

# 12

## CHAPTER

# Fast Fourier Transform

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### 12.0 Introduction

A very large class of important computational problems falls under the general rubric of *Fourier transform methods* or *spectral methods*. For some of these problems, the Fourier transform is simply an efficient computational tool for accomplishing certain common manipulations of data. In other cases, we have problems for which the Fourier transform (or the related *power spectrum*) is itself of intrinsic interest. These two kinds of problems share a common methodology.

Historically, Fourier and spectral methods have been considered a part of “signal processing,” rather than “numerical analysis” proper. There is really no justification for such a distinction. Fourier methods are commonplace in research and we will not treat them as specialized or arcane. However, we realize that many users have had relatively less experience with this field than with, say, differential equations or numerical integration. Therefore our summary of analytical results will be more complete. Numerical algorithms, per se, begin in §12.2. Various applications of Fourier transform methods are discussed in Chapter 13.

A physical process can be described either in the *time domain*, by the values of some quantity  $h$  as a function of time  $t$ , e.g.,  $h(t)$ , or else in the *frequency domain*, where the process is specified by giving its amplitude  $H$  (generally a complex number indicating phase also) as a function of frequency  $f$ , that is,  $H(f)$ , with  $-\infty < f < \infty$ . For many purposes it is useful to think of  $h(t)$  and  $H(f)$  as being two different *representations* of the same function. One goes back and forth between these two representations by means of the *Fourier transform* equations,

$$\begin{aligned}
 H(f) &= \int_{-\infty}^{\infty} h(t)e^{2\pi ift} dt \\
 h(t) &= \int_{-\infty}^{\infty} H(f)e^{-2\pi ift} df
 \end{aligned}
 \tag{12.0.1}$$

If  $t$  is measured in seconds, then  $f$  in equation (12.0.1) is in cycles per second, or Hertz (the unit of frequency). However, the equations work with other units, too. If  $h$  is a function of position  $x$  (in meters),  $H$  will be a function of inverse wavelength

(cycles per meter), and so on. If you are trained as a physicist or mathematician, you are probably more used to using *angular frequency*  $\omega$ , which is given in *radians* per second. The relation between  $\omega$  and  $f$ ,  $H(\omega)$  and  $H(f)$ , is

$$\omega \equiv 2\pi f \quad H(\omega) \equiv [H(f)]_{f=\omega/2\pi} \quad (12.0.2)$$

and equation (12.0.1) looks like this:

$$\begin{aligned} H(\omega) &= \int_{-\infty}^{\infty} h(t)e^{i\omega t} dt \\ h(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega)e^{-i\omega t} d\omega \end{aligned} \quad (12.0.3)$$

We were raised on the  $\omega$ -convention, but we changed! There are fewer factors of  $2\pi$  to remember if you use the  $f$ -convention, especially when we get to discretely sampled data in §12.1.

From equation (12.0.1) it is evident at once that Fourier transformation is a *linear* operation. The transform of the sum of two functions is equal to the sum of the transforms. The transform of a constant times a function is that same constant times the transform of the function.

In the time domain, the function  $h(t)$  may happen to have one or more special symmetries. It might be *purely real* or *purely imaginary* or it might be *even*,  $h(t) = h(-t)$ , or *odd*,  $h(t) = -h(-t)$ . In the frequency domain, these symmetries lead to relationships between  $H(f)$  and  $H(-f)$ . The following table gives the correspondence between symmetries in the two domains:

If ...	then ...
$h(t)$ is real	$H(-f) = [H(f)]^*$
$h(t)$ is imaginary	$H(-f) = -[H(f)]^*$
$h(t)$ is even	$H(-f) = H(f)$ [i.e., $H(f)$ is even]
$h(t)$ is odd	$H(-f) = -H(f)$ [i.e., $H(f)$ is odd]
$h(t)$ is real and even	$H(f)$ is real and even
$h(t)$ is real and odd	$H(f)$ is imaginary and odd
$h(t)$ is imaginary and even	$H(f)$ is imaginary and even
$h(t)$ is imaginary and odd	$H(f)$ is real and odd

In subsequent sections we shall see how to use these symmetries to increase computational efficiency.

