3\}$; $A_3 = \{-1, -2, -3\}$.
9. $A = \{0, 1, 2, 3, 4, 5, 6\}$; $B = \{4, 5, 6, 7, 8\}$; $C = \{1, 2, 3, 4, 5, 6, 7\}$; $D = \{1, 2, 3, 4, 5, 6\}$; $G = (-\infty, 4) \cup (7, +\infty)$; $H = (-\infty, 2) \cup (4, +\infty)$ etc.
10. a) $B \subset A$, $B \neq A$ etc.
11. $A_6 = C_{N^*}(B)$; $A_7 = A \vartriangle B$.
12. a) $A = \{1, 2, 5\}$, $B = E = A \cup B$; b) $A = \{1, 6, 14\}$, $B = \{1, 5, 9, 13, 14\}$, $E = \{1, 2, 5, 6, 9, 13, 14, 18, 20\}$; c) $A = \{1, 2, 3, 4, 8\}$, $B = \{1, 3, 5, 9, 10\}$, $E = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$; d) $A = \{1, 2, 4\}$, $B = \{2, 3, 5\}$; e) $A = \{1, 2, 3, 4\}$, $B = \{3, 4\}$, $C = \{2, 4, 5\}$; f) $A = \{1, 3, 4\}$, $B = \{1, 3\}$, $C = \{2, 3, 4\}$; g) 1) $A = \{1, 2, 3\}$, $B = E \setminus \{1, 2\}$; 2) $A = \{1, 2, 3, 5\}$, $B = \{2, 3, 4\}$; 3) $A = \{2, 3, 4\}$, $B = E \setminus \{4\}$; 4) $A = E \setminus \{5\}$, $B = \{2, 3, 5\}$; h) 1) $A = E$, $B = \{1, 2\}$, $C = \{2, 3\}$; 2) $A = \{1, 3\}$, $B = \{1, 2\}$, $C = \{2, 3\}$; i) $A = \{1, 2, 3, 4\}$, $B = \{1, 2, 5\}$; j) $A = E$, $B = \{2, 4, 5\}$, $C = \{3, 5, 6\}$;

k) $A = \{1,2\}$, $B = \{1,2,4\}$; l) $E = \{1,2,\dots 10\}$, $A = \{7,8,9,10\}$, $B = \{2,3,4,8,9,10\}$; m) $A = \{a,d,f,h,i\}$, $B = \{b,c,d,e,f,g,i\}$; n) $A = \{1,4,6,8,9\}$, $B = \{2,3,4,5,6,7,9\}$.

13. a) $A = \{6,10,20\}$, $B = \{-47,-8,13,22\}$ etc.; b) $A \cap B = \varnothing$ etc.; c) $A = \{0,2,3\}$, $B = \{-5,-1,1,5\}$ etc.; d) $A = \{0,2\}$, $B = \{2,4\}$ etc.; e) $A = \{-1,0,1,2\}$, $B = \{0,2\}$ etc.; f) $A = \{-4,-3,-2,-1\}$, $B = \varnothing$ etc.; g) $A = \{-1,1,2,4,5,7\}$, $B = \{1,2,4,5,7\}$ etc.; h) $A = \{-33,-18,-13,-9,-8,-6,-5,-4,-2,-1,0,2,3,7,12,27\}$, $B = \{0,2,3,7,12,27\}$ etc.; i) $A = \{2,4,6,8,10,12,14,16,18,20\}$, $B = \{2,4,6,8,12\}$ etc.; j) $A \cup B = [-7;7] \cup \{10\}$, $A \setminus B = [-7;-4) \cup (-4;7]$, $B \setminus A = \{10\}$ etc.; k) $A \cap B = \varnothing$ etc.; l) $A \cap B = \{626\}$ etc.; n) $A \cup B = \left\{\dfrac{\sqrt{2}}{4}, \dfrac{3\sqrt{2}}{4}\right\}$ etc.

14. a) $A = \{x \in \mathbf{Q} | x = (7n-4)/(n+3),\ n \leq 22,\ n \in \mathbb{N}\}$, B=M, C=$\{2,6\}$; *b*) n(D)=2497.

15. $A = \{1,2,3,4,8\}$; $B = \{1,3,5,9,10\}$.

