Applied Mathematics

Tensors: Geometry and Applications

J. M. Landsberg

Graduate Studies in Mathematics

Volume 128

American Mathematical Society

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Providence, Rhode Island

2010 *Mathematics Subject Classification.* Primary 15–01, 15A69, 68Q17, 14M17, 94A12, 94A13, 20G05, 62E10, 14N05.

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Library of Congress Cataloging-in-Publication Data

Landsberg, J. M.
Tensors : geometry and applications / J. M. Landsberg.
p. cm. — (Graduate studies in mathematics ; v. 128)
Includes bibliographical references and index.
ISBN 978-0-8218-6907-9 (alk. paper)
1. Multilinear algebra. 2. Tensor products. 3. Matrices. 4. Signal theory (Telecommunication) I. Title.

QA199.5.L36 2011
512′.5—dc23

2011021758

10 9 8 7 6 5 4 3 2 1 17 16 15 14 13 12

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Preface

Tensors are ubiquitous in the sciences. One reason for their ubiquity is that they provide a useful way to organize data. Geometry is a powerful tool for extracting information from data sets, and a beautiful subject in its own right. This book has three intended uses: as a classroom textbook, a reference work for researchers, and a research manuscript.

0.1. Usage

Classroom uses. Here are several possible courses one could give from this text:

(1) The first part of this text is suitable for an advanced course in multilinear algebra—it provides a solid foundation for the study of tensors and contains numerous applications, exercises, and examples. Such a course would cover Chapters 1–3 and parts of Chapters 4–6.

(2) For a graduate course on the geometry of tensors not assuming algebraic geometry, one can cover Chapters 1, 2, and 4–8 skipping §§2.9–12, 4.6, 5.7, 6.7 (except Pieri), 7.6 and 8.6–8.

(3) For a graduate course on the geometry of tensors assuming algebraic geometry and with more emphasis on theory, one can follow the above outline only skimming Chapters 2 and 4 (but perhaps add §2.12) and add selected later topics.

(4) I have also given a one-semester class on the complexity of matrix multiplication using selected material from earlier chapters and then focusing on Chapter 11.

(5) Similarly I have used Chapter 13 as a basis of a semester-long class on **P** v. **NP**, assuming some algebraic geometry. Here Chapter 8 is important.

(6) I have also given several intensive short classes on various topics. A short class for statisticians, focusing on cumulants and tensor decomposition, is scheduled for the near future.

Reference uses. I have compiled information on tensors in table format (e.g., regarding border rank, maximal rank, typical rank, etc.) for easy reference. In particular, Chapter 3 contains most what is known on rank and border rank, stated in elementary terms. Up until now there had been no reference for even the classical results regarding tensors. (Caveat: I do not include results relying on a metric or Hermitian metric.)

Research uses. I have tried to state all the results and definitions from geometry and representation theory needed to study tensors. When proofs are not included, references for them are given. The text includes the state of the art regarding ranks and border ranks of tensors, and explains for the first time many results and problems coming from outside mathematics in geometric language. For example, a very short proof of the well-known Kruskal theorem is presented, illustrating that it hinges upon a basic geometric fact about point sets in projective space. Many other natural subvarieties of spaces of tensors are discussed in detail. Numerous open problems are presented throughout the text.

Many of the topics covered in this book are currently very active areas of research. However, there is no reasonable reference for all the wonderful and useful mathematics that is *already* known. My goal has been to fill this gap in the literature.

0.2. Overview

The book is divided into four parts: I. First applications, multilinear algebra, and overview of results, II. Geometry and representation theory, III. More applications, and IV. Advanced topics.

Chapter 1: Motivating problems. I begin with a discussion of the *complexity of matrix multiplication*, which naturally leads to a discussion of basic notions regarding tensors (bilinear maps, rank, border rank) and the central question of determining equations that describe the set of tensors of border rank at most r. The ubiquitous problem of *tensor decomposition* is illustrated with two examples: *fluorescence spectroscopy* in chemistry and *cumulants* in statistics. A brief discussion of **P** v. **NP** and its variants is presented as a prelude to Chapter 13, where the study of symmetric tensors plays an especially important role. *Tensor product states* arising in quantum information theory and *algebraic statistics* are then introduced as they are typical of applications where one studies subvarieties of spaces of tensors. I conclude by briefly mentioning how the geometry and representation theory that occupies much of the first part of the book will be useful for future research on the motivating problems.

This chapter should be accessible to anyone who is scientifically literate.

Chapter 2: Multilinear algebra. The purpose of this chapter is to introduce the language of tensors. While many researchers using tensors often think of tensors as n-dimensional $\mathbf{a}_1 \times \cdots \times \mathbf{a}_n$-tables, I emphasize coordinate-free definitions. The coordinate-free descriptions make it easier for one to take advantage of symmetries and to apply theorems. Chapter 2 includes: numerous exercises where familiar notions from linear algebra are presented in an invariant context, a discussion of rank and border rank, and first steps towards explaining how to decompose spaces of tensors. Three appendices are included. The first contains basic definitions from algebra for reference, the second reviews Jordan and rational canonical forms. The third describes *wiring diagrams*, a pictorial tool for understanding the invariance properties of tensors and as a tool for aiding calculations.

This chapter should be accessible to anyone who has had a first course in linear algebra. It may be used as the basis of a course in multilinear algebra.

Chapter 3: Elementary results on rank and border rank. Rank and border rank are the most important properties of tensors for applications. In this chapter I report on the state of the art. When the proofs are elementary and instructional, they are included as well, otherwise they are proven later in the text. The purpose of this chapter is to provide a reference for researchers.

Chapter 4: Algebraic geometry for spaces of tensors. A central task to be accomplished in many of the motivating problems is to test if a tensor has membership in a given set (e.g., if a tensor has *rank* r). Some of these sets are defined as the zero sets of collections of polynomials, i.e., as *algebraic varieties*, while others can be expanded to be varieties by taking their *Zariski closure* (e.g., the set of tensors of border rank at most r is the Zariski closure of the set of tensors of rank at most r). I present only the essentials of projective geometry here, in order to quickly arrive at the study of groups and their modules essential to this book. Other topics in algebraic geometry are introduced as needed.

This chapter may be difficult for those unfamiliar with algebraic geometry—it is terse as numerous excellent references are available (e.g., [**157, 289**]). Its purpose is primarily to establish language. Its prerequisite is Chapter 2.

Chapter 5: Secant varieties. The notion of border rank for tensors has a vast and beautiful generalization in the context of algebraic geometry, to that of *secant varieties of projective varieties*. Many results on border rank are more easily proved in this larger geometric context, and it is easier to develop intuition regarding the border ranks of tensors when one examines properties of secant varieties in general.

The prerequisite for this chapter is Chapter 4.

Chapter 6: Exploiting symmetry: Representation theory for spaces of tensors. Representation theory provides a language for taking advantage of symmetries. Consider the space $Mat_{n\times m}$ of $n \times m$ matrices: one is usually interested in the properties of a matrix up to changes of bases (that is, the underlying properties of the linear map it encodes). This is an example of a vector space with a group acting on it. Consider polynomials on the space of matrices. The minors are the most important polynomials. Now consider the space of $n_1 \times n_2 \times \cdots \times n_k$-way arrays (i.e., a space of tensors) with $k > 2$. What are the spaces of important polynomials? Representation theory helps to give an answer.

Chapter 6 discusses representations of the group of permutations on d elements, denoted $\mathfrak{S}_d$, and of the group of invertible $n \times n$ matrices, denoted $GL_n\mathbb{C}$, and applies it to the study of homogeneous varieties. The material presented in this chapter is standard and excellent texts already exist (e.g., [**268, 135, 143**]). I focus on the aspects of representation theory useful for applications and its implementation.

The prerequisite for this chapter is Chapter 2.

Chapter 7: Tests for border rank: Equations for secant varieties. This chapter discusses the equations for secant varieties in general and gives a detailed study of the equations of secant varieties of the varieties of rank one tensors and symmetric tensors, i.e., the varieties of tensors, and symmetric tensors of border rank at most r. These are the most important objects for tensor decomposition, so an effort is made to present the state of the art and to give as many different perspectives as possible.

The prerequisite to Chapter 7 is Chapter 6.

Chapter 8: Additional varieties useful for spaces of tensors. In addition to secant varieties, there are general classes of varieties, such as *tangential varieties*, *dual varieties*, and the *Fano varieties of lines* that generalize certain attributes of tensors to a more general geometric situation. In the special cases of tensors, these varieties play a role in classifying normal forms and the study of rank. For example, dual varieties play a role in distinguishing the different *typical ranks* that can occur for tensors over the real numbers. They should also be useful for future applications. Chapter 8 discusses these as well as the *Chow variety* of polynomials that decompose to a product of linear factors. I also present differential-geometric tools for studying these varieties.

Chapter 8 can mostly be read immediately after Chapter 4.

Chapter 9: Rank. It is more natural in algebraic geometry to discuss border rank than rank because it relates to projective varieties. Yet, for applications sometimes one needs to determine the ranks of tensors. I first regard rank in a more general geometric context, and then specialize to the cases of interest for applications. Very little is known about the possible ranks of tensors, and what little is known is mostly in cases where there are normal forms, which is presented in Chapter 10. The main discussion in this chapter regards the ranks of symmetric tensors, because more is known about them. Included are the Comas-Seguir theorem classifying ranks of symmetric tensors in two variables as well as results on maximum possible rank. Results presented in this chapter indicate there is beautiful geometry associated to rank that is only beginning to be discovered.

Chapter 9 can be read immediately after Chapter 5.

Chapter 10: Normal forms for small tensors. The chapter describes the spaces of tensors admitting normal forms, and the normal forms of tensors in those spaces, as well as normal forms for points in small secant varieties.

The chapter can be read on a basic level after reading Chapter 2, but the proofs and geometric descriptions of the various orbit closures require material from other chapters.

The next four chapters deal with applications.

Chapter 11: The complexity of matrix multiplication. This chapter brings the reader up to date on what is known regarding the complexity of matrix multiplication, including new proofs of many standard results.

Much of the chapter needs only Chapter 2, but parts require results from Chapters 5 and 6.

Chapter 12: Tensor decomposition. In many applications one would like to express a given tensor as a sum of rank one tensors, or some class of simple tensors. In this chapter I focus on examples coming from signal processing and discuss two such: blind source separation and deconvolution of DS-CMDA signals. The blind source separation problem is similar to many questions arising in statistics, so I explain the larger context of the study of cumulants.

Often in applications one would like unique expressions for tensors as a sum of rank one tensors. I bring the reader up to date on what is known regarding when a unique expression is possible. A geometric proof of the often cited Kruskal uniqueness condition for tensors is given. The proof is short and isolates the basic geometric statement that underlies the result.

The chapter can be read after reading Chapters 2 and 3.

Chapter 13: P versus NP. This chapter includes an introduction to several algebraic versions of **P** and **NP**, as well as a discussion of Valiant's *holographic algorithms*. It includes a discussion of the GCT program of Mulmuley and Sohoni, which requires a knowledge of algebraic geometry and representation theory, although the rest of the chapter is elementary and only requires Chapter 2.

Chapter 14: Varieties of tensors in phylogenetics and quantum mechanics. This chapter discusses two different applications with very similar underlying mathematics. In both cases one is interested in isolating subsets (subvarieties) of spaces of tensors with certain attributes coming from physics or statistics. It turns out the resulting varieties for *phylogenetics* and *tensor network states* are strikingly similar, both in their geometry, and in the methods they were derived (via auxiliary graphs).

Much of this chapter only requires Chapter 2 as a prerequisite.

The final three chapters deal with more advanced topics.

Chapter 15: Outline of the proof of the Alexander-Hirschowitz theorem. The dimensions of the varieties of symmetric tensors of border rank at most r were determined by Alexander and Hirschowitz. A brief outline of a streamlined proof appearing in [**257**] is given here.

This chapter is intended for someone who has already had a basic course in algebraic geometry.

Chapter 16: Representation theory. This chapter includes a brief description of the rudiments of the representation theory of complex simple Lie groups and algebras. There are many excellent references for this subject, so I present just enough of the theory for our purposes: the proof of

Kostant's theorem that the ideals of homogeneous varieties are generated in degree two, the statement of the Bott-Borel-Weil theorem, and a discussion of the *inheritance* principle of Chapter 6 in a more general context.

This chapter is intended for someone who has already had a first course in representation theory.

Chapter 17: Weyman's method. The study of secant varieties of triple Segre products naturally leads to the Kempf-Weyman method for determining ideals and singularities of G-varieties. This chapter contains an exposition of the rudiments of the method, intended primarily to serve as an introduction to the book [**333**].

The prerequisites for this chapter include Chapter 16 and a first course in algebraic geometry.

0.3. Clash of cultures

In the course of preparing this book I have been fortunate to have had many discussions with computer scientists, applied mathematicians, engineers, physicists, and chemists. Often the beginnings of these conversations were very stressful to all involved. I have kept these difficulties in mind, attempting to write both to geometers and researchers in these various areas.

Tensor practitioners want practical results. To quote Rasmus Bro (personal communication): "*Practical* means that a user of a given chemical instrument in a hospital lab can push a button and right after get a result."

My goal is to initiate enough communication between geometers and scientists so that such practical results will be realized. While both groups are interested in communicating, there are language and even philosophical barriers to be overcome. The purpose of this paragraph is to alert geometers and scientists to some of the potential difficulties in communication.

To quote G. Folland [**126**] "For them [scientists], mathematics is the discipline of manipulating symbols according to certain sophisticated rules, and the external reality to which those symbols refer lies not in an abstract universe of sets but in the real-world phenomena that they are studying."

But mathematicians, as Folland observed, are Platonists, we think the things we are manipulating on paper have a higher existence. To quote Plato [**266**]: "*Let us take any common instance; there are beds and tables in the world—plenty of them, are there not?*

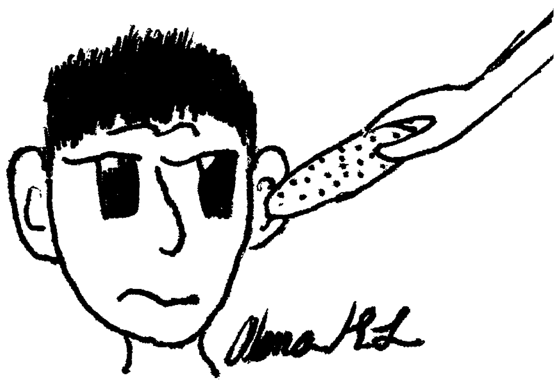
Yes. But there are only two ideas or forms of them—one the idea of a bed, the other of a table.

True. And the maker of either of them makes a bed or he makes a table for our use, in accordance with the idea—that is our way of speaking in *this and similar instances—but no artificer makes the ideas themselves: how could he?*

And what of the maker of the bed? Were you not saying that he too makes, not the idea which, according to our view, is the essence of the bed, but only a particular bed?

Yes, I did. Then if he does not make that which exists he cannot make true existence, but only some semblance of existence; and if any one were to say that the work of the maker of the bed, or of any other workman, has real existence, he could hardly be supposed to be speaking the truth."

This difference of cultures is particularly pronounced when discussing tensors: for some practitioners these are just multi-way arrays that one is allowed to perform certain manipulations on. For geometers these are spaces equipped with certain group actions. To emphasize the geometric aspects of tensors, geometers prefer to work invariantly: to paraphrase W. Fulton: "Don't use coordinates unless someone holds a pickle to your head."[1]



0.4. Further reading

For gaining a basic grasp of representation theory as used in this book, one could consult [**143, 135, 268, 170**]. The styles of these books vary significantly and the reader's taste will determine which she or he prefers. To go further with representation theory [**187**] is useful, especially for the presentation of the Weyl character formula. An excellent (and pictorial!) presentation of the implementation of the Bott-Borel-Weil theorem is in [**19**].

[1]This modification of the actual quote in tribute to my first geometry teacher, Vincent Gracchi. A problem in our 9-th grade geometry textbook asked us to determine if a 3-foot long rifle could be packed in a box of certain dimensions, and Mr. Gracchi asked us all to cross out the word 'rifle' and substitute the word 'pickle' because he "did not like guns". A big $5q + 5q$ to Mr. Gracchi for introducing his students to geometry!

For basic algebraic geometry as in Chapter 4, [**157, 289**] are useful. For the more advanced commutative algebra needed in the later chapters [**119**] is written with geometry in mind. The standard and only reference for the Kempf-Weyman method is [**333**].

The standard reference for what was known in algebraic complexity theory up to 1997 is [**54**].

0.5. Conventions, acknowledgments

0.5.1. Notations. This subsection is included for quick reference. All notations are defined properly the first time they are used in the text.

Vector spaces are usually denoted A, B, C, V, W, and A_j, and the dimensions are usually the corresponding bold letters $\mathbf{a}, \mathbf{b}, \mathbf{c}$, etc. If $v_1, \ldots, v_p \in V$, $\langle v_1, \ldots, v_p \rangle$ denotes the span of $v_1, \ldots, v_p$. If $e_1, \ldots, e_{\mathbf{v}}$ is a basis of V, $e^1, \ldots, e^{\mathbf{v}}$ denotes the dual basis of V^*. $GL(V)$ denotes the general linear group of invertible linear maps $V \to V$ and $\mathfrak{gl}(V)$ its Lie algebra. If G denotes a Lie or algebraic group, $\mathfrak{g}$ denotes its associated Lie algebra.

If $X \subset \mathbb{P}V$ is an algebraic set, then $\hat{X} \subset V$ is the cone over it, its inverse image plus 0 under $\pi : V \backslash 0 \to \mathbb{P}V$. If $v \in V$, $[v] \in \mathbb{P}V$ denotes $\pi(v)$. The linear span of a set $X \subset \mathbb{P}V$ is denoted $\langle X \rangle \subseteq V$.

For a variety X, X_{smooth} denotes its smooth points and X_{sing} denotes its singular points. $X_{general}$ denotes the set of general points of X. Sometimes I abuse language and refer to a point as a general point with respect to other data. For example, if $L \in G(k, V)$ and one is studying the pair (X, L) where $X \subset \mathbb{P}V$ is a subvariety, I will call L a general point if L is in general position with respect to X.

$\Lambda^k V$ denotes the k-th exterior power of the vector space V. The symbols $\wedge$ and $\bigwedge$ denote exterior product. $S^k V$ is the k-th symmetric power. The tensor product of $v, w \in V$ is denoted $v \otimes w \in V^{\otimes 2}$, and symmetric product has no marking, e.g., $vw = \frac{1}{2}(v \otimes w + w \otimes v)$. If $p \in S^d V$ is a homogeneous polynomial of degree d, write $p_{k,d-k} \in S^k V \otimes S^{d-k} V$ for its partial polarization and $\overline{p}$ for p considered as a d-multilinear form $V^* \times \cdots \times V^* \to \mathbb{C}$. When needed, $\circ$ is used for the symmetric product of spaces, e.g., given a subspace $W \subset S^q V$, $W \circ S^p V \subset S^{q+p} V$.