Here are some other elementary properties of the Fourier transform. (We'll use the " $\iff$ " symbol to indicate transform pairs.) If

$$h(t) \iff H(f) \quad (12.0.4)$$

is such a pair, then other transform pairs are

$$h(at) \iff \frac{1}{|a|} H\left(\frac{f}{a}\right) \quad \text{time scaling} \quad (12.0.5)$$

$$\frac{1}{|b|} h\left(\frac{t}{b}\right) \iff H(bf) \quad \text{frequency scaling} \quad (12.0.6)$$

$$h(t - t_0) \iff H(f) e^{2\pi i f t_0} \quad \text{time shifting} \quad (12.0.7)$$

$$h(t) e^{-2\pi i f_0 t} \iff H(f - f_0) \quad \text{frequency shifting} \quad (12.0.8)$$

With two functions  $h(t)$  and  $g(t)$ , and their corresponding Fourier transforms  $H(f)$  and  $G(f)$ , we can form two combinations of special interest. The *convolution* of the two functions, denoted  $g * h$ , is defined by

$$g * h \equiv \int_{-\infty}^{\infty} g(\tau) h(t - \tau) d\tau \quad (12.0.9)$$

Note that  $g * h$  is a function in the time domain and that  $g * h = h * g$ . It turns out that the function  $g * h$  is one member of a simple transform pair,

$$g * h \iff G(f)H(f) \quad \text{convolution theorem} \quad (12.0.10)$$

In other words, the Fourier transform of the convolution is just the product of the individual Fourier transforms.

The *correlation* of two functions, denoted  $\text{Corr}(g, h)$ , is defined by

$$\text{Corr}(g, h) \equiv \int_{-\infty}^{\infty} g(\tau + t) h(\tau) d\tau \quad (12.0.11)$$

The correlation is a function of  $t$ , which is called the *lag*. It therefore lies in the time domain, and it turns out to be one member of the transform pair:

$$\text{Corr}(g, h) \iff G(f)H^*(f) \quad \text{correlation theorem} \quad (12.0.12)$$

[More generally, the second member of the pair is  $G(f)H(-f)$ , but we are restricting ourselves to the usual case in which  $g$  and  $h$  are real functions, so we take the liberty of setting  $H(-f) = H^*(f)$ .] This result shows that multiplying the Fourier transform of one function by the complex conjugate of the Fourier transform of the other gives the Fourier transform of their correlation. The correlation of a function with itself is called its *autocorrelation*. In this case (12.0.12) becomes the transform pair

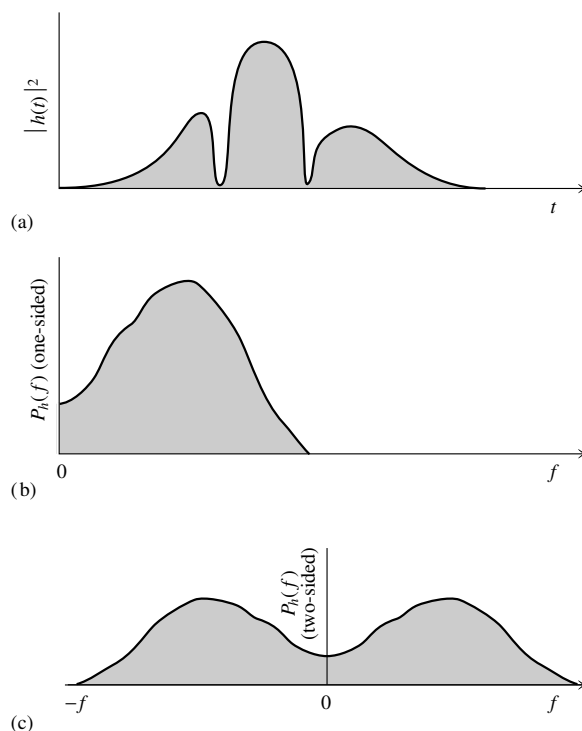
$$\text{Corr}(g, g) \iff |G(f)|^2 \quad \text{Wiener-Khinchin theorem} \quad (12.0.13)$$

The *total power* in a signal is the same whether we compute it in the time domain or in the frequency domain. This result is known as *Parseval's theorem*:

$$\text{total power} \equiv \int_{-\infty}^{\infty} |h(t)|^2 dt = \int_{-\infty}^{\infty} |H(f)|^2 df \quad (12.0.14)$$