16. $A = \{1,6,14\}$; $B = \{1,5,9,13,14\}$; $E = \{1,2,5,6,9,13,14,18,20\}$.

17. a) $A = [5,+\infty)$, $B = (1,+\infty)$, $A \subset B$; b) $A = B = [-9;-4] \cup \cup [4;9]$; c) $A = [-2;0.5(1+\sqrt{5}))$, $B = [0.5;0.5(1+\sqrt{5}))$, $B \subseteq A$; d) $A = (-\infty,-1)$, $B = A$; e) $A = B = (3/2,+\infty)$; f) $A = B = \varnothing$; g) $A = [3;\sqrt{37}/2]$, $B = (-\sqrt{37}/2;-3] \cup [3;\sqrt{37}/2)$.

18. a) $m \in \{-2,2\}$, $m \in (-\infty,-2) \cup (2,+\infty)$, $m \in (-2;2)$; b) $m \in \{-5/2,5/2\}$, $m \in (-5/2,5/2)$, $m \in (-\infty,-5/2) \cup (5/2,+\infty)$; c) $m \in \{-2\sqrt{3},2\sqrt{3}\}$, $m \in (-\infty,-2\sqrt{3}) \cup (2\sqrt{3},+\infty)$; d) $m \in \in \varnothing$, $m \in \mathbb{R}$, $m \in \varnothing$; e) $m = -2/3$, $m \neq -2/3$, $m \in \varnothing$; f) $m \in \{-12,12\}$, $m \in (-\infty,-12) \cup (12,+\infty)$, $m \in (-12;12)$; g) $m \in \left\{\dfrac{1-\sqrt{33}}{6}, \dfrac{1+\sqrt{33}}{6}\right\}$, $m \in \left(\dfrac{1-\sqrt{33}}{6}, \dfrac{1+\sqrt{33}}{6}\right)$, $m \in \left(-\infty, \dfrac{1-\sqrt{33}}{6}\right) \cup \left(\dfrac{1+\sqrt{33}}{6}, +\infty\right)$; h) $m \in \left\{\dfrac{3-2\sqrt{3}}{3}, \dfrac{3+2\sqrt{3}}{3}\right\}$, $m \in \left(\dfrac{3-2\sqrt{3}}{3}, \dfrac{3+2\sqrt{3}}{3}\right)$, $m \in \left(-\infty, \dfrac{3-2\sqrt{3}}{3}\right) \cup \left(\dfrac{3+2\sqrt{3}}{3}, +\infty\right)$.

19. a) $n(A) = 47$; b) $n(A) = 82$; c) $n(A) = 4$; d) $n(A) = 16$; e) $n(A) = 4$; f) $n(A) = 2$.

20. a) $A \cap B \cap C = \{60t - 17 | t \in I\!N^*\}$; b) $A \cap B \cap C = \{200\}$.

21. a) $A \cap B = \{37, 79\}$; b) $A \cap B = \{37, 79\}$; c) $A \cap B = \{6k+1 | k \in \in \mathbf{Z}, k \in [0; 166]\}$.

24. a) $\{(2,2), (2,3), (3,2), (3,3)\}$; b) $\{(3,3)\}$; c) $\{(1,3)\}$; d) $\{(2,4), (3,4), (2,5), (3,5)\}$; e) $\{(1,4), (2,4), (3,4), (4,4)\}$; f) $\{(1,2), (1,3), (1,4), (4,4)\}$; g) $\{(1,4)\}$ etc.

26. a) $A = (-\infty, -2) \cup [3; 4)$, $B = [3; 4]$ etc.; b) $A = B$ etc.; c) $A = [2; 3]$, $B = [-3; -2]$ etc.; d) $A = (-2; -1) \cup (2, +\infty)$, $B = = (-\infty, -1) \cup (3, +\infty)$ etc.; e) $A = \{0, -11\}$, $B = \{-4/5, 0, 6/5\}$ etc.; f) $A = (-\infty, -1) \cup (0; 4)$, $B = (-1; 3)$ etc.; g) $A = \varnothing$, $B = \varnothing$ etc.; h) $A = \{7, 35/3\}$; $B = \{-219/8, 7\}$ etc.; i) $A = \{7/4\} = B$ etc.; j) $A = (-\infty, -1) \cup (0, +\infty)$, $B = (-\infty, -2/5) \cup [4, +\infty)$ etc.; k) $A = = (-\infty, 1] \cup [3/2, +\infty)$, $B = I\!R$ etc.; l) $A = [0, 1/3]$, $B = [-1; 1]$ etc.; m) $A = (-2; 3)$, $B = (-\infty, -2) \cup (3, +\infty)$ etc.; n) $A = (-\infty, 2] \cup \cup [4, +\infty)$, $B = (-\infty, 2)$ etc.