$\mathfrak{S}_d$ denotes the group of permutations on d elements. To a partition $\pi = (p_1, \ldots, p_r)$ of d, i.e., a set of integers $p_1 \geq p_2 \geq \cdots \geq p_r$, $p_i \in \mathbb{Z}_+$, such that $p_1 + \cdots + p_r = d$, $[\pi]$ denotes the associated irreducible $\mathfrak{S}_d$-module, and $S_\pi V$ denotes the associated irreducible $GL(V)$-module. I write $|\pi| = d$ and $\ell(\pi) = r$.

0.5.2. Layout. All theorems, propositions, remarks, examples, etc., are numbered together within each section; for example, Theorem 1.3.2 is the second numbered item in Section 1.3. Equations as well as figures are numbered sequentially within each section. I have included hints for selected exercises, those marked with the symbol ⊚ at the end, which is meant to be suggestive of a life preserver.

0.5.3. Acknowledgments. This project started as a collaboration with Jason Morton, who contributed significantly to the writing and editing of Chapters 2, 4, 5, and 6. The book has greatly benefited from his input. The first draft of this book arose out of a series of lectures B. Sturmfels invited me to give for his working group at UC Berkeley in Spring 2006. I then gave a graduate class at Texas A&M University in Fall 2007 on the complexity of matrix multiplication and a class on complexity theory in spring 2009. J. Morton and I gave lectures from the notes at a summer graduate workshop at MSRI in July 2008, as well as several lectures at a follow-up research workshop at AIM in July 2008. I also gave a GNSAGA lecture series in Florence, Italy in June 2009 on secant varieties, a short course on the geometry of tensors in June 2010 in Nordfjordeid, Norway, and a one-semester graduate class on the same subject at Texas A&M, Fall 2010. It is a pleasure to thank the students in these classes as well as my hosts B. Sturmfels, MSRI, AIM, G. Ottaviani and K. Ranestad. Much of the material in this book comes from joint work with L. Manivel, G. Ottaviani, and J. Weyman. It is a pleasure to thank these three collaborators for significant help at each step along the way. Other material comes from joint work with the (at the time) post-docs J. Buczyński, J. Morton, S. Norine, and Z. Teitler, who have also significantly helped with the book. I also thank my students and post-docs A. Boralevi, L. Nguyen, L. Oeding, Y. Qi, D. The, M. Yang, and K. Ye for their useful comments and questions, as well as R. Bardeli, M. Bläser, P. Bürgisser, E. Briand, J-Y Cai, P. Comon, L. De Lathauwer, M. Dillon, M. Eastwood, J. von zur Gathen, J. Grochow, C. Hammond, L-H Lim, G. Paouris, and A. Stegeman for useful comments and patiently answering my questions. It would not have been possible to write this book without such wonderful help. I was very fortunate to have careful and thoughtful anonymous referees who made numerous outstanding suggestions for improving the manuscript for which I am truly grateful. Finally I thank my family for their help, support and understanding.

Part 1

Motivation from applications, multilinear algebra, and elementary results

Chapter 1

Introduction

The workhorse of scientific computation is matrix multiplication. In many applications one would like to multiply large matrices, ten thousand by ten thousand or even larger. The standard algorithm for multiplying $n \times n$ matrices uses on the order of n^3 arithmetic operations, whereas addition of matrices only uses n^2. For a $10,000 \times 10,000$ matrix this means 10^{12} arithmetic operations for multiplication compared with 10^8 for addition. Wouldn't it be great if all matrix operations were as easy as addition? As "pie in the sky" as this wish sounds, it might not be far from reality. After reviewing the standard algorithm for comparison, §1.1 begins with Strassen's algorithm for multiplying 2×2 matrices using seven multiplications. As shocking as this algorithm may be already, it has an even more stunning consequence: $n \times n$ matrices can be multiplied by performing on the order of $n^{2.81}$ arithmetic operations. This algorithm is implemented in practice for multiplication of large matrices. More recent advances have brought the number of operations needed even closer to the n^2 of addition.

To better understand Strassen's algorithm, and to investigate if it can be improved, it helps to introduce the language of *tensors*, which is done in §1.2. In particular, the *rank* and *border rank* of a tensor are the standard measures of its complexity.

The problem of determining the complexity of matrix multiplication can be rephrased as the problem of decomposing a particular tensor (the matrix multiplication operator) according to its rank. *Tensor decomposition* arises in numerous application areas: locating the area causing epileptic seizures in a brain, determining the compounds in a solution using fluorescence spectroscopy, and data mining, to name a few. In each case, researchers compile data into a "multi-way array" and isolate essential features of the data by decomposing the corresponding tensor into a sum of rank one tensors. Chapter 12 discusses several examples of tensor decomposition arising in wireless communication. In §1.3, I briefly discuss two examples of tensor decomposition: *fluorescence spectroscopy* in chemistry, and *blind source separation*. Blind source separation (BSS) was proposed in 1982 as a way to study how, in vertebrates, the brain detects motion from electrical signals sent by tendons (see [**98**, p. 3]). Since then numerous researchers have applied BSS in many fields, in particular engineers in signal processing. A key ingredient of BSS comes from statistics, the *cumulants* defined in §12.1 and also discussed briefly in §1.3. P. Comon utilized cumulants in [**96**], initiating *independent component analysis* (ICA), which has led to an explosion of research in signal processing.

In a letter addressed to von Neumann from 1956, see [**294**, Appendix], Gödel attempted to describe the apparent difference between intuition and systematic problem solving. His ideas, and those of others at the time, evolved to become the complexity classes **NP** (modeling intuition, or theorem proving) and **P** (modeling problems that could be solved systematically in a reasonable amount of time, like proof checking). See Chapter 14 for a brief history and explanation of these classes. Determining if these two classes are indeed distinct has been considered a central question in theoretical computer science since the 1970s. It is now also considered a central question in mathematics. I discuss several mathematical aspects of this problem in Chapter 13. In §1.4, I briefly discuss an algebraic variant due to L. Valiant: determine whether or not the *permanent* of a matrix of size m can be computed as the determinant of a matrix of size m^c for some constant c. (It is known that a determinant of size on the order of $m^2 2^m$ will work, see Chapter 13.)

A new approach to statistics via algebraic geometry has been proposed in the past 10 years. This *algebraic statistics* [**265, 259**] associates geometric objects (algebraic varieties) to sets of tables having specified statistical attributes. In a similar vein, since the 1980s, in physics, especially quantum information theory, certain types of subsets of the Hilbert space describing all possible quantum states of a crystal lattice have been studied. These subsets are meant to model the states that are physically reasonable. These so-called *tensor network states* go under many names in the literature (see §1.5.2) and have surprising connections to both algebraic statistics and complexity theory. Chapter 14 discusses these two topics in detail. They are introduced briefly in §1.5.

The purpose of this chapter is to introduce the reader to the problems mentioned above and to motivate the study of the geometry of tensors that underlies them. I conclude the introduction in §1.6 by mentioning several key ideas and techniques from geometry and representation theory that are essential for the study of tensors.

1.1. The complexity of matrix multiplication

1.1.1. The standard algorithm. Let A, B be 2×2 matrices

$$A = \begin{pmatrix} a_1^1 & a_2^1 \\ a_1^2 & a_2^2 \end{pmatrix}, \quad B = \begin{pmatrix} b_1^1 & b_2^1 \\ b_1^2 & b_2^2 \end{pmatrix}.$$

The usual algorithm to calculate the matrix product $C = AB$ is

$$\begin{aligned} c_1^1 &= a_1^1 b_1^1 + a_2^1 b_1^2, \\ c_2^1 &= a_1^1 b_2^1 + a_2^1 b_2^2, \\ c_1^2 &= a_1^2 b_1^1 + a_2^2 b_1^2, \\ c_2^2 &= a_1^2 b_2^1 + a_2^2 b_2^2. \end{aligned}$$

It requires 8 multiplications and 4 additions to execute, and applied to $n \times n$ matrices, it uses n^3 multiplications and $n^3 - n^2$ additions.

1.1.2. Strassen's algorithm for multiplying 2×2 matrices using only seven scalar multiplications [301]. Set

$$\begin{aligned} I &= (a_1^1 + a_2^2)(b_1^1 + b_2^2), \\ II &= (a_1^2 + a_2^2) b_1^1, \\ III &= a_1^1 (b_2^1 - b_2^2), \\ IV &= a_2^2(-b_1^1 + b_1^2), \\ V &= (a_1^1 + a_2^1) b_2^2, \\ VI &= (-a_1^1 + a_1^2)(b_1^1 + b_2^1), \\ VII &= (a_2^1 - a_2^2)(b_1^2 + b_2^2). \end{aligned} \tag{1.1.1}$$

Exercise 1.1.2.1: Show that if $C = AB$, then

$$\begin{aligned} c_1^1 &= I + IV - V + VII, \\ c_1^2 &= II + IV, \\ c_2^1 &= III + V, \\ c_2^2 &= I + III - II + VI. \end{aligned}$$

Remark 1.1.2.2. According to P. Bürgisser (personal communication), Strassen was actually attempting to prove such an algorithm did not exist when he arrived at it by process of elimination. In fact he initially was working over $\mathbb{Z}_2$ (where a systematic study was feasible), and then realized that by carefully choosing signs the algorithm works over an arbitrary field. We will see in §5.2.2 why the algorithm could have been anticipated using elementary algebraic geometry.

Remark 1.1.2.3. In fact there is a nine-parameter family of algorithms for multiplying 2×2 matrices using just seven scalar multiplications. See (2.4.5).

1.1.3. Fast multiplication of $n \times n$ matrices. In Strassen's algorithm, the entries of the matrices need not be scalars—they could themselves be matrices. Let A, B be 4×4 matrices, and write

$$A = \begin{pmatrix} a_1^1 & a_2^1 \\ a_1^2 & a_2^2 \end{pmatrix}, \quad B = \begin{pmatrix} b_1^1 & b_2^1 \\ b_1^2 & b_2^2 \end{pmatrix},$$

where a_j^i, b_j^i are 2×2 matrices. One may apply Strassen's algorithm to get the blocks of $C = AB$ in terms of the blocks of A, B by performing 7 multiplications of 2×2 matrices. Since one can apply Strassen's algorithm to each block, one can multiply 4×4 matrices using $7^2 = 49$ multiplications instead of the usual $4^3 = 64$. If A, B are $2^k \times 2^k$ matrices, one may multiply them using 7^k multiplications instead of the usual 8^k.

The total number of arithmetic operations for matrix multiplication is bounded by the number of multiplications, so counting multiplications is a reasonable measure of complexity. For more on how to measure complexity see Chapter 11.

Even if n is not a power of two, one can still save multiplications by enlarging the matrices with blocks of zeros to obtain matrices whose size is a power of two. Asymptotically, one can multiply $n \times n$ matrices using approximately $n^{\log_2(7)} \simeq n^{2.81}$ arithmetic operations. To see this, let $n = 2^k$ and write $7^k = (2^k)^a$ so $k\log_2 7 = ak\log_2 2$ so $a = \log_2 7$.

Definition 1.1.3.1. The *exponent* ω of matrix multiplication is

$$\omega := \inf\{h \in \mathbb{R} \mid Mat_{n\times n} \text{ may be multiplied} \\ \text{using } O(n^h) \text{ arithmetic operations}\}.$$

Strassen's algorithm shows that $\omega \leq \log_2(7) < 2.81$. Determining ω is a central open problem in complexity theory. The current "world record" is $\omega < 2.38$; see [**103**]. I present several approaches to this problem in Chapter 11.

Strassen's algorithm is actually implemented in practice when large matrices must be multiplied. See §11.1 for a brief discussion.

1.2. Definitions from multilinear algebra

1.2.1. Linear maps. In this book I generally work over the complex numbers $\mathbb{C}$. The definitions below are thus presented for complex vector spaces; however the identical definitions hold for real vector spaces (just adjusting the ground field where necessary). Let $\mathbb{C}^n$ denote the vector space of n-tuples of complex numbers, i.e., if $v \in \mathbb{C}^n$, write the vector v as $v = (v_1, \dots, v_n)$ with $v_j \in \mathbb{C}$. The vector space structure of $\mathbb{C}^n$ means that for $v, w \in \mathbb{C}^n$ and $\alpha \in \mathbb{C}$, $v + w = (v_1 + w_1, \dots, v_n + w_n) \in \mathbb{C}^n$ and $\alpha v = (\alpha v_1, \dots, \alpha v_n) \in \mathbb{C}^n$. A map $f : \mathbb{C}^n \to \mathbb{C}^m$ is *linear* if $f(v + \alpha w) = f(v) + \alpha f(w)$ for all $v, w \in \mathbb{C}^n$ and $\alpha \in \mathbb{C}$. In this book vector spaces will generally be denoted by capital letters A, B, C, V, W, with the convention $\dim A = \mathbf{a}$, $\dim B = \mathbf{b}$, etc. I will generally reserve the notation $\mathbb{C}^n$ for an n-dimensional vector space equipped with a basis as above. The reason for making this distinction is that the geometry of many of the phenomena we will study is more transparent if one does not make choices of bases.

If A is a vector space, let $A^* := \{f : A \to \mathbb{C} \mid f \text{ is linear}\}$ denote the *dual vector space*. If $\alpha \in A^*$ and $b \in B$, one can define a linear map $\alpha \otimes b : A \to B$ by $a \mapsto \alpha(a)b$. Such a linear map has *rank one*. The *rank* of a linear map $f : A \to B$ is the smallest r such that there exist $\alpha_1, \dots, \alpha_r \in A^*$ and $b_1, \dots, b_r \in B$ such that $f = \sum_{i=1}^r \alpha_i \otimes b_i$. (See Exercise 2.1(4) for the equivalence of this definition with other definitions of rank.)

If $\mathbb{C}^{2*}$ and $\mathbb{C}^3$ are equipped with bases (e^1, e^2), (f_1, f_2, f_3) respectively, and $A : \mathbb{C}^2 \to \mathbb{C}^3$ is a linear map given with respect to this basis by a matrix

$$\begin{pmatrix} a^1_1 & a^1_2 \\ a^2_1 & a^2_2 \\ a^3_1 & a^3_2 \end{pmatrix},$$

then A may be written as the tensor

$$A = a^1_1 e^1 \otimes f_1 + a^1_2 e^2 \otimes f_1 + a^2_1 e^2 \otimes f_1 + a^2_2 e^2 \otimes f_2 + a^3_1 e^1 \otimes f_3 + a^3_2 e^2 \otimes f_3$$

and there exists an expression $A = \alpha^1 \otimes b_1 + \alpha^2 \otimes b_2$ because A has rank at most two.

Exercise 1.2.1.1: Find such an expression explicitly when the matrix of A is $\left(\begin{smallmatrix} 1 & 2 \\ -1 & 0 \\ 3 & 1 \end{smallmatrix}\right)$ XS. ⊚

1.2.2. Bilinear maps. Matrix multiplication is an example of a *bilinear map*, that is, a map $f : A \times B \to C$, where A, B, C are vector spaces and for each fixed element $b \in B$, $f(\cdot, b) : A \to C$ is linear and similarly for each fixed element of A. Matrix multiplication of square matrices is a bilinear map:

$$M_{n,n,n} : \mathbb{C}^{n^2} \times \mathbb{C}^{n^2} \to \mathbb{C}^{n^2}. \tag{1.2.1}$$

If $\alpha \in A^*$, $\beta \in B^*$ and $c \in C$, the map $\alpha \otimes \beta \otimes c : A \times B \to C$ defined by $(a, b) \mapsto \alpha(a)\beta(b)c$ is a bilinear map. For any bilinear map $T : A \times B \to C$, one can represent it as a sum

$$T(a,b) = \sum_{i=1}^{r} \alpha^i(a)\beta^i(b)c_i \tag{1.2.2}$$

for some r, where $\alpha^i \in A^*$, $\beta^i \in B^*$, and $c_i \in C$.

Definition 1.2.2.1. For a bilinear map $T : A \times B \to C$, the minimal number r over all such presentations (1.2.2) is called the *rank* of T and denoted $\mathbf{R}(T)$.

This notion of rank was first defined by F. Hitchcock [**164**] in 1927.

1.2.3. Aside: Differences between linear and multilinear algebra. Basic results from linear algebra are that "rank equals row rank equals column rank", i.e., for a linear map $f : A \to B$, $\mathrm{rank}(f) = \dim f(A) = \dim f^T(B^*)$. Moreover, the maximum possible rank is $\min\{\dim A, \dim B\}$, and for "most" linear maps $A \to B$ the maximum rank occurs. Finally, if $\mathrm{rank}(f) > 1$, the expression of f as a sum of rank one linear maps is never unique; there are parameters of ways of doing so.

We will see that all these basic results fail in multilinear algebra. Already for bilinear maps, $f : A \times B \to C$, the rank of f is generally different from $\dim f(A \times B)$, the maximum possible rank is generally greater than $\max\{\dim A, \dim B, \dim C\}$, and "most" bilinear maps have rank *less than* the maximum possible. Finally, in many cases of interest, the expression of f as a sum of rank one linear maps *is* unique, which turns out to be crucial for applications to signal processing and medical imaging.

1.2.4. Rank and algorithms. If T has rank r, it can be executed by performing r scalar multiplications (and $\mathcal{O}(r)$ additions). Thus the rank of a bilinear map gives a measure of its complexity.

In summary, $\mathbf{R}(M_{n,n,n})$ measures the number of multiplications needed to compute the product of two $n \times n$ matrices and gives a measure of its complexity. Strassen's algorithm shows $\mathbf{R}(M_{2,2,2}) \leq 7$. S. Winograd [**334**] proved $\mathbf{R}(M_{2,2,2}) = 7$. The exponent ω of matrix multiplication may be rephrased as

$$\omega = \underline{\lim}_{n\to\infty} \log_n(\mathbf{R}(M_{n,n,n})).$$

Already for 3×3 matrices, all that is known is $19 \leq \mathbf{R}(M_{3,3,3}) \leq 23$ [**29, 199**]. The best asymptotic lower bound is $\frac{5}{2}m^2 - 3m \leq \mathbf{R}(M_{m,m,m})$ [**28**]. See Chapter 11 for details.