Frequently one wants to know “how much power” is contained in the frequency interval between  $f$  and  $f + df$ . In such circumstances, one does not usually distinguish between positive and negative  $f$ , but rather regards  $f$  as varying from 0 (“zero frequency” or D.C.) to  $+\infty$ . In such cases, one defines the *one-sided power spectral density (PSD)* of the function  $h$  as

$$P_h(f) \equiv |H(f)|^2 + |H(-f)|^2 \quad 0 \leq f < \infty \quad (12.0.15)$$



**Figure 12.0.1.** Normalizations of one- and two-sided power spectra. The area under the square of the function, (a), equals the area under its one-sided power spectrum at positive frequencies, (b), and also equals the area under its two-sided power spectrum at positive and negative frequencies, (c).

so that the total power is just the integral of  $P_h(f)$  from  $f = 0$  to  $f = \infty$ . When the function  $h(t)$  is real, the two terms in (12.0.15) are equal, so  $P_h(f) = 2|H(f)|^2$ . Be warned that one occasionally sees PSDs defined without this factor two. These, strictly speaking, are called *two-sided power spectral densities*, but some books are not careful about stating whether one- or two-sided is to be assumed. We will always use the one-sided density given by equation (12.0.15). Figure 12.0.1 contrasts the two conventions.

If the function  $h(t)$  goes endlessly from  $-\infty < t < \infty$ , then its total power and power spectral density will, in general, be infinite. Of interest then is the (*one- or two-sided*) *power spectral density per unit time*. This is computed by taking a long, but finite, stretch of the function  $h(t)$ , computing its PSD [that is, the PSD of a function that equals  $h(t)$  in the finite stretch but is zero everywhere else], and then dividing the resulting PSD by the length of the stretch used. Parseval's theorem in this case states that the integral of the one-sided PSD-per-unit-time over positive frequency is equal to the mean square amplitude of the signal  $h(t)$ .

You might well worry about how the PSD-per-unit-time, which is a function of frequency  $f$ , converges as one evaluates it using longer and longer stretches of data. This interesting question is the content of the subject of “power spectrum estimation” and will be considered below in §13.4 – §13.7. A crude answer for now is, the PSD-per-unit-time converges to finite values at all frequencies *except* those where  $h(t)$  has a discrete sine-wave (or cosine-wave) component of finite amplitude. At

those frequencies, it becomes a delta-function, i.e., a sharp spike, whose width gets narrower and narrower, but whose area converges to be the mean square amplitude of the discrete sine or cosine component at that frequency.

We have by now stated all of the analytical formalism that we will need in this chapter, with one exception: In computational work, especially with experimental data, we are almost never given a continuous function  $h(t)$  to work with, but are given, rather, a list of measurements of  $h(t_i)$  for a discrete set of  $t_i$ 's. The profound implications of this seemingly trivial fact are the subject of §12.1.

### 12.0.1 Higher-Order Statistics

The Wiener-Khinchin theorem, equation (12.0.13), along with the definition (12.0.11), shows that the power spectrum of a function is fully equivalent to the function's *two-point statistic*, that is, the expectation value of the product of the function at two different points separated by  $t$ . One can correspondingly define *higher-order statistics* in both the time and Fourier domains. For example, a function's *three-point correlation* is

$$\text{Corr3}(g, g, g) \equiv \int_{-\infty}^{\infty} g(\tau)g(\tau + t_1)g(\tau + t_2) d\tau \quad (12.0.16)$$

a function of the two variables  $t_1$  and  $t_2$ . The two-dimensional Fourier transform (§12.5) of equation (12.0.16) over  $t_1$  and  $t_2$  is called the *bispectrum*, a function of two frequencies  $f_1$  and  $f_2$ .

Higher-order statistics, including the bispectrum, can make visible non-Gaussian and nonlinear phenomena to which two-point statistics (and thus power spectra) are blind. However, they have the disadvantages of being often difficult to interpret and, because of the high powers of the signal that enter, highly susceptible to noise. On these grounds, we advise caution. Useful, if sometimes overly enthusiastic, references are [1-3].

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