2. Relations, functions

2. 1) $G_\alpha = \{(2,5),(2,7)\}$; 2) $G_\alpha = \{(4,1),(4,3),(6,1),(6,3),(8,1),(8,3)\}$; 3) $G_\alpha=\{(8,1),(8,3),(8,5),(8,7)\}$; 4) $G_\alpha=\{(2,1),(2,3)\}$; 5) $G_\alpha=\{(2,1),(2,3),(2,5),(2,7),(4,1),(6,1),(8,1)\}$; 6) $G_\alpha=\{(8,7)\}$; 7) $G_\alpha = \{(4,1),(4,3),(6,1),(6,3),(8,1),(8,3)\}$; 8) $G_\alpha = \{(2,1),(2,3),(2,5),(4,1),(4,3),(4,5),(6,1),(6,3),(6,5),(8,1),(8,3),(8,5),(8,7)\}$.
3. 1) $G_\alpha = \{(1,8),(2,7),(4,5)\}$; 2) $G_\alpha = \{(1,1),(2,3),(3,5),(4,7)\}$; 3) $G_\alpha = \{(3,1)\}$; 4) $G_\alpha = \{(4,1)\}$; 5) $G_\alpha=\{(1,1),(1,3),(1,5),(1,7),(1,8),(2,8),(3,3),(4,8)\}$; 6) $G_\alpha = \{(1,7),(2,3)\}$; 7) $G_\alpha = = \{(1,5),(1,8),(2,1),(2,7),(3,3),(4,5),(4,8)\}$; 8) $G_\alpha = \{(1,1),(2,1),(3,1),(3,3),(4,1),(4,3)\}$.
4. 1) $x+y=6$; 2) $y=x+1$; 3) $x<y$; 4) max $(x,y)>4$; 5) min $(x,y)=2$; 6) cmmdc $(x,y)=2$; 7) x is even or $y=6$; 8) $x=y^2$.
5. 1) transitive; 2) symmetrical; 3) symmetrical and transitive; 4) reflexive, transitive; 5) reflexive; 6) reflexive, symmetrical; 7), 8) reflexive, symmetrical, transitive; 9) reflexive, transitive; 10) transitive etc.
6. 1) $\delta_\alpha = \rho_\alpha = \mathbb{N}$, symmetrical; 2) $\delta_\alpha = \rho_\alpha = \mathbb{N}$, reflexive; 3) $\delta_\alpha = \rho_\alpha = \mathbb{N}$, symmetrical, antireflexive; 4) $\delta_\alpha = \{1,4,9,\dots,n^2,\dots\}$,$\rho_\alpha = \mathbb{N}$ anti-symmetrical; 5)) $\delta_\alpha = \rho_\alpha = \mathbb{N}$,, reflexive, symmetrical, transitive etc. 6) $\delta_\alpha = \rho_\alpha = \{1,2,3,5,6,10,15,30\}$, antireflexive, symmetrical; *7)* $\delta_\alpha = \mathbb{N}, \rho_\alpha = \{3,4,5,\dots\}$, antireflexive, symmetrical; 8) $\delta_\alpha = \rho_\alpha = \mathbb{N}$, reflexive, anti-symmetrical, transitive; 9) $\delta_\alpha = \mathbb{N}, \rho_\alpha = \{3,4,5,\dots\}$, antireflexive, anti-symmetrical, transitive; 10) $\delta_\alpha = \mathbb{N}, \rho_\alpha = \{0,2,4,\dots\}$, anti-symmetrical; 11) $\delta_\alpha = \rho_\alpha = \mathbb{N}$, reflexive, symmetrical, transitive; *12)* $\delta_\alpha = \rho_\alpha = \mathbb{N}$, symmetrical.
9. 1) $\hat{a} = \{a, e/a\}, a \in \mathbb{R}, a \neq \frac{e}{a}$; 2) $a=\sqrt{e}$, we have $\hat{a} = \{\sqrt{e}\}$.
10. $\hat{a} = \{x \in \mathbb{R} \mid x = a + 2k\pi \text{ or } x = \pi - a + 2m\pi, k, m \in \mathbb{Z}\}$.
11. a) yes;