The rank of a bilinear map and the related notions of *symmetric rank*, *border rank*, and *symmetric border rank* will be central to this book so I introduce these additional notions now.

1.2.5. Symmetric rank of a polynomial. Given a quadratic polynomial $p(x) = ax^2 + bx + c$, algorithms to write p as a sum of two squares $p = (\alpha x + \beta)^2 + (\gamma x + \delta)^2$ date back at least to ancient Babylon 5,000 years ago [**186**, Chap 1].

In this book it will be more convenient to deal with homogeneous polynomials. It is easy to convert from polynomials to homogeneous polynomials; one simply adds an extra variable, say y, and if the highest degree monomial appearing in a polynomial is d, one multiplies each monomial in the expression by the appropriate power of y so that the monomial has degree d. For example, the above polynomial becomes $p(x,y) = ax^2 + bxy + cy^2 = (\alpha x + \beta y)^2 + (\gamma x + \delta y)^2$. See §2.6.5 for more details.

A related notion to the rank of a bilinear map is the *symmetric rank* of a homogeneous polynomial. If P is homogeneous of degree d in n variables, it may always be written as a sum of d-th powers. Define the *symmetric rank* of P, $\mathbf{R}_S(P)$, to be the smallest r such that P is expressible as the sum of r d-th powers.

For example, a general homogeneous polynomial of degree 3 in two variables, $P = ax^3 + bx^2y + cxy^2 + ey^3$, where a, b, c, e are constants, will be the sum of two cubes (see Exercise 5.3.2.3).

1.2.6. Border rank and symmetric border rank. Related to the notions of rank and symmetric rank, and of equal importance for applications, are those of *border rank* and *symmetric border rank* defined below, respectively denoted $\mathrm{Brank}(T)$, $\underline{\mathbf{R}}_S(P)$. Here is an informal example to illustrate symmetric border rank.

Example 1.2.6.1. While a general homogeneous polynomial of degree three in two variables is a sum of two cubes, it is *not* true that every cubic polynomial is either a cube or the sum of two cubes. For an example, consider

$$P = x^3 + 3x^2y.$$

P is not the sum of two cubes. (To see this, write $P = (sx + ty)^3 + (ux + vy)^3$ for some constants s, t, u, v, equate coefficients and show there is no solution.) However, it is the limit as $\epsilon \to 0$ of polynomials P_ϵ that are sums of two cubes, namely

$$P_\epsilon := \frac{1}{\epsilon}((\epsilon - 1)x^3 + (x + \epsilon y)^3).$$

This example dates back at least to Terracini nearly 100 years ago. Its geometry is discussed in Example 5.2.1.2.

Definition 1.2.6.2. The *symmetric border rank* of a homogeneous polynomial P, $\underline{\mathbf{R}}_S(P)$, is the smallest r such that there exists a sequence of polynomials P_ϵ, each of rank r, such that P is the limit of the P_ϵ as ϵ tends to zero. Similarly, the *border rank* $\underline{\mathbf{R}}(T)$ of a bilinear map $T : A \times B \to C$ is the smallest r such that there exists a sequence of bilinear maps T_ϵ, each of rank r, such that T is the limit of the T_ϵ as ϵ tends to zero.

Thus the polynomial P of Example 1.2.6.1 has symmetric border rank two and symmetric rank three.

The border rank bounds the rank as follows: if T can be approximated by a sequence T_ϵ where one has to divide by ϵ^q to obtain the limit (i.e., take q derivatives in the limit), then $\mathbf{R}(T) \leq q^2\underline{\mathbf{R}}(T)$; see [**54**, pp. 379–380] for a more precise explanation. A similar phenomenon holds for the symmetric border rank.

By the remark above, the exponent ω of matrix multiplication may be rephrased as

$$\omega = \underline{\lim}_{n\to\infty} \log_n(\underline{\mathbf{R}}(M_{n,n,n})).$$

In [**201**] it was shown that Brank$(M_{2,2,2}) = 7$. For 3×3 matrices, all that is known is $14 \leq \underline{\mathbf{R}}(M_{3,3,3}) \leq 21$, and for $n \times n$ matrices, $\frac{3n^2}{2} + \frac{n}{2} - 1 \leq \underline{\mathbf{R}}(M_{n,n,n})$ [**220**]. See Chapter 11 for details.

The advantage of border rank over rank is that the set of bilinear maps of border rank at most r can be described as the zero set of a collection of polynomials, as discussed in the next paragraph.

Remark 1.2.6.3. Border rank was first used in the study of matrix multiplication in [**26**]. The term "border rank" first appeared in the paper [**27**].

1.2.7. Our first spaces of tensors and varieties inside them. Let $A^*{\otimes}B$ denote the vector space of linear maps $A \to B$. The set of linear maps of rank at most r will be denoted $\hat{\sigma}_r = \hat{\sigma}_{r,A^*\otimes B}$. This set is the zero set of a collection of homogeneous polynomials on the vector space $A^*{\otimes}B$. Explicitly, if we choose bases and identify $A^*{\otimes}B$ with the space of $\mathbf{a} \times \mathbf{b}$ matrices, $\hat{\sigma}_r$ is the set of matrices whose $(r+1) \times (r+1)$ minors are all zero. In particular there is a simple test to see if a linear map has rank at most r.

A subset of a vector space defined as the zero set of a collection of homogeneous polynomials is called an *algebraic variety*.

Now let $A^*{\otimes}B^*{\otimes}C$ denote the vector space of bilinear maps $A \times B \to C$. This is our first example of a space of *tensors*, defined in Chapter 2, beyond the familiar space of linear maps. Expressed with respect to bases, a bilinear map is a "three-dimensional matrix" or array.

The set of bilinear maps of rank at most r is not an algebraic variety, i.e., it is not the zero set of a collection of polynomials. However the set of bilinear maps of border rank at most r is an algebraic variety. The set of bilinear maps $f : A \times B \to C$ of border rank at most r will be denoted $\hat{\sigma}_r = \hat{\sigma}_{r,A^*\otimes B^*\otimes C}$. It is the zero set of a collection of homogeneous polynomials on the vector space $A^*\otimes B^*\otimes C$.

In principle, to test for membership of a bilinear map T in $\hat{\sigma}_r$, one could simply evaluate the defining equations of $\hat{\sigma}_r$ on T and see if they are zero. However, unlike the case of linear maps, defining equations for $\hat{\sigma}_{r,A^*\otimes B^*\otimes C}$ are *not* known in general. Chapter 7 discusses what is known about them.

More generally, if $A_1, \dots, A_n$ are vector spaces, one can consider the *multilinear maps* $A_1 \times \cdots \times A_n \to \mathbb{C}$, the set of all such forms a vector space, which is a space of tensors and is denoted $A_1^*\otimes \cdots \otimes A_n^*$. The rank of an element of $A_1^*\otimes \cdots \otimes A_n^*$ is defined similarly to the definitions above, see §2.4.1. In bases one obtains the set of $\mathbf{a}_1 \times \cdots \times \mathbf{a}_n$-way arrays. Adopting a coordinate-free perspective on these arrays will make it easier to isolate the properties of tensors of importance.

1.3. Tensor decomposition

One of the most ubiquitous applications of tensors is to *tensor decomposition*. In this book I will use the term *CP decomposition* for the decomposition of a tensor into a sum of rank one tensors. Other terminology used (and an explantion of the origin of the term “CP”) is given in the introduction to Chapter 13.

We already saw an example of this problem in §1.1, where we wanted to write a bilinear map $f : A \times B \to C$ as a sum of maps of the form $(v, w) \mapsto \alpha(v)\beta(w)c$ where $\alpha \in A^*$, $\beta \in B^*$, $c \in C$.

In applications, one collects data arranged in a multidimensional array T. Usually the science indicates that this array, considered as a tensor, should be “close” to a tensor of small rank, say r. The problems researchers face are i) to find the correct value of r and/or ii) to find a tensor of rank r that is “close” to T. Subsection 1.3.1 presents an example arising in chemistry, taken from the book [**40**].

A central problem in signal processing is *source separation*. An often used metaphor to explain the problem is the “cocktail party problem” where a collection of people in a room is speaking. Several receivers are set up, recording the conversations. What they actually record is not the individual conversations, but all of them mixed together and the goal is to recover what each individual is saying. Remarkably, this “unmixing” can often be accomplished using a CP decomposition. This problem relates to larger questions in statistics and is discussed in detail in Chapter 12, and briefly in §1.3.2.

1.3.1. Tensor decomposition and fluorescence spectroscopy. I samples of solutions are analyzed, each contains different chemicals diluted at different concentrations. The first goal is to determine the number r of different chemicals present. In [**40**, §10.2] there are four such, *dihydroxybenzene, tryptophan, phenylalanine* and *DOPA*. Each sample, say sample number i, is successively excited by light at J different wavelengths. For every excitation wavelength one measures the emitted spectrum. Say the intensity of the fluorescent light emitted is measured at K different wavelengths. Hence for every i, one obtains a $J \times K$ table of excitation-emission matrices.

Thus the data one is handed is an $I \times J \times K$ array. In bases, if $\{e_i\}$ is a basis of $\mathbb{C}^I$, $\{h_j\}$ a basis of $\mathbb{C}^J$, and $\{g_k\}$ a basis of $\mathbb{C}^K$, then $T = \sum_{ijk} T_{ijk} e_i \otimes h_j \otimes g_k$. A first goal is to determine r such that

$$T \simeq \sum_{f=1}^{r} a_f \otimes b_f \otimes c_f,$$

where each f represents a substance. Writing $a_f = \sum a_{i,f} e_i$, then $a_{i,f}$ is the concentration of the f-th substance in the i-th sample, and similarly using the given bases of $\mathbb{R}^J$ and $\mathbb{R}^K$, $c_{k,f}$ is the fraction of photons the f-th substance emits at wavelength k, and $b_{j,f}$ is the intensity of the incident light at excitation wavelength j multiplied by the absorption at wavelength j.

There will be noise in the data, so T will actually be of generic rank, but there will be a very low rank tensor $\tilde{T}$ that closely approximates it. (For all complex spaces of tensors, there is a rank that occurs with probability one which is called the *generic rank*, see Definition 3.1.4.2.) There is no metric naturally associated to the data, so the meaning of "approximation" is not clear. In [**40**], one proceeds as follows to find r. First of all, r is assumed to be very small (at most 7 in their exposition). Then for each r_0, $1 \leq r_0 \leq 7$, one assumes $r_0 = r$ and applies a numerical algorithm that attempts to find the r_0 *components* (i.e., rank one tensors) that $\tilde{T}$ would be the sum of. The values of r_0 for which the algorithm does not converge quickly are thrown out. (The authors remark that this procedure is not mathematically justified, but seems to work well in practice. In the example, these discarded values of r_0 are too large.) Then, for the remaining values of r_0, one looks at the resulting tensors to see if they are reasonable physically. This enables them to remove values of r_0 that are too small. In the example, they are left with $r_0 = 4, 5$.

Now assume r has been determined. Since the value of r is relatively small, up to trivialities, the expression of $\tilde{T}$ as the sum of r rank one elements will be unique, see §3.3. Thus, by performing the decomposition of $\tilde{T}$, one recovers the concentration of each of the r substances in each solution by determining the vectors a_f as well as the individual excitation and emission spectra by determining the vectors b_f.

1.3.2. Cumulants. In statistics one collects data in large quantities that are stored in a multidimensional array, and attempts to extract information from the data. An important sequence of quantities to extract from data sets are *cumulants*, the main topic of the book [**232**]. This subsection is an introduction to cumulants, which is continued in Chapter 12.

Let $\mathbb{R}^{\mathbf{v}}$ have a probability measure $d\mu$, i.e., a reasonably nice measure such that $\int_{\mathbb{R}^{\mathbf{v}}} d\mu = 1$. A measurable function $f : \mathbb{R}^{\mathbf{v}} \to \mathbb{R}$ is called a *random variable* in the statistics literature. (The reason for this name is that if we pick a "random point" $x \in \mathbb{R}^{\mathbf{v}}$, according to the probability measure μ, f determines a "random value" $f(x)$.) For a random variable f, write $E\{f\} := \int_{\mathbb{R}^{\mathbf{v}}} f(x) d\mu$. $E\{f\}$ is called the *expectation* or the *mean* of the function f.

For example, consider a distribution of mass in space with coordinates x^1, x^2, x^3 and the density given by a probability measure μ. (Each coordinate function may be considered a random variable.) Then the integrals $m^j := E\{x^j\} := \int_{\mathbb{R}^3} x^j d\mu$ give the coordinates of the center of mass (called the *mean*).

More generally, define the *moments* of random variables x^j:

$$m_x^{i_i,\dots,i_p} := E\{x^{i_1} \cdots x^{i_p}\} = \int_{\mathbb{R}^{\mathbf{v}}} x^{i_1} \cdots x^{i_p} d\mu.$$

Remark 1.3.2.1. In practice, the integrals are approximated by discrete statistical data taken from samples.

A first measure of the dependence of functions x^j is given by the quantities

$$\kappa^{ij} = \kappa_x^{ij} := m^{ij} - m^i m^j = \int_{\mathbb{R}^{\mathbf{v}}} x^i x^j d\mu - \left(\int_{\mathbb{R}^{\mathbf{v}}} x^j d\mu\right)\left(\int_{\mathbb{R}^{\mathbf{v}}} x^j d\mu\right),$$

called *second order cumulants* or the *covariance matrix*.

One says that the functions x^j are *statistically independent at order two* if $\kappa_x^{ij} = 0$ for all $i \neq j$. To study statistical independence, it is better not to consider the κ_x^{ij} individually, but to form a symmetric matrix $\kappa_{2,x} = \kappa_2(x) \in S^2\mathbb{R}^{\mathbf{v}}$. If the measurements depend on $r < \mathbf{v}$ independent events, the rank of this matrix will be r. (In practice, the matrix will be "close" to a matrix of rank r.) The matrix $\kappa_2(x)$ is called a *covariance matrix*. One can define higher order cumulants to obtain further measurements of statistical independence. For example, consider

$$\kappa^{ijk} = m^{ijk} - (m^i m^{jk} + m^j m^{ik} + m^k m^{ij}) + 2m^i m^j m^k. \tag{1.3.1}$$

We may form a third order symmetric tensor from these quantities, and similarly for higher orders.

Cumulants of a set of random variables (i.e., functions on a space with a probability measure) give an indication of their mutual statistical dependence, and higher-order cumulants of a single random variable are some measure of its non-Gaussianity.

Definition 1.3.2.2. In probability, two events A, B are independent if $Pr(A \wedge B) = Pr(A)Pr(B)$, where $Pr(A)$ denotes the probability of the event A. If x is a random variable, one can compute $Pr(x \leq a)$. Two random variables x, y are *statistically independent* if $Pr(\{x \leq a\} \wedge \{y \leq b\}) = Pr(\{x \leq a\})Pr(\{y \leq b\})$ for all $a, b \in \mathbb{R}_+$. The statistical independence of random variables $x^1, \ldots, x^m$ is defined similarly.

An important property of cumulants, explained in §12.1, is:

If the x^i are determined by r statistically independent quantities, then $\mathbf{R}_S(\kappa_p(x)) \leq r$ (with equality generally holding) for all $p \geq 2$.

We will apply this observation in the next subsection.

1.3.3. Blind source separation. A typical application of blind source separation (BSS) is as follows: Big Brother would like to determine the location of pirate radio transmissions in Happyville. To accomplish this, antennae are set up at several locations to receive the radio signals. How can one determine the location of the sources from the signals received at the antennae?

Let $y^j(t)$ denote the measurements at the antennae at time t. Assume there is a relation of the form

$$\begin{pmatrix} y^1(t) \\ \vdots \\ y^m(t) \end{pmatrix} = \begin{pmatrix} a_1^1 \\ \vdots \\ a_1^m \end{pmatrix} x^1(t) + \cdots + \begin{pmatrix} a_r^1 \\ \vdots \\ a_r^m \end{pmatrix} x^r(t) + \begin{pmatrix} v^1(t) \\ \vdots \\ v^m(t) \end{pmatrix} \tag{1.3.2}$$

which we write in vector notation as

$$y = Ax + v. \tag{1.3.3}$$

Here A is a fixed $m \times r$ matrix, $v = v(t) \in \mathbb{R}^m$ is a vector-valued function representing the noise, and $x(t) = (x^1(t), \ldots, x^r(t))^T$ represents the statistically independent functions of t that correspond to the locations of the sources. ("T" denotes transpose.) The $v^i(t)$ are assumed (i) to be independent random variables which are independent of the $x^j(t)$, (ii) $E\{v^i\} = 0$, and (iii) the moments $E\{v^{i_1} \cdots v^{i_p}\}$ are bounded by a small constant.

One would like to recover $x(t)$, plus the matrix A, from knowledge of the function $y(t)$ alone. Note that the x^j are like eigenvectors, in the sense that they are only well defined up to scale and permutation, so "recover" means modulo this ambiguity. And "recover" actually means recover approximate samplings of x from samplings of y.

At first this task appears impossible. However, note that if we have such an equation as (1.3.3), then we can compare the quantities κ_y^{ij} with κ_x^{st} because, by the linearity of the integral,

$$\kappa_y^{ij} = A_s^i A_t^j \kappa_x^{st} + \kappa_v^{ij}.$$

By the assumption of statistical independence, $\kappa_x^{st} = \delta^{st}\kappa_x^{ss}$, where δ^{st} is the Kronecker delta, and κ_v^{ij} is small, so we ignore it to obtain a system of $\binom{m}{2}$ linear equations for $mr+r$ unknowns. If we assume $m > r$ (this case is called an *overdetermined mixture*), this count is promising. However, although a rank r symmetric $m \times m$ matrix may be written as a sum of rank one symmetric matrices, that sum is never unique. (An algebraic geometer would recognize this fact as the statement that the secant varieties of quadratic Veronese varieties are degenerate.)

Thus we turn to the order three symmetric tensor (cubic polynomial) $\kappa_3(y)$, and attempt to decompose it into a sum of r cubes in order to recover the matrix A. Here the situation is much better: as long as the rank is less than generic, with probability one there will be a unique decomposition (except in $S^3\mathbb{C}^5$ where one needs two less than generic), see Theorem 12.3.4.3. Once one has the matrix A, one can recover the x^j at the sampled times. What is even more amazing is that this algorithm will work in principle even if the number of sources is greater than the number of functions sampled, i.e., if $r > m$ (this is called an *underdetermined mixture*)—see §12.1.