b) $\hat{a} = \{a, 2-a\}$, $\hat{1} = \{1\}$, $\hat{8} = \{8\}$.

12. 1), 2) $\delta_\alpha = \rho_\alpha = I\!N$, $\alpha \circ \alpha = \alpha$, $\alpha \circ \alpha^{-1} = \alpha^{-1} \circ \alpha = I\!N^2$; 3) $\delta_\alpha = \rho_\alpha = I\!R$, $\alpha^{-1} = \alpha$, $\alpha \circ \alpha = \alpha^{-1} \circ \alpha = \alpha \circ \alpha^{-1} = I\!R^2$; 4) $\delta_\alpha = \rho_\alpha = I\!R$, $\alpha \circ \alpha = \{(x,y) \in I\!R^2 | 4x \geq 9y\}$, $\alpha \circ \alpha^{-1} = \alpha^{-1} \circ \alpha = I\!R^2$; 5) $\delta_\alpha = \left[-\frac{\pi}{2}; \frac{\pi}{2}\right]$, $\rho_\alpha = \left[-1; \frac{\pi}{2}\right]$, $\alpha \circ \alpha = \{(x,y) \in I\!R^2 | \sin(\sin x) \leq y\}$, $\alpha \circ \alpha^{-1} = \left[-\frac{\pi}{2}; \frac{\pi}{2}\right]^2$, $\alpha^{-1} \circ \alpha = \{(x,y) \in I\!R^2 | x, y \in [-1; \pi/2]\}$.

15. $\varphi(R) = [-1;1]$; $\varphi((0;\pi)) = (0;1]$; $\varphi^{-1}([-1;0]) = \bigcup\limits_{k\in \mathbf{Z}} [(2k-1)\pi, 2k\pi]$; $\left\{(-1)^n \frac{\pi}{6} + n\pi | n \in \mathbf{Z}\right\}$; $\{\pi/2 + 2k\pi | k \in \mathbf{Z}\}$; $\varnothing$.

16. B; $\{b,d\}$; $\{b,c,d\}$; $\{a,b,e\}$; $\{3,5,7,4,6\}$; $\{6,9\}$; $\{2,8\}$.

18. $\{1,4,9,16,25,36,49,64,81,100\}$; $\{1,2,3\}$.

23. 1) bijective; 2) not bijective; 3) bijective; 4) not bijective; 5) bijective: 6) not bijective; 7) bijective; 8) not bijective; 9) injective; 10) not injective; 11) injective; 12) surjective; 13) not surjective; 14) surjective; 15) surjective.

29. 1) $u(x) = 4x - 9$, $v(x) = 7x^5 + 4$, $f(x) = (v \circ u)(x)$; 2) $u(x) = x^2 + 3x$, $v(x) = x^{2/3} + x^{1/3} - 7$, $f(x) = (v \circ u)(x)$; 3) $u(x) = x^2 - 3$, $v(x) = 1/\sqrt{x}$, $f(x) = (v \circ u)(x)$ etc.

30. 1) $(f \circ g)(x) = x + 1 = (g \circ f)(x)$; 4; 4; 2) $(f \circ g)(x) = (x-3)^2 + 8$, $(g \circ f)(x) = x^2 + 5$; 8; 14; 3) $(f \circ g)(x) = f(x) = g(x)$; 3; 3; 4) $(f \circ g)(x) = x^2 + 2x$, $(g \circ f)(x) = x^2$; 12; 9; 5) $(f \circ g)(x) = 2x^4 - 4x^2 + 3$, $(g \circ f)(x) = 2x^4 + 4x^2$; 129; 360; 6) $(f \circ g)(x) = x^6 = (g \circ f)(x)$; 729; 729; 7) $(f \circ g)(x) = 4x^4$, $(g \circ f)(x) = -2x^4 - 8x^3 - 12x^2 - 8x - 3$; 4; -1; 8) $(f \circ g)(x) = 3x^2 - 18x + 29$, $(g \circ f)(x) = 3x^2 - 1$; 50; 2 etc.

31. 1) 9; 2) 3; 3) 8; 4) 4; 5) 12; 6) 16; 7) -9; 8) -7; 9) 2.25; 10) 0,75; 11) $6+4\sqrt{2}$; 12) $27+18\sqrt{2}$; 13) $9c^2$; 14) $3c-3$; 15) $9c^2-18c+9$; 16) $9c^2 - 18c + 9$.

32. 1) yes; 2) no; 3) yes; 4) yes; 5) yes; 6) no; 7) no; 8) no.

33. 1) $f^{-1}(4) = 3$; 2) $f^{-1}(6) = 0.5$; 3) $f^{-1}(b) = a$; 4) $f^{-1}(2) = a + 1$; 5) $f^{-1}(p) = m + n$.

34. 1) $f^{-1}(x) = \dfrac{2}{x-1}$; 2) $f^{-1}(x) = \dfrac{1}{x-2}$; 3) $f^{-1}(x) = \dfrac{1}{(x-2)^2}$;
4) $f^{-1}(x) = \dfrac{1}{\sqrt{x}}$; 5) $f^{-1}(x) = \dfrac{4\sqrt{x}}{1-\sqrt{x}}$; 6) $f^{-1}(x) = \dfrac{x^2}{x^2-1}$;
7) $f^{-1}(x) = \dfrac{x^2+1}{1-x^2}$; 8) $f^{-1}(x) = (x^2+2)^2 - 2$; 9) $f^{-1}(x) = \dfrac{3+\sqrt{x}}{1-\sqrt{x}}$;
10) $f^{-1}(x) = \dfrac{4(\sqrt{x}+2)^2}{1-(\sqrt{x}+2)^2}$.

36. 1) $(f \circ g)(x) = \begin{cases} x, & x \geq 2, \\ 4-x, & x < 2, \end{cases}$ $(g \circ f)(x) = \begin{cases} x, & x \geq 1, \\ 2-x, & x < 1. \end{cases}$

2) $$(f \circ g)(x) = \begin{cases} (4x-2)^2 - 1, & x \leq 0, \\ (3x^2-2)^2 - 1, & x \in (0; \sqrt{2/3}], \\ -5(3x-2)^2 - 1, & x > \sqrt{2/3}; \end{cases}$$

$$(g \circ f)(x) = \begin{cases} 3(x^2-1) - 2, & x < -1, \\ 4(x^2-1) - 2, & -1 \leq x \leq 0, \\ 4(-5x-1) - 2, & x > 0. \end{cases}$$