Example 1.3.3.1 (BSS was inspired by nature)**.** How does our central nervous system detect where a muscle is and how it is moving? The muscles send electrical signals through two types of transmitters in the tendons, called primary and secondary, as the first type sends stronger signals. There are two things to be recovered, the function $p(t)$ of angular position and $v(t) = \frac{dp}{dt}$ of angular speed. (These are to be measured at any given instant so your central nervous system can't simply "take a derivative".) One might think one type of transmitter sends information about $v(t)$ and the other about $p(t)$, but the opposite was observed, there is some kind of mixing: say the signals sent are respectively given by functions $f_1(t), f_2(t)$. Then it was observed there is a matrix A, such that

$$\begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} v(t) \\ p(t) \end{pmatrix}$$

and the central nervous system somehow decodes $p(t), v(t)$ from $f_1(t), f_2(t)$. This observation is what led to the birth of blind source separation, see [**98**, §1.1.1].

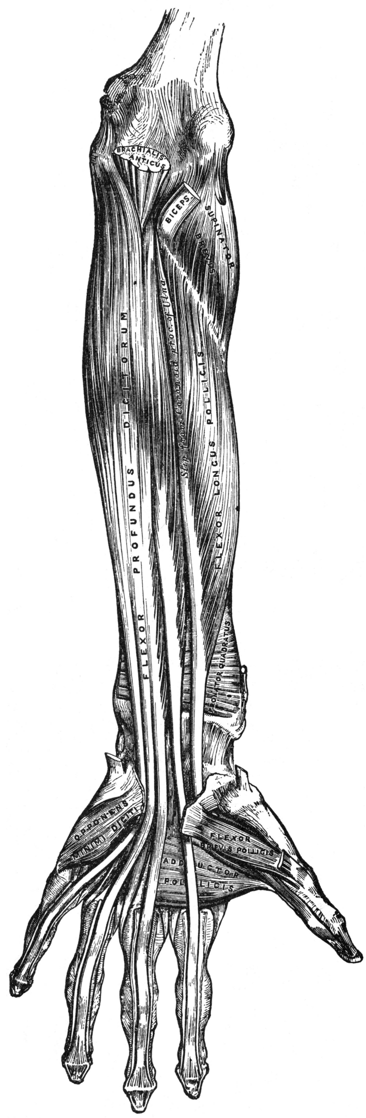


Figure 1.3.1. The brain uses source separation to detect where muscles are and how they are moving.

1.3.4. Other uses of tensor decomposition. Tensor decomposition occurs in numerous areas, here are just a few:

- An early use of tensor decomposition was in the area of *psychometrics*, which sought to use it to help evaluate intelligence and other personality characteristics. Early work in the area includes [**311, 71, 158**].
- In geophysics; for example the interpretation of magnetotelluric data for one-dimensional and two-dimensional regional structures; see, e.g., [**335**] and the references therein.
- In interpreting magnetic resonance imaging (MRI); see, e.g., [**284, 125**], and the references therein. One such use is in determining the location in the brain that is causing epileptic seizures in a patient. Another is the diagnosis and management of strokes.

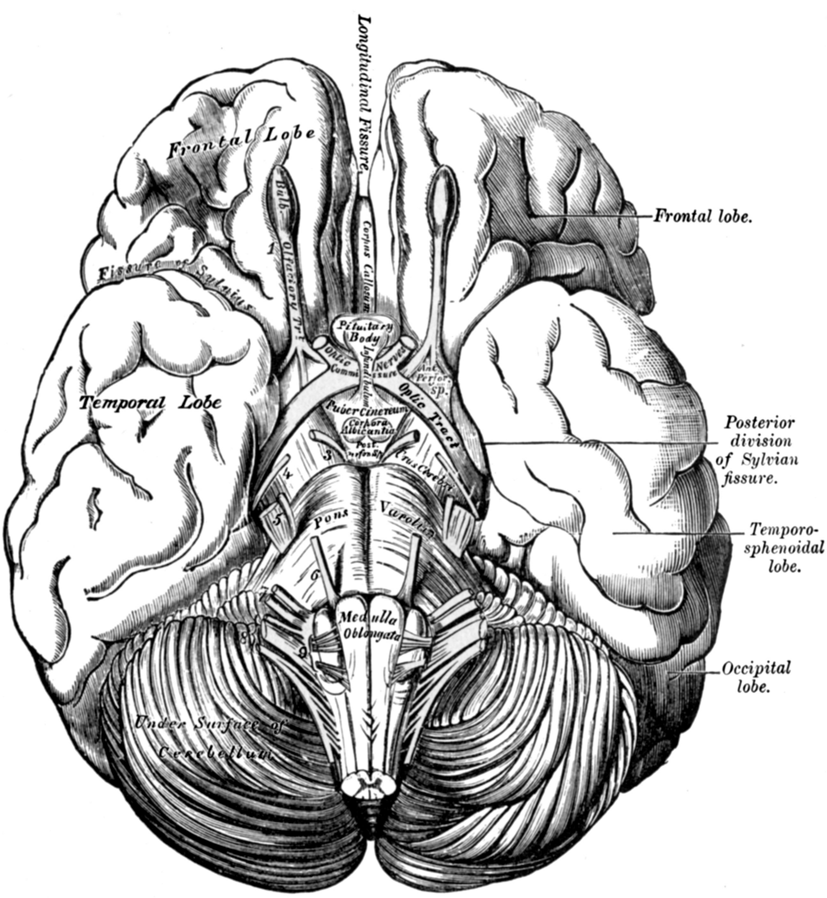


Figure 1.3.2. When locating sources of epileptic seizures in order to remove them, unique up to finite decomposition is not enough.

- In data-mining, see, e.g., [**219**].
- In numerical analysis. Thanks to the convergence of Fourier series in $L^2(\mathbb{R}^n)$, one has $L^2(\mathbb{R}^n) = L^2(\mathbb{R}) \otimes \cdots \otimes L^2(\mathbb{R})$, and one often approximates a function of n variables by a finite sum of products of functions of one variable, generalizing classical separation of variables. See, e.g., [**155**] and the references therein.

1.3.5. Practical issues in tensor decomposition. Four issues to deal with are *existence*, *uniqueness*, *executing the decomposition* and *noise*. I now discuss each briefly. In this subsection I use "tensor" for tensors, symmetric tensors, and partially symmetric tensors, "rank" for rank and symmetric rank etc.

Existence. In many tensor decomposition problems, the first issue to resolve is to determine the rank of the tensor T one is handed. In cases where one has explicit equations for the tensors of border rank r, if T solves the equations, then with probability one, it is of rank at most r. (For symmetric tensors of small rank, it is always of rank at most r; see Theorem 3.5.2.2.)

Uniqueness. In the problems coming from spectroscopy and signal processing, one is also concerned with uniqueness of the decomposition. If the rank is sufficiently small, uniqueness is assured with probability one, see §3.3.3. Moreover there are explicit tests one can perform on any given tensor to be assured of uniqueness, see §3.3.2.

Performing the decomposition. In certain situations there are algorithms that exactly decompose a tensor, see, e.g., §3.5.3—these generally are a consequence of having equations that test for border rank. In most situations one uses numerical algorithms, which is an area of active research outside the scope of this book. See [**97, 188**] for surveys of decomposition algorithms.

Noise. In order to talk about noise in data, one must have a distance function. The properties of tensors discussed in this book are defined independently of any distance function, and there are no natural distance functions on spaces of tensors (but rather classes of such). For this reason I do not discuss noise or approximation in this book. In specific applications there are often distance functions that are natural, based on the science, but often in the literature such functions are chosen by convenience. R. Bro (personal communication) remarks that assuming that the noise has a certain behavior (iid and Gaussian) can determine a distance function.

1.4. P v. NP and algebraic variants

1.4.1. Computing the determinant of a matrix. Given an $n \times n$ matrix, how do you compute its determinant? Perhaps in your first linear algebra class you learned the Laplace expansion—given a 2×2 matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

its determinant is $ad - bc$, then given an $n \times n$ matrix $X = (x^i_j)$,

$$\det(x^i_j) = x^1_1 \Delta_{1,1} - x^1_2 \Delta_{1,2} + \cdots + (-1)^{n-1} x^1_n \Delta_{1,n},$$

where $\Delta_{i,j}$ is the determinant of the $(n-1) \times (n-1)$ matrix obtained by deleting the i-th row and j-th column of X. This algorithm works fine for 3×3 and 4×4 matrices, but by the time one gets to 10×10 matrices, it becomes a bit of a mess.

Aside 1.4.1.1. Let's pause for a moment to think about exactly *what* we are computing, i.e., the meaning of $\det(X)$. The first thing we learn is that X is invertible if and only if its determinant is nonzero. To go deeper, we first need to remind ourselves of the meaning of an $n \times n$ matrix X: what does it *represent*? It may represent many different things—a table of data, a bilinear form, a map between two different vector spaces of the same dimension, but one gets the most meaning of the determinant when X represents a linear map from a vector space to itself. Then, $\det(X)$ represents the product of the eigenvalues of the linear map, or equivalently, the measure of the change in oriented volume the linear map effects on n-dimensional parallelepipeds with one vertex at the origin. These interpretations will be important for what follows (see, e.g., §13.4.2), but for now we continue our computational point of view.

The standard formula for the determinant is

$$\det(X) = \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma) x^1_{\sigma(1)} \cdots x^n_{\sigma(n)}, \tag{1.4.1}$$

where $\mathfrak{S}_n$ is the *group* of all permutations on n elements and $\operatorname{sgn}(\sigma)$ is the sign of the permutation $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$. The formula is a sum of $n!$ terms, so for a 10×10 matrix, one would have to perform $9(10!) \simeq 10^7$ multiplications and $10! - 1$ additions to compute the determinant.

The standard method for computing the determinant of large matrices is *Gaussian elimination*. Recall that if X is an upper triangular matrix, its determinant is easy to compute: it is just the product of the diagonal entries. On the other hand, $\det(gX) = \det(g)\det(X)$, so multiplying X by a matrix with determinant one will not change its determinant. To perform Gaussian elimination one chooses a sequence of matrices with determinant one to multiply X by until one obtains an upper triangular matrix whose determinant is then easy to compute. Matrix multiplication, even of arbitrary matrices, uses on the order of n^3 scalar multiplications (actually less, as discussed in §1.1), but even executed naïvely, one uses approximately n^4

multiplications. Thus for a 10×10 matrix one has 10^4 for Gaussian elimination applied naïvely versus 10^7 for (1.4.1) applied naïvely. This difference in complexity is discussed in detail in Chapter 13.

The determinant of a matrix is unchanged by the following operations:

$$X \mapsto gXh,$$
$$X \mapsto gX^T h,$$

where g, h are $n \times n$ matrices with $\det(g)\det(h) = 1$. In 1897 Frobenius [**132**] proved these are exactly the symmetries of the determinant. On the other hand, a random homogeneous polynomial of degree n in n^2 variables will have hardly any symmetries, and one might attribute the facility of computing the determinant to this large group of symmetries.

We can make this more precise as follows: first note that polynomials with small formulas, such as $f(x^i_j) = x^1_1 x^2_2 \cdots x^n_n$ are easy to compute.

Let $\mathfrak{b} \subset Mat_{n\times n}$ denote the subset of upper triangular matrices, and $G(\det_n)$ the symmetry group of the determinant. Observe that $\det_n|_{\mathfrak{b}}$ is just the polynomial f above, and that $G(\det_n)$ can move any matrix into $\mathfrak{b}$ relatively cheaply. This gives a geometric proof that the determinant is easy to compute.

One can ask: what other polynomials are easy to compute? More precisely, what polynomials with no known small formulas are easy to compute? An example of one such is given in §13.5.3.

1.4.2. The permanent and counting perfect matchings. The *marriage problem*: n people are attending a party where the host has n different flavors of ice cream for his guests. Not every guest likes every flavor. We can make a *bipartite graph*, a graph with one set of nodes the set of guests, and another set of nodes the set of flavors, and then draw an edge joining any two nodes that are compatible. A *perfect matching* is possible if everyone can get a suitable dessert. The host is curious as to how many different perfect matchings are possible.

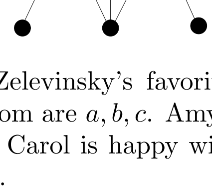


Figure 1.4.1. Andrei Zelevinsky's favorite bipartite graph: top nodes are A, B, C, bottom are a, b, c. Amy is allergic to chocolate, Bob insists on banana, Carol is happy with banana or chocolate. Only Amy likes apricot.

Given a bipartite graph on (n, n) vertices one can check if the graph has a perfect matching in polynomial time [**156**]. However, there is no known polynomial time algorithm to count the number of perfect matchings.

Problems such as the marriage problem appear to require a number of arithmetic operations that grows exponentially with the size of the data in order to solve them, however a proposed solution can be verified by performing a number of arithmetic operations that grows polynomialy with the size of the data. Such problems are said to be of class **NP**. (See Chapter 13 for precise definitions.)

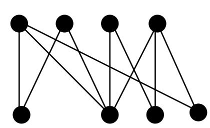
Form an *incidence matrix* $X = (x^i_j)$ for a bipartite graph by letting the upper index correspond to one set of nodes and the lower index to the other. One then places a 1 in the (i, j)-th slot if there is an edge joining the corresponding nodes and a zero if there is not.

Define the *permanent* of an $n \times n$ matrix $X = (x^i_j)$ by

$$\mathrm{perm}_n(X) := \sum_{\sigma \in \mathfrak{S}_n} x^1_{\sigma(1)} x^2_{\sigma(2)} \cdots x^n_{\sigma(n)}, \tag{1.4.2}$$

and observe the similarities with (1.4.1).

Exercise 1.4.2.1: Verify directly that the permanent of the incidence matrix for the following graph indeed equals its number of perfect matchings:



Exercise 1.4.2.2: Show that if X is an incidence matrix for an (n, n)-bipartite graph Γ, that the number of perfect matchings of Γ is given by the permanent.

Remark 1.4.2.3. The term "permanent" was first used by Cauchy [**82**] to describe a general class of functions with properties similar to the permanent.

1.4.3. Algebraic variants of P v. NP. Matrix multiplication, and thus computing the determinant of a matrix, can be executed by performing a number of arithmetic operations that is polynomial in the size of the data. (If the data size is of order $m = n^2$, then one needs roughly $m^{\frac{3}{2}} = n^3$ operations, or roughly n^4 if one wants an algorithm without divisions or decisions, see §13.4.2.) Roughly speaking, such problems are said to be of class **P**, or are *computable in polynomial time*.

L. Valiant [**312**] had the following idea: Let $P(x^1, \dots, x^{\mathbf{v}})$ be a homogeneous polynomial of degree m in $\mathbf{v}$ variables. We say P is an *affine* *projection* of a determinant of size n if there exists an affine linear function $f : \mathbb{C}^{\mathbf{v}} \to Mat_n(\mathbb{C})$ such that $P = \det \circ f$. Write $dc(P)$ for the smallest n such that P may be realized as the affine projection of a determinant of size n.

Comparing the two formulas, the difference between the permanent and the determinant is "just" a matter of a few minus signs. For example, one can convert the permanent of a 2×2 matrix to a determinant by simply changing the sign of one of the off-diagonal entries. Initially there was hope that one could do something similar in general, but these hopes were quickly dashed. The next idea was to write the permanent of an $m \times m$ matrix as the determinant of a larger matrix.

Using the sequence $(\det_n)$, Valiant defined an algebraic analog of the class $\mathbf{P}$, denoted $\mathbf{VP}_{ws}$, as follows: a sequence of polynomials (p_m) where p_m is a homogeneous polynomial of degree $d(m)$ in $\mathbf{v}(m)$ variables, with $d(m), \mathbf{v}(m)$ polynomial functions of m, is in the class $\mathbf{VP}_{ws}$ if $dc(p_m) = \mathcal{O}(m^c)$ for some constant c. (For an explanation of the cryptic notation $\mathbf{VP}_{ws}$, see §13.3.3.)

Conjecture 1.4.3.1 (Valiant)**.** *The function* $dc(\mathrm{perm}_m)$ *grows faster than any polynomial in* m*, i.e.,* $(\mathrm{perm}_m) \notin \mathbf{VP}_{ws}$.

How can one determine $dc(P)$? I discuss geometric approaches towards this question in Chapter 13. One, that goes back to Valiant and has been used in [**329, 235, 234**], involves studying the local differential geometry of the two sequences of zero sets of the polynomials $\det_n$ and perm_m. Another, due to Mulmuley and Sohoni [**244**], involves studying the algebraic varieties in the vector spaces of homogenous polynomials obtained by taking the orbit closures of the polynomials $\det_n$ and perm_m in the space of homogeneous polynomials of degree n (resp. m) in n^2 (resp. m^2) variables. This approach is based in algebraic geometry and representation theory.

1.5. Algebraic statistics and tensor networks

1.5.1. Algebraic statistical models. As mentioned above, in statistics one is handed data, often in the form of a multidimensional array, and is asked to extract meaningful information from the data. Recently the field of *algebraic statistics* has arisen.

Instead of asking: *What is the meaningful information to be extracted from this data?* one asks: *How can I partition the set of* **all** *arrays of a given size into subsets of data sets sharing similar attributes?*

Consider a weighted 6-sided die, and for $1 \leq j \leq 6$, let p_j denote the probability that j is thrown, so $0 \leq p_j \leq 1$ and $\sum_j p_j = 1$. We record the information in a vector $p \in \mathbb{R}^6$. Now say we had a second die, say 20-sided, with probabilities q_s, $0 \leq q_s \leq 1$ and $q_1 + \cdots + q_{20} = 1$. Now if we throw the dice together, assuming the events are independent, the probability of throwing i for the first and s for the second is simply $p_i q_s$. We may form a 6×20 matrix $x = (x_{i,s}) = (p_i q_s)$ recording all the possible throws with their probabilities. Note that $x_{i,s} \geq 0$ for all i, s and $\sum_{i,s} x_{i,s} = 1$. The matrix x has an additional property: *x has rank one.*

Were the events not independent, we would not have had this additional constraint. Consider the set $\{T \in \mathbb{R}^6 \otimes \mathbb{R}^{20} \mid T_{i,s} \geq 0,\ \sum_{i,s} T_{i,s} = 1\}$. This is the set of all discrete probability distributions on $\mathbb{R}^6 \otimes \mathbb{R}^{20}$, and the set of the previous paragraph is this set intersected with the set of rank one matrices.

Now say some gamblers were cheating with κ sets of dice, each with different probabilities. They watch to see how bets are made and then choose one of the sets accordingly. Now we have probabilities $p_{i,u}$, $q_{s,u}$, and a $6 \times 20 \times \kappa$ array $z_{i,s,u}$ with $\operatorname{rank}(z) = 1$, in the sense that if we consider z as a bilinear map, it has rank one.

Say that we cannot observe the betting. Then, to obtain the probabilities of what we can observe, we must sum over all the κ possibilities. We end up with an element of $\mathbb{R}^6 \otimes \mathbb{R}^{20}$, with entries $r_{i,s} = \sum_u p_{i,u} q_{i,u}$. That is, we obtain a 6×20 matrix of probabilities of rank (at most) κ, i.e., an element of $\hat{\sigma}_{\kappa, \mathbb{R}^6 \otimes \mathbb{R}^{20}}$. The set of all such distributions is the set of matrices of $\mathbb{R}^6 \otimes \mathbb{R}^{20}$ of rank at most κ intersected with $PD_{6,20}$.

This is an example of a *Bayesian network* . In general, one associates a graph to a collection of random variables having various conditional dependencies and then from such a graph, one defines sets (varieties) of distributions. More generally an *algebraic statistical model* is the intersection of the probability distributions with a closed subset defined by some dependence relations. Algebraic statistical models are discussed in Chapter 14.

A situation discussed in detail in Chapter 14 concerns algebraic statistical models arising in phylogeny: Given a collection of species, say humans, monkeys, gorillas, orangutans, etc., all of which are assumed to have evolved from some common ancestor, ideally we might like to reconstruct the corresponding evolutionary tree from sampling DNA. Assuming we can only measure the DNA of existing species, this will not be completely possible, but it might be possible, e.g., to determine which pairs are most closely related.

One might imagine, given the numerous possibilities for evolutionary trees, that there would be a horrific amount of varieties to find equations for. A major result of E. Allmann and J. Rhodes states that this is not the case.

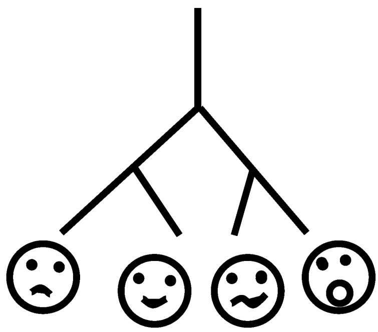


Figure 1.5.1. Evolutionary tree; extinct ancestor gave rise to 4 species.

Theorem 1.5.1.1 ([**9, 8**]). *Equations for the algebraic statistical model associated to any bifurcating evolutionary tree can be determined explicitly from equations for* $\hat{\sigma}_{4,\mathbb{C}^4\otimes\mathbb{C}^4\otimes\mathbb{C}^4}$.

See §14.2.3 for a more precise statement. Moreover, motivated by Theorem 1.5.1.1 (and the promise of a hand-smoked alaskan salmon), equations for $\hat{\sigma}_{4,\mathbb{C}^4\otimes\mathbb{C}^4\otimes\mathbb{C}^4}$ were recently found by S. Friedland; see §3.9.3 for the equations and salmon discussion.

The geometry of this field has been well developed and exposed; see [**259, 168**]. I include a discussion in Chapter 14.

1.5.2. Tensor network states. Tensors describe states of quantum mechanical systems. If a system has n particles, its state is an element of $V_1 \otimes \cdots \otimes V_n$, where V_j is a Hilbert space associated to the j-th particle. In many-body physics, in particular solid state physics, one wants to simulate quantum states of thousands of particles, often arranged on a regular lattice (e.g., atoms in a crystal). Due to the exponential growth of the dimension of $V_1 \otimes \cdots \otimes V_n$ with n, any naïve method of representing these tensors is intractable on a computer. *Tensor network states* were defined to reduce the complexity of the spaces involved by restricting to a subset of tensors that is physically reasonable, in the sense that the corresponding spaces of tensors are only "locally" entangled because interactions (entanglement) in the physical world appear to just happen locally.

Such spaces have been studied since the 1980s. These spaces are associated to graphs, and go under different names: *tensor network states*, *finitely correlated states (FCS)*, *valence-bond solids (VBS)*, *matrix product states (MPS)*, *projected entangled pairs states (PEPS)*, and *multi-scale entanglement renormalization ansatz states (MERA)*; see, e.g., [**280, 124, 173, 123, 323, 90**] and the references therein. I will use the term tensor network states.

The topology of the tensor network is often modeled to mimic the physical arrangement of the particles and their interactions.

Just as phylogenetic trees, and more generally Bayes models, use graphs to construct varieties in spaces of tensors that are useful for the problem at hand, in physics one uses graphs to construct varieties in spaces of tensors that model the "feasible" states. The precise recipe is given in §14.1, where I also discuss geometric interpretations of the tensor network states arising from chains, trees and loops. The last one is important for physics; large loops are referred to as "1-D systems with periodic boundary conditions" in the physics literature and are the prime objects people use in practical simulations today.

To entice the reader uninterested in physics, but perhaps interested in complexity, here is a sample result:

Proposition 1.5.2.1 ([**216**]). *Tensor networks associated to graphs that are triangles consist of matrix multiplication (up to relabeling) and its degenerations.*

See Proposition 14.1.4.1 for a more precise statement. Proposition 1.5.2.1 leads to a surprising connection between the study of tensor network states and the geometric complexity theory program mentioned above and discussed in §13.6.

1.6. Geometry and representation theory

So far we have seen the following:

- A key to determining the complexity of matrix multiplication will be to find explicit equations for the set of tensors in $\mathbb{C}^{n^2}\otimes\mathbb{C}^{n^2}\otimes\mathbb{C}^{n^2}$ of border rank at most r in the range $\frac{3}{2}n^2 \leq r \leq n^{2.38}$.
- For fluorescence spectroscopy and other applications, one needs to determine the ranks of tensors (of small rank) in $\mathbb{R}^I\otimes\mathbb{R}^J\otimes\mathbb{R}^K$ and to decompose them. Equations will be useful both for determining what the rank of an approximating tensor should be and for developing explicit algorithms for tensor decomposition.
- To study cumulants and blind source separation, one is interested in the analogous questions for symmetric tensor rank.
- In other applications, such as the GCT program for the Mulmuley-Sohoni variant of **P** v. **NP**, and in algebraic statistics, a central goal is to find the equations for other algebraic varieties arising in spaces of tensors.

To find equations, we will exploit the symmetries of the relevant varieties. We will also seek geometric descriptions of the varieties. For instance, Example 1.2.6.1 admits the simple interpretation that the limit of the sequence of secant lines is a tangent line as illustrated in Figure 1.6.1. This interpretation is explained in detail in Chapter 5, and exploited many times in later chapters.

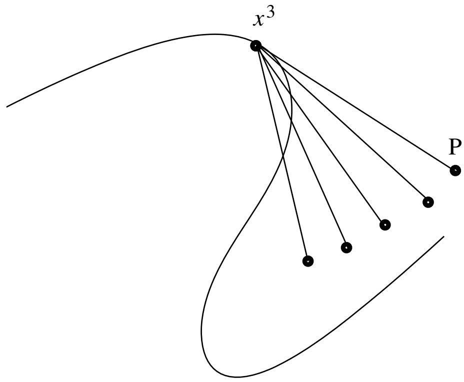


Figure 1.6.1. Unlabeled points are various P_ϵ's lying on secant lines to the curve of perfect cubes.

Given a multidimensional array in $\mathbb{R}^{\mathbf{a}}\otimes\mathbb{R}^{\mathbf{b}}\otimes\mathbb{R}^{\mathbf{c}}$, its rank and border rank will be unchanged if one changes bases in $\mathbb{R}^{\mathbf{a}}$, $\mathbb{R}^{\mathbf{b}}$, $\mathbb{R}^{\mathbf{c}}$. Another way to say this is that the properties of rank and border rank are *invariant* under the action of the group of changes of bases $G := GL_{\mathbf{a}} \times GL_{\mathbf{b}} \times GL_{\mathbf{c}}$. When one looks for the defining equations of $\hat{\sigma}_r$, the space of equations will also be invariant under the action of G. *Representation theory* provides a way to organize all polynomials into sums of subspaces invariant under the action of G. It is an essential tool for the study of equations.

Chapter 2

Multilinear algebra

This chapter approaches multilinear algebra from a geometric perspective. If $X = (a^i_s)$ is a matrix, one is not so much interested in the collection of numbers that make up X, but what X *represents* and what qualitative information can be extracted from the entries of X. For this reason and others, in §2.3 an invariant definition of tensors is given and its utility is explained, especially in terms of groups acting on spaces of tensors. Before that, the chapter begins in §2.1 with a collection of exercises to review facts from linear algebra that will be important in what follows. For those readers needing a reminder of the basic definitions in linear algebra, they are given in an appendix, §2.9. Basic definitions regarding group actions that will be needed throughout are stated in §2.2. In §2.4, rank and border rank of tensors are defined, Strassen's algorithm is revisited, and basic results about rank are established. A more geometric perspective on the matrix multiplication operator in terms of contractions is given in §2.5. Among subspaces of spaces of tensors, the *symmetric* and *skew-symmetric* tensors discussed in §2.6 are distinguished, not only because they are the first subspaces one generally encounters, but all other natural subspaces may be built out of symmetrizations and skew-symmetrizations. As a warm up for the detailled discussions of polynomials that appear later in the book, polynomials on the space of matrices are discussed in §2.7. In §2.8, $V^{\otimes 3}$ is decomposed as a $GL(V)$-module, which serves as an introduction to Chapter 6.

There are three appendices to this chapter. As mentioned above, in §2.9 basic definitions are recalled for the reader's convenience. §2.10 reviews Jordan and rational canonical forms. Wiring diagrams, a useful pictorial tool for studying tensors, are introduced in §2.11.

I work over the field $\mathbb{C}$. Unless otherwise noted, A_j, A, B, C, V, and W are finite-dimensional complex vector spaces respectively of dimensions $\mathbf{a}_j, \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{v}$, and $\mathbf{w}$. The dual vector space to V (see §1.2.1) is denoted V^*.

2.1. Rust removal exercises

For $\alpha \in V^*$, let $\alpha^\perp := \{v \in V \mid \alpha(v) = 0\}$, and for $w \in V$, let $\langle w \rangle$ denote the span of w.

Definition 2.1.0.2. Let $\alpha \in V^*$, and $w \in W$. Consider the linear map $\alpha \otimes w : V \to W$ defined by $v \mapsto \alpha(v)w$. A linear map of this form is said to be of *rank one*.

Observe that

$$\ker(\alpha \otimes w) = \alpha^\perp, \qquad \operatorname{image}(\alpha \otimes w) = \langle w \rangle.$$

(1) Show that if one chooses bases of V and W, the matrix representing $\alpha \otimes w$ has rank one.

(2) Show that every rank one $n \times m$ matrix is the product of a column vector with a row vector. To what extent is this presentation unique?

(3) Show that a nonzero matrix has rank one if and only if all its 2×2 minors are zero. ⊚

(4) Show that the following definitions of the *rank* of a linear map $f : V \to W$ are equivalent:
   (a) $\dim f(V)$.
   (b) $\dim V - \dim(\ker(f))$.
   (c) The smallest r such that f is the sum of r rank one linear maps.
   (d) The smallest r such that any matrix representing f has all size $r + 1$ minors zero.
   (e) There exist choices of bases in V and W such that the matrix of f is $\begin{pmatrix} \mathrm{Id}_r & 0 \\ 0 & 0 \end{pmatrix}$, where the blocks in the previous expression come from writing $\dim V = r + (\dim V - r)$ and $\dim W = r + (\dim W - r)$, and Id_r is the $r \times r$ identity matrix.

(5) Given a linear subspace $U \subset V$, define $U^\perp \subset V^*$, the *annihilator of* U, by $U^\perp := \{\alpha \in V^* \mid \alpha(u) = 0 \ \forall u \in U\}$. Show that $(U^\perp)^\perp = U$.

(6) Show that for a linear map $f : V \to W$, $\ker f = (\operatorname{image} f^T)^\perp$. (See 2.9.1.6 for the definition of the transpose $f^T : W^* \to V^*$.) This is sometimes referred to as the *fundamental theorem of linear algebra*. It implies $\operatorname{rank}(f) = \operatorname{rank}(f^T)$, i.e., that for a matrix, *row rank equals column rank*, as was already seen in Exercise (4) above.

(7) Let V denote the vector space of 2×2 matrices. Take a basis

$$v_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \ v_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ v_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \ v_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Fix $a, b, c, d \in \mathbb{C}$ and let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Write out a 4×4 matrix expressing the linear map

$$\begin{aligned} L_A : V &\to V, \\ X &\mapsto AX \end{aligned}$$

that corresponds to left multiplication by A. Write the analogous matrix for right multiplication. For which matrices A are the two induced linear maps the same?

(8) Given a 2×2 matrix A, write out a 4×4 matrix expressing the linear map

$$\begin{aligned} ad(A) : V &\to V, \\ X &\mapsto AX - XA. \end{aligned}$$

What is the largest possible rank of this linear map?

(9) Let A be a 3×3 matrix and write out a 9×9 matrix representing the linear map $L_A : Mat_{3\times 3} \to Mat_{3\times 3}$.

(10) Choose new bases such that the matrices of Exercises (7) and (9) become block diagonal (i.e., the only nonzero entries occur in 2×2 (resp. 3×3) blocks centered about the diagonal). What will the $n \times n$ case look like?

(11) A vector space V admits a *direct sum decomposition* $V = U \oplus W$ if $U, W \subset V$ are linear subspaces and if for all $v \in V$ there exist unique $u \in U$ and $w \in W$ such that $v = u + w$. Show that a necessary and sufficient condition to have a direct sum decomposition $V = U \oplus W$ is that $\dim U + \dim W \geq \dim V$ and $U \cap W = (0)$. Similarly, show that another necessary and sufficient condition to have a direct sum decomposition $V = U \oplus W$ is that $\dim U + \dim W \leq \dim V$ and $\mathrm{span}\{U, W\} = V$.

(12) Let $S^2\mathbb{C}^n$ denote the vector space of symmetric $n \times n$ matrices. Calculate $\dim S^2\mathbb{C}^n$. Let $\Lambda^2\mathbb{C}^n$ denote the vector space of skew-symmetric $n \times n$ matrices. Calculate its dimension, and show that there is a direct sum decomposition

$$Mat_{n\times n} = S^2(\mathbb{C}^n) \oplus \Lambda^2(\mathbb{C}^n).$$

(13) Let $v_1, \dots, v_{\mathbf{v}}$ be a basis of V, and let $\alpha^i \in V^*$ be defined by $\alpha^i(v_j) = \delta^i_j$. (Recall that a linear map is uniquely specified by prescribing the image of a basis.) Show that $\alpha^1, \dots, \alpha^{\mathbf{v}}$ is a basis for V^*, called the *dual basis* to $v_1, \dots, v_n$. In particular, $\dim V^* = \mathbf{v}$.

(14) Define, in a coordinate-free way, an injective linear map $V \to (V^*)^*$. (Note that the map would not necessarily be surjective if V were an infinite-dimensional vector space.)

(15) A *filtration* of a vector space is a sequence of subspaces $0 \subset V_1 \subset V_2 \subset \cdots \subset V$. Show that a filtration of V naturally induces a filtration of V^*.

(16) Show that by fixing a basis of V, one obtains an identification of the group of invertible endomorphisms of V, denoted $GL(V)$, and the set of bases of V.

2.2. Groups and representations

A significant part of our study will be to exploit symmetry to better understand tensors. The set of symmetries of any object forms a *group*, and the realization of a group as a group of symmetries is called a *representation* of a group. The most important group in our study will be $GL(V)$, the group of invertible linear maps $V \to V$, which forms a group under the composition of mappings.

2.2.1. The group $GL(V)$. If one fixes a reference basis, $GL(V)$ is the group of changes of bases of V. If we use our reference basis to identify V with $\mathbb{C}^{\mathbf{v}}$ equipped with its standard basis, $GL(V)$ may be identified with the set of invertible $\mathbf{v} \times \mathbf{v}$ matrices. I sometimes write $GL(V) = GL_{\mathbf{v}}$ or $GL_{\mathbf{v}}\mathbb{C}$ if V is $\mathbf{v}$-dimensional and comes equipped with a basis. I emphasize $GL(V)$ as a group rather than the invertible $\mathbf{v} \times \mathbf{v}$ matrices because it not only acts on V, but on many other spaces constructed from V.

Definition 2.2.1.1. Let W be a vector space, let G be a group, and let $\rho : G \to GL(W)$ be a *group homomorphism* (see §2.9.2.4). (In particular, $\rho(G)$ is a subgroup of $GL(W)$.) A group homomorphism $\rho : G \to GL(W)$ is called a *(linear) representation* of G. One says *G acts on W*, or that W is a *G-module*.

For $g \in GL(V)$ and $v \in V$, write $g \cdot v$, or $g(v)$ for the action. Write $\circ$ for the composition of maps.

Example 2.2.1.2. Here are some actions: $g \in GL(V)$ acts on

(1) V^* by $\alpha \mapsto \alpha \circ g^{-1}$.

(2) $\mathrm{End}(V)$ by $f \mapsto g \circ f$.

(3) A second action on $\mathrm{End}(V)$ is by $f \mapsto g \circ f \circ g^{-1}$.

(4) The vector space of homogeneous polynomials of degree d on V for each d by $P \mapsto g \cdot P$, where $g \cdot P(v) = P(g^{-1}v)$. Note that this agrees with (1) when $d = 1$.

(5) Let $V = \mathbb{C}^2$, so the standard action of $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL(V)$ on $\mathbb{C}^2$ is $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$. Then GL_2 also acts on $\mathbb{C}^3$ by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} a^2 & ac & c^2 \\ 2ab & ad+bc & 2cd \\ b^2 & bd & d^2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

The geometry of this action is explained by Exercise 2.6.2(3).

These examples give group homomorphisms $GL(V) \to GL(V^*)$, $GL(V) \to GL(\mathrm{End}(V))$ (two different ones) and $GL(V) \to GL(S^dV^*)$, where S^dV^* denotes the vector space of homogeneous polynomials of degree d on V.

Exercise 2.2.1.3: Verify that each of the above examples is indeed an action, e.g., show that $(g_1g_2) \cdot \alpha = g_1(g_2 \cdot \alpha)$.

Exercise 2.2.1.4: Let $\dim V = 2$ and choose a basis of V so that $g \in GL(V)$ may be written as $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Write out the 4×4 matrices for Examples 2.2.1.2(2) and 2.2.1.2(3).

2.2.2. Modules and submodules. If W is a G-module and there is a linear subspace $U \subset W$ such that $g \cdot u \in U$ for all $g \in G$ and $u \in U$, then one says U is a *G-submodule* of W.