3. Elements of combinatorics

1. 42. **2.** 1140. **3.** 364. **4.** 124. **5.** $30! - 2 \cdot 29!$ **6.** 2520. **7.** 204. **8.** 2027025. **9.** 105. **10.** $30^2 \cdot 10^4$. **11.** $n \cdot m \cdot k$. **12.** $2(P_5)^2 = 2 \cdot (120)^2$. **13.** $2(P_6)^2 = 2 \cdot (720)^2$. **14.** $\overline{A_3^7} = 3^7 \cdot 15$. **15.** $A_3^2 \cdot A_5^4 = 120$. **16.** $4 \cdot 7^3 = 1372$. **18.** $C_{20}^6 \cdot \overline{P}_6$; $C_{20}^3 \cdot C_{20}^2 \cdot C_{20}^1$. **19.** 62. **20.** $0.5 C_4^2 \cdot C_{12}^6 = 2772$. **36.** 4. **37.** 5. **38.** 5. **39.** 5. **40.** 3. **41.** 14. **42.** 7. **43.** 5. **44.** 7. **45.** 10. **46.** 7. **47.** 4. **48.** 11. **49.** 5. **50.** 7. **51.** 8. **52.** 4. **53.** 10. **54.** 10. **55.** 8. **56.** 9. **57.** 3. **58.** 3. **59.** 13. **60.** 10. **61.** 9. **62.** 9. **63.** 8. **64.** 8. **65.** 9. **66.** 10. **67.** 1; 3. **68.** $\varnothing$. **69.** 3. **70.** 9; 10. **71.** 19. **72.** 10. **73.** (5,3). **74.** (8,3). **75.** (7,3). **76.** (7,3). **77.** (34,14). **78.** (15,7). **79.** (8,3). **80.** (10,4). **81.** $\left(\frac{2}{C_n^k}, \frac{k(k-1)(2k-n)}{n(n-1)^2}\right)$. **82.** $\varnothing$. **83.** $\{3,4,5,6,7,8,9\}$. **84.** $\{1,2,3,4,5,6\}$. **85.** $\{4,5,6,7,8,9,10,11\}$. **86.** $\{x > 11 | x \in \mathbb{N}\}$. **87.** $\{7 \le x < 11 | x \in \mathbb{N}\}$. **88.** $\{1 \le x \le 10 | x \in \in \mathbb{N}\}$. **89.** $\{9 < x < 18, x \in \mathbb{N}\}$. **90.** $\{5,6,7\}$. **91.** $\{0,1,2,3,4,5\}$. **92.** $\{11,12,13,14,15,16,17,18\}$. **93.** $\{6,7,8,9\}$. **94.** $\{x > 14 | x \in \in \mathbb{N}\}$. **95.** $\{x \ge 2 | x \in \mathbb{N}\}$. **96.** $\{x \ge 12 | x \in \mathbb{N}\}$. **97.** $\{5,6,\dots\}$. **98.** $\{10\}$. **99.** $\{3,4,\dots,13\}$. **100.** $\{2,3,4,\dots,9\}$. **101.** $\{n \ge 8 | n \in \in \mathbb{N}\}$. **102.** $\{5,6,7,8,9,10\}$. **103.** $\{1,2,3,\}$. **104.** $\{2 \le x \le 36 | x \in \in \mathbb{N}\}$. **105.** $\varnothing$. **106.** $\varnothing$. **107.** $1120x^7 \cdot \sqrt[3]{x}$. **108.** $70x^4y^4$. **109.** 12. 110. 240. 111. $70(1-x^2)^2$. 112. $m = \frac{22k+15}{7}$, the smallest value $k = 6$, then $m = 21$. $C_{16}^6 \,.\, x^3$. 114. $T_6 = C_{15}^6$. 115. $T_5 = C_{12}^5 \,.\, 6^{-7}$. 116. $T_9 = C_{17}^8$. **117.** $T_{1378} = C_{1989}^{1377} \cdot 2^{1377}$. **118.** $T_3 = 55z^{50/3}$. **119.** $C_{35}^{32} b^6 a^{-12}$. **120.** $C_{12}^4 = T_4$. **121.** $84x^{-5/2}$. **122.** $(2,3,5)$. **123.** 7. **124.** $T_{14+1} = 36C_{24}^{10}$. **125.** 26. **126.** $\{T_9, T_{10}, T_{11}\}$ și $\{T_{14}, T_{15}, T_{16}\}$. **127.** $T_2 = C_{18}^2 \cdot x^{6.5} = 153x^{6.5}$. **128.** 2. **129.** $5/8 < x < 20/21$. **130.** $\{T_0, T_4, T_8\}$. **131.** 1. **132.** $T_7 = 28ab^2$. **133.** $\{-1,2\}$. **134.** $\{0,2\}$. **135.** $T_{1421} = = C_{1988}^{1420} \cdot 2^{1420}$. **136.** $\{T_1, T_{16}, T_{31}\}$.

137. $n = 8$, we have $T_1 = y^4$, $T_5 = \frac{35}{8} y$; $n = 4$, we have $T_1 = y^2$.

138. 10.

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In this book, you will find algebra exercises and problems, grouped by chapters, intended for higher grades in high schools or middle schools of general education. Its purpose is to facilitate training in mathematics for students in all high school categories, but can be equally helpful in a standalone workout. The book can also be used as an extracurricular source, as the reader shall find enclosed important theorems and formulas, standard definitions and notions that are not always included in school textbooks.

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