Exercise 2.2.2.1: Using Exercise 2.2.1.4 show that both actions on $\mathrm{End}(V)$ have nontrivial submodules; in the first case (when $\dim V = 2$) one can find two-dimensional subspaces preserved by $GL(V)$ and in the second there is a unique one-dimensional subspace and a unique three-dimensional subspace preserved by $GL(V)$.⊚

A module is *irreducible* if it contains no nonzero proper submodules. For example, the action 2.2.1.2(3) restricted to the trace-free linear maps is irreducible.

If $Z \subset W$ is a subset and a group G acts on W, one says Z is *invariant* under the action of G if $g \cdot z \in Z$ for all $z \in Z$ and $g \in G$.

2.2.3. Exercises.

(1) Let $\mathfrak{S}_n$ denote the group of permutations of $\{1, \dots, n\}$ (see Definition 2.9.2.2). Endow $\mathbb{C}^n$ with a basis. Show that the action of $\mathfrak{S}_n$ on $\mathbb{C}^n$ defined by permuting basis elements, i.e., given $\sigma \in \mathfrak{S}_n$ and a basis $e_1, \dots, e_n$, $\sigma \cdot e_j = e_{\sigma(j)}$, is not irreducible.⊚

(2) Show that the action of GL_n on $\mathbb{C}^n$ is irreducible.

(3) Show that the map $GL_p \times GL_q \to GL_{pq}$ given by (A, B) acting on a $p \times q$ matrix X by $X \mapsto AXB^{-1}$ is a linear representation.

(4) Show that the action of the group of invertible upper triangular matrices on $\mathbb{C}^n$ is not irreducible. ⊚

(5) Let Z denote the set of rank one $p \times q$ matrices inside the vector space of $p \times q$ matrices. Show Z is invariant under the action of $GL_p \times GL_q \subset GL_{pq}$.

2.3. Tensor products

In physics, engineering and other areas, tensors are often defined to be multidimensional arrays. Even a linear map is often defined in terms of a matrix that represents it in a given choice of basis. In what follows I give more invariant definitions, and with a good reason.

Consider for example the space of $\mathbf{v} \times \mathbf{v}$ matrices, first as representing the space of linear maps $V \to W$, where $\mathbf{w} = \mathbf{v}$. Then given $f : V \to W$, one can always make changes of bases in V and W such that the matrix of f is of the form

$$\begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}$$

where the blocking is $(k, \mathbf{v} - k) \times (k, \mathbf{v} - k)$, so there are only $\mathbf{v}$ different such maps up to equivalence. On the other hand, if the space of $\mathbf{v} \times \mathbf{v}$ matrices represents the linear maps $V \to V$, then one can only have Jordan (or rational) canonical form, i.e., there are parameters worth of distinct matrices up to equivalence.

Because it will be essential to keep track of group actions in our study, I give basis-free definitions of linear maps and tensors.

2.3.1. Definitions.

Notation 2.3.1: Let $V^*{\otimes}W$ denote the vector space of linear maps $V \to W$. With this notation, $V{\otimes}W$ denotes the linear maps $V^* \to W$.

The space $V^*{\otimes}W$ may be thought of in four different ways: as the space of linear maps $V \to W$, as the space of linear maps $W^* \to V$ (using the isomorphism determined by taking transpose), as the dual vector space to $V{\otimes}W^*$, by Exercise 2.3.2(3) below, and as the space of bilinear maps $V \times W^* \to \mathbb{C}$. If one chooses bases and represents $f \in V^*{\otimes}W$ by a $\mathbf{v} \times \mathbf{w}$ matrix $X = (f^i_s)$, the first action is multiplication by a column vector $v \mapsto Xv$, the second by right multiplication by a row vector $\beta \mapsto \beta X$, the third by, given a $\mathbf{w} \times \mathbf{v}$ matrix $Y = (g^s_i)$, taking $\sum_{i,s} f^i_s g^s_i$, and the fourth by $(v, \beta) \mapsto f^i_s v_i \beta^s$.

Exercise 2.3.1.1: Show that the rank one elements in $V{\otimes}W$ span $V{\otimes}W$. More precisely, given bases (v_i) of V and (w_s) of W, show that the $\mathbf{vw}$ vectors $v_i{\otimes}w_s$ provide a basis of $V{\otimes}W$.

Exercise 2.3.1.2: Let $v_1, \dots, v_n$ be a basis of V with dual basis $\alpha^1, \dots, \alpha^n$. Write down an expression for a linear map as a sum of rank one maps $f : V \to V$ such that each v_i is an eigenvector with eigenvalue λ_i, that is $f(v_i) = \lambda_i v_i$ for some $\lambda_i \in \mathbb{C}$. In particular, write down an expression for the identity map (the case of all $\lambda_i = 1$).

Definition 2.3.1.3. Let $V_1, \dots, V_k$ be vector spaces. A function

$$f : V_1 \times \cdots \times V_k \to \mathbb{C} \tag{2.3.1}$$

is *multilinear* if it is linear in each factor V_ℓ. The space of such multilinear functions is denoted $V_1^*{\otimes}V_2^*{\otimes} \cdots \otimes V_k^*$ and called the *tensor product* of the vector spaces $V_1^*, \dots, V_k^*$. Elements $T \in V_1^*{\otimes} \cdots \otimes V_k^*$ are called *tensors*. The integer k is sometimes called the *order* of T. The sequence of natural numbers $(\mathbf{v}_1, \dots, \mathbf{v}_k)$ is sometimes called the *dimensions* of T.

More generally, a function

$$f : V_1 \times \cdots \times V_k \to W \tag{2.3.2}$$

is *multilinear* if it is linear in each factor V_ℓ. The space of such multilinear functions is denoted $V_1^*{\otimes}V_2^*{\otimes} \cdots {\otimes}V_k^*{\otimes}W$ and called the *tensor product* of $V_1^*, \dots, V_k^*, W$.

If $f : V_1 \times V_2 \to W$ is bilinear, define the left kernel

$$\mathrm{Lker}(f) = \{v \in V_1 \mid f(v_1, v_2) = 0 \ \forall v_2 \in V_2\}$$

and similarly for the right kernel $\mathrm{Rker}(f)$. For multilinear maps one analogously defines the *i-th kernel*.

When studying tensors in $V_1{\otimes} \cdots \otimes V_n$, introduce the notation $V_{\hat{j}} := V_1{\otimes} \cdots \otimes V_{j-1}{\otimes}V_{j+1}{\otimes} \cdots \otimes V_n$. Given $T \in V_1{\otimes} \cdots \otimes V_n$, write $T(V_j^*) \subset V_{\hat{j}}$ for the image of the linear map $V_j^* \to V_{\hat{j}}$.

Definition 2.3.1.4. Define the *multilinear rank* (sometimes called the *duplex rank* or *Tucker rank*) of $T \in V_1{\otimes} \cdots \otimes V_n$ to be the n-tuple of natural numbers $\mathrm{R}_{\mathrm{multlin}}(T) := (\dim T(V_1^*), \dots, \dim T(V_n^*))$.

The number $\dim(T(V_j^*))$ is sometimes called the *mode j rank* of T.

Write $V^{\otimes k} := V{\otimes} \cdots \otimes V$ where there are k copies of V in the tensor product.

Remark 2.3.1.5. Some researchers like to picture tensors given in bases in terms of *slices*. Let A have basis $a_1, \dots, a_{\mathbf{a}}$ and similarly for B, C, let

$T \in A\otimes B\otimes C$, so in bases $T = T^{i,s,u} a_i\otimes b_s\otimes c_u$. Then one forms an $\mathbf{a} \times \mathbf{b} \times \mathbf{c}$ rectangular solid table whose entries are the $T^{i,s,u}$. This solid is then decomposed into *modes* or *slices*; e.g., consider T as a collection of $\mathbf{a}$ matrices of size $\mathbf{b} \times \mathbf{c}$: $(T^{1,s,u}), \dots, (T^{\mathbf{a},s,u})$, which might be referred to as *horizontal slices* (e.g. [**188**, p. 458]), or a collection of $\mathbf{b}$ matrices $(T^{i,1,u}), \dots, (T^{i,\mathbf{b},u})$ called *lateral slices*, or a collection of $\mathbf{c}$ matrices called *frontal slices*. When two indices are fixed, the resulting vector in the third space is called a *fiber*.

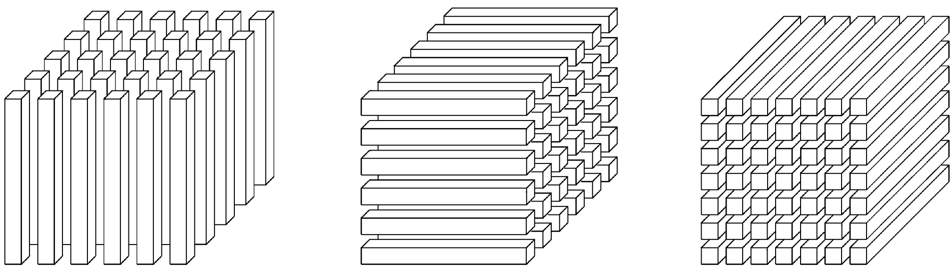


Figure 2.3.1. Slices of a three-way tensor. (Tamara G. Kolda and Brett W. Bader, *Tensor decompositions and applications*, SIAM Rev. 51 (2009), no. 3, 455–500.

2.3.2. Exercises.

(1) Write out the slices of the 2×2 matrix multiplication operator $M \in A\otimes B\otimes C = (U^*\otimes V)\otimes(V^*\otimes W)\otimes(W^*\otimes U)$ with respect to the basis $a_1 = u^1\otimes v_1, a_2 = u^1\otimes v_2, a_3 = u^2\otimes v_1, a_4 = u^2\otimes v_2$ of A and the analogous bases for B, C.

(2) Verify that the space of multilinear functions (2.3.2) is a vector space.

(3) Given $\alpha \in V^*$, $\beta \in W^*$, allow $\alpha\otimes\beta \in V^*\otimes W^*$ to act on $V\otimes W$ by, for $v \in V$, $w \in W$, $\alpha\otimes\beta(v\otimes w) = \alpha(v)\beta(w)$ and extending linearly. Show that this identification defines an isomorphism $V^*\otimes W^* \cong (V\otimes W)^*$.

(4) Show that $V\otimes\mathbb{C} \simeq V$.

(5) Show that for each $I \subset \{1, \dots, k\}$ with complementary index set I^c, there are canonical identifications of $V_1^*\otimes\cdots\otimes V_k^*$ with the space of multilinear maps $V_{i_1} \times \cdots \times V_{i_{|I|}} \to V^*_{i^c_1}\otimes\cdots\otimes V^*_{i^c_{k-|I|}}$.

(6) A bilinear map $f : U\times V \to \mathbb{C}$ is called a *perfect pairing* if $\mathrm{Lker}(f) = \mathrm{Rker}(f) = 0$. Show that if f is a perfect pairing, it determines an identification $U \simeq V^*$.

(7) Show that the map $\mathrm{tr} : V^* \times V \to \mathbb{C}$ given by $(\alpha, v) \mapsto \alpha(v)$ is bilinear and thus is well defined as a linear map $V^* \otimes V \to \mathbb{C}$. For $f \in V^* \otimes V$, show that $\mathrm{tr}(f)$ agrees with the usual notion of the trace of a linear map, that is, the sum of the eigenvalues or the sum of the entries on the diagonal in any matrix expression of f.

(8) Show that $\mathrm{tr} \in V \otimes V^*$, when considered as a map $V \to V$, is the identity map, i.e., that tr and Id_V are the same tensors. Write out this tensor with respect to any choice of basis of V and dual basis of V^*.

(9) A linear map $p : V \to V$ is a *projection* if $p^2 = p$. Show that if p is a projection, then $\mathrm{tr}(p) = \dim(\mathrm{image}(p))$.

(10) Show that a basis of V induces a basis of $V^{\otimes d}$ for each d.

(11) Determine $\dim(V_1 \otimes \cdots \otimes V_k)$, in terms of $\dim V_i$.

2.4. The rank and border rank of a tensor

2.4.1. The rank of a tensor. Given $\beta_1 \in V_1^*, \ldots, \beta_k \in V_k^*$, define an element $\beta_1 \otimes \cdots \otimes \beta_k \in V_1^* \otimes V_2^* \otimes \cdots \otimes V_k^*$ by

$$\beta_1 \otimes \cdots \otimes \beta_k(u_1, \ldots, u_k) = \beta_1(u_1) \cdots \beta_k(u_k). \tag{2.4.1}$$

Definition 2.4.1.1. An element of $V_1^* \otimes V_2^* \otimes \cdots \otimes V_k^*$ is said to have *rank one* if it may be written as in (2.4.1).

Note that the property of having rank one is independent of any choices of basis. (The vectors in each space used to form the rank one tensor will usually not be basis vectors, but linear combinations of them.)

Definition 2.4.1.2. Define the *rank* of a tensor $T \in V_1 \otimes V_2 \otimes \cdots \otimes V_k$, denoted $\mathbf{R}(T)$, to be the minimum number r such that $T = \sum_{u=1}^{r} Z_u$ with each Z_u rank one.

Note that the rank of a tensor is unchanged if one makes changes of bases in the vector spaces V_i. Rank is sometimes called *outer product rank* in the tensor literature.

2.4.2. Exercises on ranks of tensors. In what follows, A, B, and C are vector spaces with bases respectively $a_1, \ldots, a_{\mathbf{a}}$, $b_1, \ldots, b_{\mathbf{b}}$, and $c_1, \ldots, c_{\mathbf{c}}$.

(1) Show that $V_1^* \otimes V_2^* \otimes \cdots \otimes V_k^*$ is spanned by its rank one elements.

(2) Compute the ranks of the following tensors: $T_1 = a_1 \otimes b_1 \otimes c_1 + a_1 \otimes b_2 \otimes c_1$, $T_2 = a_1 \otimes b_1 \otimes c_1 + a_2 \otimes b_2 \otimes c_1$, $T_3 = a_1 \otimes b_1 \otimes c_1 + a_2 \otimes b_2 \otimes c_2$.

(3) Show that for T_2 above, $\dim(T_2(C^*)) \neq \dim(T_2(B^*))$.

(4) For $T \in A \otimes B \otimes C$, show that $\mathbf{R}(T) \geq \dim T(A^*)$.

(5) Show that for all $T \in V_1 \otimes \cdots \otimes V_k$, $\mathbf{R}(T) \leq \prod_j (\dim V_j)$.

(6) Show that if $T \in A_1^* \otimes \cdots \otimes A_n^*$, then the multilinear rank $(\mathbf{b}_1, \ldots, \mathbf{b}_n)$ of T satisfies $\mathbf{b}_i \leq \min(\mathbf{a}_i, \prod_{j \neq i} \mathbf{a}_j)$, and that equality holds for generic tensors, in the sense that equality will hold for most small perturbations of any tensor.

By Exercise 2.4.2(3), one sees that the numbers that coincided for linear maps fail to coincide for tensors, i.e., the analog of the fundamental theorem of linear algebra is false for tensors.

2.4.3. $GL(V)$ acts on $V^{\otimes d}$. The group $GL(V)$ acts on $V^{\otimes d}$. The action on rank one elements is, for $g \in GL(V)$ and $v_1 \otimes \cdots \otimes v_d \in V^{\otimes d}$,

$$g \cdot (v_1 \otimes \cdots \otimes v_d) = (g \cdot v_1) \otimes \cdots \otimes (g \cdot v_d) \tag{2.4.2}$$

and the action on $V^{\otimes d}$ is obtained by extending this action linearly.

Similarly, $GL(V_1) \times \cdots \times GL(V_k)$ acts on $V_1 \otimes \cdots \otimes V_k$.

Exercises 2.4.3.1:

(1) Let $\dim V = 2$ and give V basis e_1, e_2 and dual basis α^1, α^2. Let $g \in GL(V)$ be $g = \alpha^1 \otimes (e_1 + e_2) + \alpha^2 \otimes e_2$. Compute $g \cdot (e_1 \otimes e_1 \otimes e_2 + e_1 \otimes e_2 \otimes e_2)$.

(2) For fixed $g \in GL(V)$, show that the map $V^{\otimes d} \to V^{\otimes d}$ given on rank one elements by (2.4.2) is well defined (i.e., independent of our choice of basis).

(3) Let $V = \mathbb{C}^2$ and let $d = 3$. The action of $GL(V)$ on $V^{\otimes 3}$ determines an embedding $GL(V) \subset GL(\mathbb{C}^8) = GL(V^{\otimes 3})$. Fix a basis of V, and write an 8×8 matrix expressing the image of $GL(V)$ in GL_8 with respect to the induced basis (as in Exercise 2.3.2.10).

2.4.4. Strassen's algorithm revisited. The standard algorithm for the multiplication of 2×2 matrices may be expressed in terms of tensors as follows. Let V_1, V_2, V_3 each denote the space of 2×2 matrices. Give V_1 the standard basis a^i_j for the matrix with a 1 in the (i, j)-th slot and zeros elsewhere and let α^i_j denote the dual basis element of V_1^*. Similarly for V_2, V_3. Then the standard algorithm is:

$$\begin{aligned} M_{2,2,2} = {} & a^1_1 \otimes b^1_1 \otimes c^1_1 + a^1_2 \otimes b^2_1 \otimes c^1_1 + a^2_1 \otimes b^1_1 \otimes c^2_1 + a^2_2 \otimes b^2_1 \otimes c^2_1 \\ & + a^1_1 \otimes b^1_2 \otimes c^1_2 + a^1_2 \otimes b^2_2 \otimes c^1_2 + a^2_1 \otimes b^1_2 \otimes c^2_2 + a^2_2 \otimes b^2_2 \otimes c^2_2. \end{aligned} \tag{2.4.3}$$

Strassen's algorithm is

$$
\begin{aligned}
M_{2,2,2} = {} & (a_1^1 + a_2^2)\otimes(b_1^1 + b_2^2)\otimes(c_1^1 + c_2^2) \\
& + (a_1^2 + a_2^2)\otimes b_1^1\otimes(c_1^2 - c_2^2) \\
& + a_1^1\otimes(b_2^1 - b_2^2)\otimes(c_2^1 + c_2^2) \\
& + a_2^2\otimes(-b_1^1 + b_1^2)\otimes(c_1^2 + c_1^1) \\
& + (a_1^1 + a_2^1)\otimes b_2^2\otimes(-c_1^1 + c_2^1) \\
& + (-a_1^1 + a_1^2)\otimes(b_1^1 + b_2^1)\otimes c_2^2 \\
& + (a_2^1 - a_2^2)\otimes(b_1^2 + b_2^2)\otimes c_1^1.
\end{aligned} \tag{2.4.4}
$$

Exercise 2.4.4.1: Verify that (2.4.4) and (2.4.3) are indeed the same tensors.

Remark 2.4.4.2. To present Strassen's algorithm this way, solve for the coefficients of the vector equation, set each Roman numeral in (1.1.1) to a linear combination of the c_j^i and set the sum of the terms equal to (2.4.3).

Strassen's algorithm for matrix multiplication using seven multiplications is far from unique. Let $t \in \mathbb{R}$ be a constant. One also has

$$
\begin{aligned}
M_{2,2,2} = {} & (a_1^1 + a_2^2 + ta_1^2)\otimes(b_1^1 - tb_1^2 + b_2^2)\otimes(c_1^1 + c_2^2) \\
& + (a_1^2 + a_2^2 + ta_1^2)\otimes(b_1^1 - tb_1^2)\otimes(c_1^2 - c_2^2) \\
& + a_1^1\otimes(b_2^1 - tb_2^2 - b_2^2)\otimes(c_2^1 + c_2^2) \\
& + (a_2^2 + ta_1^2)\otimes(-b_1^1 + tb_1^2 + b_1^2)\otimes(c_1^2 + c_1^1) \\
& + (a_1^1 + a_2^1 + ta_1^1)\otimes b_2^2\otimes(-c_1^1 + c_2^1) \\
& + (-a_1^1 + a_1^2)\otimes(b_1^1 - tb_1^2 + b_2^1 - tb_2^2)\otimes c_2^2 \\
& + (a_2^1 + ta_1^1 - a_2^2 - ta_1^2)\otimes(b_1^2 + b_2^2)\otimes c_1^1.
\end{aligned} \tag{2.4.5}
$$

In fact there is a nine-parameter family of algorithms for $M_{2,2,2}$, each using seven multiplications. The geometry of this family is explained in §2.5.

An expression of a bilinear map $T \in V_1^*\otimes V_2^*\otimes V_3$ as a sum of rank one elements may be thought of as an algorithm for executing it. The number of rank one elements in the expression is the number of multiplications needed to execute the algorithm. The rank of the tensor T therefore gives an upper bound on the number of multiplications needed to execute the corresponding bilinear map using a best possible algorithm.

2.4.5. Border rank of a tensor. Chapter 4 is dedicated to the study of algebraic varieties, which are the zero sets of polynomials. In particular, if a sequence of points is in the zero set of a collection of polynomials, any limit point for the sequence must be in the zero set. In our study of tensors of a given rank r, we will also study limits of such tensors.

Consider the tensor

$$T = a_1 \otimes b_1 \otimes c_1 + a_1 \otimes b_1 \otimes c_2 + a_1 \otimes b_2 \otimes c_1 + a_2 \otimes b_1 \otimes c_1. \tag{2.4.6}$$

One can show that the rank of T is three, but T can be approximated as closely as one likes by tensors of rank two. To see this let:

$$T(\epsilon) = \frac{1}{\epsilon}[(\epsilon - 1)a_1 \otimes b_1 \otimes c_1 + (a_1 + \epsilon a_2) \otimes (b_1 + \epsilon b_2) \otimes (c_1 + \epsilon c_2)]. \tag{2.4.7}$$

Definition 2.4.5.1. A tensor T has *border rank* r if it is a limit of tensors of rank r but is not a limit of tensors of rank s for any $s < r$. Let $\underline{\mathbf{R}}(T)$ denote the border rank of T.

Note that $\mathbf{R}(T) \geq \underline{\mathbf{R}}(T)$.

For example, the sequence (2.4.7) shows that T of (2.4.6) has border rank at most two, and it is not hard to see that its border rank is exactly two.

The border rank admits an elegant geometric interpretation which I discuss in detail in §5.1. Intuitively, $T(\epsilon)$ is a point on the line spanned by the two tensors $a_1 \otimes b_1 \otimes c_1$ and $z(\epsilon) := (a_1 + \epsilon a_2) \otimes (b_1 + \epsilon b_2) \otimes (c_1 + \epsilon c_2)$ inside the set of rank one tensors. Draw $z(\epsilon)$ as a curve, for $\epsilon \neq 0$, $T(\epsilon)$ is a point on the secant line through $z(0)$ and $z(\epsilon)$, and in the limit, one obtains a point on the tangent line to $z(0) = a_1 \otimes b_1 \otimes c_1$

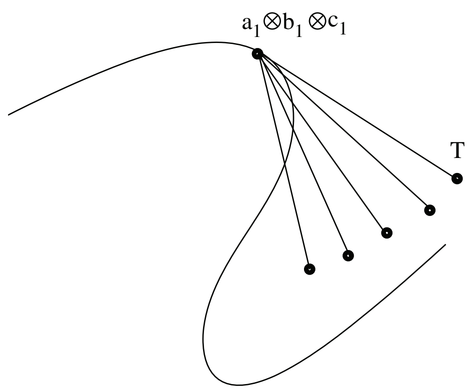


Figure 2.4.1. Unlabeled points are various T_ϵ's lying on secant lines to the curve.

An especially important question is: *What is the border rank of the matrix multiplication operator?* All that is known is that $\underline{\mathbf{R}}(M_{2\times 2}) = 7$, that $14 \leq \underline{\mathbf{R}}(M_{3\times 3}) \leq 21$, and $\frac{3}{2}m^2 + \frac{m}{2} - 1 \leq \underline{\mathbf{R}}(M_{m\times m}) \leq m^{2.38}$; see Chapter 11.

2.5. Examples of invariant tensors

Certain tensors, viewed as multilinear maps, commute with the action of the group of changes of bases, i.e., as tensors, they are *invariant* with respect to the group action. Matrix multiplication is one such, as I explain below.

2.5.1. Contractions of tensors. There is a natural bilinear map

$\mathrm{Con} : (V_1 \otimes \cdots \otimes V_k) \times (V_k^* \otimes U_1 \otimes \cdots \otimes U_m) \to V_1 \otimes \cdots \otimes V_{k-1} \otimes U_1 \otimes \cdots \otimes U_m$

given by $(v_1 \otimes \cdots \otimes v_k, \alpha \otimes b_1 \otimes \cdots \otimes b_m) \mapsto \alpha(v_k) v_1 \otimes \cdots \otimes v_{k-1} \otimes b_1 \otimes \cdots \otimes b_m$, called a *contraction*. One can view the contraction operator Con as an element of

$$(V_1 \otimes \cdots \otimes V_k)^* \otimes (V_k^* \otimes U_1 \otimes \cdots \otimes U_m)^* \otimes (V_1 \otimes \cdots \otimes V_{k-1} \otimes U_1 \otimes \cdots \otimes U_m).$$

If $T \in V_1 \otimes \cdots \otimes V_k$ and $S \in U_1 \otimes \cdots \otimes U_m$, and for some fixed i, j there is an identification $V_i \simeq U_j^*$, one may contract $T \otimes S$ to obtain an element of $V_{\hat{i}} \otimes U_{\hat{j}}$ which is sometimes called the *(i, j)-mode product of T and S*.

Exercise 2.5.1.1: Show that if $f : V \to V$ is a linear map, i.e., $f \in V^* \otimes V$, then the trace of f corresponds to Con above.

In other words (recalling the convention that repeated indices are to be summed over) Exercise 2.5.1.1 says:

Con, Id, *and* trace *are all the same tensors. If (a_i) is a basis of A with dual basis (α^i), then they all correspond to the tensor $\alpha^i \otimes a_i$.*

2.5.2. Matrix multiplication as a tensor. Let A, B, and C be vector spaces of dimensions $\mathbf{a}, \mathbf{b}, \mathbf{c}$, and consider the matrix multiplication operator $M_{\mathbf{a},\mathbf{b},\mathbf{c}}$ that composes a linear map from A to B with a linear map from B to C to obtain a linear map from A to C. Let $V_1 = A^* \otimes B$, $V_2 = B^* \otimes C$, $V_3 = A^* \otimes C$, so $M_{\mathbf{a},\mathbf{b},\mathbf{c}} \in V_1^* \otimes V_2^* \otimes V_3$. On rank one elements:

$$\begin{aligned} M_{\mathbf{a},\mathbf{b},\mathbf{c}} : (A^* \otimes B) \times (B^* \otimes C) &\to A^* \otimes C, \\ (\alpha \otimes b) \times (\beta \otimes c) &\mapsto \beta(b) \alpha \otimes c \end{aligned} \tag{2.5.1}$$

and $M_{\mathbf{a},\mathbf{b},\mathbf{c}}$ is defined on all elements by extending the definition on rank one elements bilinearly. In other words, as a tensor,

$$\begin{aligned} M_{\mathbf{a},\mathbf{b},\mathbf{c}} = \mathrm{Id}_A \otimes \mathrm{Id}_B \otimes \mathrm{Id}_C &\in (A^* \otimes B)^* \otimes (B^* \otimes C)^* \otimes (A^* \otimes C) \\ &= A \otimes B^* \otimes B \otimes C^* \otimes A^* \otimes C, \end{aligned}$$

and it is clear from the expression (2.4.3) that it may be viewed as any of the three possible contractions: as a bilinear map $A^* \otimes B \times B^* \otimes C \to A^* \otimes C$, or as a bilinear map $A \otimes C^* \times B^* \otimes C \to A \otimes B^*$ or $A \otimes C^* \times A^* \otimes B \to C^* \otimes B$. When $A = B = C$, this gives rise to a symmetry under the action of the group $\mathfrak{S}_3$ of permutations on three elements, which is often exploited in the study of the operator.

Exercise 2.5.2.1: Show that $M_{\mathbf{a},\mathbf{b},\mathbf{c}}$, viewed as a trilinear form in bases, takes a triple of matrices (X, Y, Z) to $\mathrm{trace}(XYZ)$, and hence is invariant under changes in bases in A, B and C. The nine-parameter family of algorithms for $M_{2,2,2}$ is the action of $SL(A)\times SL(B)\times SL(C)$ on the expression. (The action of the scalars times the identity will not effect the expression meaningfully as we identify $\lambda v\otimes w = v\otimes(\lambda w)$ for a scalar λ.)

Remark 2.5.2.2. The above exercise gives rise to a nine-parameter family of expressions for $M_{2,2,2}$ as a sum of seven rank one tensors. One could ask if there are any other expressions for $M_{2,2,2}$ as a sum of seven rank one tensors. In [**108**] it is shown that there are no other such expressions.

2.5.3. Another $GL(V)$-invariant tensor.

Recall from above that as tensors Con, tr and Id_V are the same. In Chapter 6 we will see that Id_V and its scalar multiples are the only $GL(V)$-invariant tensors in $V\otimes V^*$. The space $V\otimes V\otimes V^*\otimes V^* = \mathrm{End}(V\otimes V)$, in addition to the identity map $\mathrm{Id}_{V\otimes V}$, has another $GL(V)$-invariant tensor. As a linear map it is simply

$$\begin{aligned} \sigma : V\otimes V &\to V\otimes V, \\ a\otimes b &\mapsto b\otimes a. \end{aligned} \tag{2.5.2}$$

2.6. Symmetric and skew-symmetric tensors

2.6.1. The spaces S^2V and Λ^2V.

Recall the map (2.5.2), $\sigma : V^*\otimes V^* \to V^*\otimes V^*$. (Note that here we look at it on the dual space.) Consider $V^{\otimes 2} = V\otimes V$ with basis $\{v_i\otimes v_j,\ 1\le i,j\le n\}$. The subspaces defined by

$$\begin{aligned} S^2V &:= \mathrm{span}\{v_i\otimes v_j + v_j\otimes v_i,\ 1\le i,j\le n\} \\ &= \mathrm{span}\{v\otimes v \mid v\in V\} \\ &= \{X\in V\otimes V \mid X(\alpha,\beta) = X(\beta,\alpha)\ \forall \alpha,\beta\in V^*\} \\ &= \{X\in V\otimes V \mid X\circ\sigma = X\}, \\ \Lambda^2V &:= \mathrm{span}\{v_i\otimes v_j - v_j\otimes v_i,\ 1\le i,j\le n\} \\ &= \mathrm{span}\{v\otimes w - w\otimes v \mid v,w\in V\}, \\ &= \{X\in V\otimes V \mid X(\alpha,\beta) = -X(\beta,\alpha)\ \forall \alpha,\beta\in V^*\} \\ &= \{X\in V\otimes V \mid X\circ\sigma = -X\} \end{aligned}$$

are respectively the spaces of *symmetric* and *skew-symmetric* 2-tensors of V. In the fourth lines we are considering X as a map $V^*\otimes V^*\to\mathbb{C}$. The second description of these spaces implies that if $T\in S^2V$ and $g\in GL(V)$, then (using (2.4.2)) $g\cdot T\in S^2V$ and similarly for Λ^2V. That is, they are invariant under linear changes of coordinates, i.e., they are $GL(V)$-submodules of $V^{\otimes 2}$.

For $v_1, v_2 \in V$, define $v_1v_2 := \frac{1}{2}(v_1 \otimes v_2 + v_2 \otimes v_1) \in S^2V$ and $v_1 \wedge v_2 := \frac{1}{2}(v_1 \otimes v_2 - v_2 \otimes v_1) \in \Lambda^2 V$.

2.6.2. Exercises.

(1) Show that the four descriptions of S^2V all agree. Do the same for the four descriptions of $\Lambda^2 V$.

(2) Show that

$$V \otimes V = S^2V \oplus \Lambda^2 V. \tag{2.6.1}$$

By the remarks above, this direct sum decomposition is invariant under the action of $GL(V)$, cf. Exercise 2.1.12. One says that $V^{\otimes 2}$ *decomposes* as a $GL(V)$-module to $\Lambda^2 V \oplus S^2 V$.

(3) Show that the action of GL_2 on $\mathbb{C}^3$ of Example 2.2.1.2(5) is the action induced on $S^2\mathbb{C}^2$ from the action on $\mathbb{C}^2 \otimes \mathbb{C}^2$.

(4) Show that no proper linear subspace of S^2V is invariant under the action of $GL(V)$; i.e., S^2V is an irreducible submodule of $V^{\otimes 2}$.

(5) Show that $\Lambda^2 V$ is an irreducible $GL(V)$-submodule of $V^{\otimes 2}$.

(6) Define maps

$$\begin{aligned} \pi_S : V^{\otimes 2} &\to V^{\otimes 2}, \\ X &\mapsto \frac{1}{2}(X + X \circ \sigma), \end{aligned} \tag{2.6.2}$$

$$\begin{aligned} \pi_\Lambda : V^{\otimes 2} &\to V^{\otimes 2}, \\ X &\mapsto \frac{1}{2}(X - X \circ \sigma). \end{aligned} \tag{2.6.3}$$

Show that $\pi_S(V^{\otimes 2}) = S^2V$ and $\pi_\Lambda(V^{\otimes 2}) = \Lambda^2 V$.

(7) What is $\ker \pi_S$?

Notational warning. Above I used $\circ$ as composition. It is also used in the literature to denote symmetric product as defined below. To avoid confusion I reserve $\circ$ for composition of maps with the exception of taking the symmetric product of spaces, e.g., $S^dV \circ S^\delta V = S^{d+\delta}V$.

2.6.3. Symmetric tensors S^dV. Let $\pi_S : V^{\otimes d} \to V^{\otimes d}$ be the map defined on rank one elements by

$$\pi_S(v_1 \otimes \cdots \otimes v_d) = \frac{1}{d!} \sum_{\sigma \in \mathfrak{S}_d} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(d)},$$

where $\mathfrak{S}_d$ denotes the group of permutations of d elements.

Exercises 2.6.3.1:

1. Show that π_S agrees with (2.6.2) when $d = 2$.
2. Show that $\pi_S(\pi_S(X)) = \pi_S(X)$, i.e., that π_S is a projection operator (cf. Exercise 2.3.2.9).

Introduce the notation $v_1 v_2 \cdots v_d := \pi_S(v_1 \otimes v_2 \otimes \cdots \otimes v_d)$.

Definition 2.6.3.2. Define

$$S^d V := \pi_S(V^{\otimes d}),$$

the *d-th symmetric power of* V.

Note that

$$\begin{aligned} S^d V &= \{X \in V^{\otimes d} \mid \pi_S(X) = X\} \\ &= \{X \in V^{\otimes d} \mid X \circ \sigma = X \ \forall \sigma \in \mathfrak{S}_d\}. \end{aligned} \tag{2.6.4}$$

Exercise 2.6.3.3: Show that in bases, if $u \in S^p \mathbb{C}^r$, $v \in S^q \mathbb{C}^r$, the symmetric tensor product $uv \in S^{p+q}\mathbb{C}^r$ is

$$(uv)^{i_1,\dots,i_{p+q}} = \frac{1}{(p+q)!} \sum u^I v^J,$$

where the summation is over $I = i_1, \dots, i_p$ with $i_1 \leq \cdots \leq i_p$ and analogously for J.

Exercise 2.6.3.4: Show that $S^d V \subset V^{\otimes d}$ is invariant under the action of $GL(V)$.

Exercise 2.6.3.5: Show that if $e_1, \dots, e_{\mathbf{v}}$ is a basis of V, then $e_{j_1} e_{j_2} \cdots e_{j_d}$, for $1 \leq j_1 \leq \cdots \leq j_d \leq \mathbf{v}$ is a basis of $S^d V$. Conclude that $\dim S^d \mathbb{C}^{\mathbf{v}} = \binom{\mathbf{v}+d-1}{d}$.

2.6.4. $S^k V^*$ as the space of homogeneous polynomials of degree k on V. The space $S^k V^*$ was defined as the space of symmetric k-linear forms on V. It may also be considered as the space of homogeneous polynomials of degree k on V. Namely, given a multilinear form $\overline{Q}$, the map $x \mapsto \overline{Q}(x, \dots, x)$ is a polynomial mapping of degree k. The process of passing from a homogeneous polynomial to a multilinear form is called *polarization*. For example, if Q is a homogeneous polynomial of degree two on V, define the bilinear form $\overline{Q}$ by the equation

$$\overline{Q}(x, y) = \frac{1}{2}[Q(x + y) - Q(x) - Q(y)].$$

For general symmetric multilinear forms, the polarization identity is

$$\overline{Q}(x_1, \dots, x_k) = \frac{1}{k!} \sum_{I \subset [k], I \neq \emptyset} (-1)^{k-|I|} Q\left(\sum_{i \in I} x_i\right). \tag{2.6.5}$$

Here $[k]=\{1,\dots,k\}$. Since Q and $\overline{Q}$ are really the same object, I generally will not distinguish them by different notation.

Example 2.6.4.1. For a cubic polynomial in two variables $P(s,t)$, one obtains the cubic form

$$\begin{aligned}&\overline{P}((s_1,t_1),(s_2,t_2),(s_3,t_3))\\&\quad=\frac{1}{6}[P(s_1+s_2+s_3,t_1+t_2+t_3)-P(s_1+s_2,t_1+t_2)-P(s_1+s_3,t_1+t_3)\\&\qquad-P(s_2+s_3,t_2+t_3)+P(s_1,t_1)+P(s_2,t_2)+P(s_3,t_3)]\end{aligned}$$

so for, e.g., $P=s^2t$ one obtains $\overline{P}=\frac{1}{3}(s_1s_2t_3+s_1s_3t_2+s_2s_3t_1)$.

From this perspective, the contraction map is

$$\begin{aligned}V^*\times S^dV&\to S^{d-1}V,\\(\alpha,P)&\mapsto P(\alpha,\cdot);\end{aligned}\tag{2.6.6}$$

if one fixes α, this is just the partial derivative of P in the direction of α.

Exercise 2.6.4.2: Prove the above assertion by choosing coordinates and taking $\alpha=x_1$.

2.6.5. Polynomials and homogeneous polynomials. In this book I generally restrict the study of polynomials to homogeneous polynomials—this is no loss of generality, as there is a bijective map

$$S^d\mathbb{C}^m\to\{\text{polynomials of degree at most } d \text{ in } m-1 \text{ variables}\}$$

by setting $x_m=1$; i.e., let $I=(i_i,\dots,i_m)$ be a multi-index and write $|I|=i_1+\cdots+i_m$,

$$\sum_{|I|=d}a_{i_1,\dots,i_m}x_1^{i_1}\cdots x_{m-1}^{i_{m-1}}x_m^{i_m}\mapsto\sum_{|I|=d}a_{i_1,\dots,i_m}x_1^{i_1}\cdots x_{m-1}^{i_{m-1}}.$$

2.6.6. Symmetric tensor rank.

Definition 2.6.6.1. Given $\phi\in S^dV$, define the *symmetric tensor rank* of ϕ, $\mathbf{R}_S(\phi)$, to be the smallest r such that $\phi=v_1^d+\cdots+v_r^d$ for some $v_j\in V$. Define the *symmetric tensor border rank* of ϕ, $\underline{\mathbf{R}}_S(\phi)$, to be the smallest r such that ϕ is a limit of symmetric tensors of symmetric tensor rank r.

Exercise 2.6.6.2: Show that for any $\phi\in S^d\mathbb{C}^n$, $\mathbf{R}_S(\phi)\le\binom{n+d-1}{d}$. ⊚

There is a natural inclusion $S^dV\subset S^sV\otimes S^{d-s}V$ given by partial polarization. Write $\phi_{s,d-s}\in S^sV\otimes S^{d-s}V$ for the image of $\phi\in S^dV$. Thinking of $S^sV\otimes S^{d-s}V$ as a space of linear maps $S^sV^*\to S^{d-s}V$, $\phi_{s,d-s}(\alpha_1\cdots\alpha_s)=\overline{\phi}(\alpha_1,\dots,\alpha_s,\cdot,\dots,\cdot)$.

Exercise 2.6.6.3: Show that if $\underline{\mathbf{R}}_S(\phi)\le k$, then $\operatorname{rank}(\phi_{s,d-s})\le k$ for all s.

Remark 2.6.6.4. Exercise 2.6.6.3 provides a test for symmetric tensor border rank that dates back to Macaulay [**224**].

Exercise 2.6.6.5: Considering $S^dV \subset V^{\otimes d}$, show that, for $\phi \in S^dV \subset V^{\otimes d}$, $\mathbf{R}_S(\phi) \geq \mathbf{R}(\phi)$ and $\underline{\mathbf{R}}_S(\phi) \geq \underline{\mathbf{R}}(\phi)$. Pierre Comon has conjectured that equality holds, see [**102**, §4.1] and §5.7.2.

More generally one can define the *partially symmetric rank* of partially symmetric tensors. We will not dwell much on this since this notion will be superceded by the notion of X-rank in Chapter 5. The term INDSCAL is used for the partially symmetric rank of elements of $S^2W \otimes V$.

2.6.7. Alternating tensors.

Define a map

$$\pi_\Lambda : V^{\otimes k} \to V^{\otimes k}, \tag{2.6.7}$$

$$v_1 \otimes \cdots \otimes v_k \mapsto v_1 \wedge \cdots \wedge v_k := \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} (\operatorname{sgn}(\sigma)) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)}, \tag{2.6.8}$$

where $\operatorname{sgn}(\sigma) = \pm 1$ denotes the sign of the permutation σ (see Remark 2.9.2.3). Denote the image by $\Lambda^k V$, called the space of *alternating k-tensors*, and note that it agrees with our previous definition of $\Lambda^2 V$ when $k = 2$.

In particular

$$\Lambda^k V = \{X \in V^{\otimes k} \mid X \circ \sigma = \operatorname{sgn}(\sigma) X \ \forall \sigma \in \mathfrak{S}_k\}.$$

Exercise 2.6.7.1: Show that $v_1 \wedge \cdots \wedge v_k = 0$ if and only if $v_1, \dots, v_k$ are linearly dependent.⊚

2.6.8. Tensor, symmetric and exterior algebras.

Definition 2.6.8.1. For a vector space V, define $V^{\otimes} := \bigoplus_{k \geq 0} V^{\otimes k}$, the *tensor algebra of V*. The multiplication is given by defining the product of $v_1 \otimes \cdots \otimes v_s$ with $w_1 \otimes \cdots \otimes w_t$ to be $v_1 \otimes \cdots \otimes v_s \otimes w_1 \otimes \cdots \otimes w_t$ and extending linearly.

Definition 2.6.8.2. Define the *exterior algebra* $\Lambda^\bullet V = \bigoplus_k \Lambda^k V$ and the *symmetric algebra* $S^\bullet V := \bigoplus_d S^d V$ by defining the multiplications $\alpha \wedge \beta := \pi_\Lambda(\alpha \otimes \beta)$ for $\alpha \in \Lambda^s V$, $\beta \in \Lambda^t V$ and $\alpha\beta := \pi_S(\alpha \otimes \beta)$ for $\alpha \in S^s V$, $\beta \in S^t V$.

Note the following: (i) $\Lambda^1 V = S^1 V = V$, (ii) the multiplication $S^s V \times S^t V \to S^{s+t} V$, when considering $S^k V$ as the space of homogeneous polynomials on V^*, corresponds to the multiplication of polynomials, and (iii) these are both associative algebras with units respectively $1 \in S^0 V$, $1 \in \Lambda^0 V$.

2.6.9. Contractions preserve symmetric and skew-symmetric tensors. Recall (§2.5.1) the contraction

$$
\begin{aligned}
(2.6.9) \qquad & V^* \times V^{\otimes k} \to V^{\otimes k-1}, \\
& (\alpha, v_1 \otimes \cdots \otimes v_k) \mapsto \alpha(v_1) v_2 \otimes \cdots \otimes v_k.
\end{aligned}
$$

Here we could have just as well defined contractions on any of the factors. This contraction preserves the subspaces of symmetric and skew-symmetric tensors, as you verify in Exercise 2.6.10(3).

Remark 2.6.9.1. The *first fundamental theorem of invariant theory* (see, e.g., [**268**, p. 388]) states that the only $GL(V)$-invariant operators are of the form (2.6.9), and the only $SL(V)$-invariant operators are these and contractions with the volume form. (Here $SL(V)$ is the group of invertible endomorphisms of determinant one, see Exercises 2.6.12 and 2.6.13.)

For a pairing $V^* \times V \otimes W \to W$, I sometimes let $\alpha \lrcorner T$ denote the contraction of $\alpha \in V^*$ and $T \in V \otimes W$.

2.6.10. Exercises.

(1) Show that the subspace $\Lambda^k V \subset V^{\otimes k}$ is invariant under the action of $GL(V)$.

(2) Show that a basis of V induces a basis of $\Lambda^k V$. Using this induced basis, show that, if $\dim V = \mathbf{v}$, then $\dim \Lambda^k V = \binom{\mathbf{v}}{k}$. In particular, $\Lambda^{\mathbf{v}} V \simeq \mathbb{C}$, $\Lambda^l V = 0$ for $l > \mathbf{v}$, and $S^3 V \oplus \Lambda^3 V \neq V^{\otimes 3}$ when $\mathbf{v} > 1$.

(3) Calculate, for $\alpha \in V^*$, $\alpha \lrcorner (v_1 \cdots v_k)$ explicitly and show that it indeed is an element of $S^{k-1} V$, and similarly for $\alpha \lrcorner (v_1 \wedge \cdots \wedge v_k)$.

(4) Show that the composition $(\alpha \lrcorner) \circ (\alpha \lrcorner) : \Lambda^k V \to \Lambda^{k-2} V$ is the zero map.

(5) Show that if $V = A \oplus B$, then there is an induced direct sum decomposition $\Lambda^k V = \Lambda^k A \oplus (\Lambda^{k-1} A \otimes \Lambda^1 B) \oplus (\Lambda^{k-2} A \otimes \Lambda^2 B) \oplus \cdots \oplus \Lambda^k B$ as a $GL(A) \times GL(B)$-module.

(6) Show that a subspace $A \subset V$ determines a well-defined induced filtration of $\Lambda^k V$ given by $\Lambda^k A \subset \Lambda^{k-1} A \wedge \Lambda^1 V \subset \Lambda^{k-2} A \wedge \Lambda^2 V \subset \cdots \subset \Lambda^k V$. If $P_A := \{g \in GL(V) \mid g \cdot v \in A \ \forall v \in A\}$, then each filtrand is a P_A-submodule.

(7) Show that if V is equipped with a *volume form*, i.e., a nonzero element $\phi \in \Lambda^{\mathbf{v}} V$, then one obtains an identification $\Lambda^k V \simeq \Lambda^{\mathbf{v}-k} V^*$. ⊚

(8) Show that $V^* \simeq \Lambda^{\mathbf{v}-1} V \otimes \Lambda^{\mathbf{v}} V^*$ as $GL(V)$-modules. ⊚

(9) Show that the tensor, symmetric, and exterior algebras are associative.

2.6.11. Induced linear maps. Tensor product and the symmetric and skew-symmetric constructions are *functorial.* This essentially means: given a linear map $f : V \to W$ there are induced linear maps $f^{\otimes k} : V^{\otimes k} \to W^{\otimes k}$ given by $f^{\otimes k}(v_1 \otimes \cdots \otimes v_k) = f(v_1) \otimes \cdots \otimes f(v_k)$. These restrict to give well-defined maps $f^{\wedge k} : \Lambda^k V \to \Lambda^k W$ and $f^{\circ k} : S^k V \to S^k W$.

Definition 2.6.11.1. Given a linear map $f : V \to V$, the induced map $f^{\wedge \mathbf{v}} : \Lambda^{\mathbf{v}} V \to \Lambda^{\mathbf{v}} V$ is called the *determinant* of f.

Example 2.6.11.2. Let $\mathbb{C}^2$ have basis e_1, e_2. Say $f : \mathbb{C}^2 \to \mathbb{C}^2$ is represented by the matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with respect to this basis, i.e., $f(e_1) = ae_1 + be_2$, $f(e_2) = ce_1 + de_2$. Then

$$\begin{aligned} f(e_1 \wedge e_2) &= (ae_1 + be_2) \wedge (ce_1 + de_2) \\ &= (ad - bc) e_1 \wedge e_2. \end{aligned}$$

Sometimes one fixes a given element of $\Lambda^{\mathbf{v}} V^*$ and calls it the determinant det. Which meaning of the word is used should be clear from the context.

To see these induced maps explicitly, let $\alpha^1, \ldots, \alpha^{\mathbf{v}}$ be a basis of V^* and $w_1, \ldots, w_{\mathbf{w}}$ a basis of W. Write $f = f^s_j \alpha^j \otimes w_s$ in this basis. Then

$$f^{\otimes 2} = f^s_j f^t_k \alpha^j \otimes \alpha^k \otimes w_s \otimes w_t, \tag{2.6.10}$$

$$f^{\wedge 2} = (f^s_j f^t_k - f^s_k f^t_j)(\alpha^j \wedge \alpha^k) \otimes (w_s \wedge w_t), \tag{2.6.11}$$

$$\begin{aligned} f^{\wedge p} = \sum_{\sigma \in \mathfrak{S}_p} & \operatorname{sgn}(\sigma) f^{s_1}_{i_{\sigma(1)}} \cdots f^{s_p}_{i_{\sigma(p)}} \alpha^{i_{\sigma(1)}} \\ & \wedge \cdots \wedge \alpha^{i_{\sigma(p)}} \otimes w_{s_1} \wedge \cdots \wedge w_{s_p}. \end{aligned} \tag{2.6.12}$$

The $GL(V)$-module isomorphism from Exercise 2.6.10(8) shows that a linear map $\phi : \Lambda^{\mathbf{v}-1} V \to \Lambda^{\mathbf{v}-1} W$, where $\dim V = \dim W = \mathbf{v}$, induces a linear map $V^* \otimes \Lambda^{\mathbf{v}} V \to W^* \otimes \Lambda^{\mathbf{v}} W$, i.e., a linear map $W \otimes \Lambda^{\mathbf{v}} W^* \to V \otimes \Lambda^{\mathbf{v}} V^*$. If $\phi = f^{\wedge(\mathbf{v}-1)}$ for some linear map $f : V \to W$, and f is invertible, then the induced linear map is $f^{-1} \otimes \det(f)$. If f is not invertible, then $f^{\wedge(\mathbf{v}-1)}$ has rank one. If $\operatorname{rank}(f) \leq \mathbf{v} - 2$, then $f^{\wedge(\mathbf{v}-1)}$ is zero. An advantage of $f^{\wedge \mathbf{v}-1}$ over f^{-1} is that it is defined even if f is not invertible, which is exploited in §7.6.

2.6.12. Exercises on induced linear maps and the determinant.

(1) Verify that if f has rank $\mathbf{v} - 1$, then $f^{\wedge(\mathbf{v}-1)}$ has rank one, and if $\operatorname{rank}(f) \leq \mathbf{v} - 2$, then $f^{\wedge(\mathbf{v}-1)}$ is zero. ⊚

(2) Show more generally that if f has rank r, then $\operatorname{rank}(f^{\wedge s}) = \binom{r}{s}$.

(3) Show that the eigenvalues of $f^{\wedge k}$ are the k-th elementary symmetric functions (see §6.11) of the eigenvalues of f.

(4) Given $f : V \to V$, $f^{\wedge \mathbf{v}}$ is a map from a one-dimensional vector space to itself, and thus multiplication by some scalar. Show that if one chooses a basis for V and represents f by a matrix, the scalar representing $f^{\wedge \mathbf{v}}$ is the determinant of the matrix representing f.

(5) Given $f : V \to V$, assume that V admits a basis of eigenvectors for f. Show that $\Lambda^k V$ admits a basis of eigenvectors for $f^{\wedge k}$ and find the eigenvectors and eigenvalues for $f^{\wedge k}$ in terms of those for f. In particular, show that the coefficient of $t^{\mathbf{v}-k}$ in $\det(f - t\,\mathrm{Id})$, the characteristic polynomial of f, is $(-1)^k \operatorname{tr}(f^{\wedge k})$, where tr is defined in Exercise 2.3.2(7).

(6) Let $f : V \to W$ be invertible, with $\dim V = \dim W = \mathbf{v}$. Verify that $f^{\wedge \mathbf{v}-1} = f^{-1} \otimes \det(f)$ as asserted above. ⊚

(7) Fix $\det \in \Lambda^{\mathbf{v}} V^*$. Let

$$SL(V) := \{g \in GL(V) \mid g \cdot \det = \det\}. \tag{2.6.13}$$

Show that $SL(V)$ is a group, called the *Special Linear group*. Show that if one fixes a basis $\alpha^1, \dots, \alpha^{\mathbf{v}}$ of V^* such that $\det = \alpha^1 \wedge \cdots \wedge \alpha^{\mathbf{v}}$, and uses this basis and its dual to express linear maps $V \to V$ as $\mathbf{v} \times \mathbf{v}$ matrices, that $SL(V)$ becomes the set of matrices with determinant one (where one takes the usual determinant of matrices).

(8) Given n-dimensional vector spaces E, F, fix an element $\Omega \in \Lambda^n E^* \otimes \Lambda^n F$. Since $\dim(\Lambda^n E^* \otimes \Lambda^n F) = 1$, Ω is unique up to scale. Then given a linear map $f : V \to W$, one may write $f^{\wedge n} = c_f \Omega$, for some constant c_f. Show that if one chooses bases $e_1, \dots, e_n$ of E, $f_1, \dots, f_n$ of F such that $\Omega = e_1 \wedge \cdots \wedge e_n \otimes f_1 \wedge \cdots \wedge f_n$, and expresses f as a matrix M_f with respect to these bases, then $c_f = \det(M_f)$.

(9) Note that Ω determines a vector $\Omega^* \in \Lambda^n E \otimes \Lambda^n F^*$ by $\langle \Omega^*, \Omega \rangle = 1$. Recall that $f : V \to W$ determines a linear map $f^T : W^* \to V^*$. Use Ω^* to define $\det_{f^T}$. Show $\det_f = \det_{f^T}$.

2.6.13. The group $\mathfrak{S}_d$ acts on $V^{\otimes d}$. Let $\mathfrak{S}_d$ denote the symmetric group on d elements (see Definition 2.9.2.2). $\mathfrak{S}_d$ acts on $V^{\otimes d}$ by, for $v_1, \dots, v_d \in V$,

$$\sigma(v_1 \otimes \cdots \otimes v_d) = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(d)}$$

and extending linearly. Looking at symmetric tensors from this perspective yields:

$$S^d V = \{T \in V^{\otimes d} \mid \sigma \cdot T = T \ \forall \sigma \in \mathfrak{S}_d\}.$$

This description is slightly different from (2.6.4) as before elements of $\mathfrak{S}_d$ acted on $V^{*\otimes d}$ and it was not explicitly mentioned that the elements were part of an $\mathfrak{S}_d$ action. In words:

S^dV is the subspace of $V^{\otimes d}$ whose elements are invariant under the action of $\mathfrak{S}_d$.

Exercise 2.6.13.1: Show that if $g \in GL(V)$ and $\sigma \in \mathfrak{S}_d$, then, for all $T \in V^{\otimes d}$, $g \cdot \sigma \cdot T = \sigma \cdot g \cdot T$.

Because it will be important in Chapter 6, I record the result of Exercise 2.6.13.1:

The actions of $GL(V)$ and $\mathfrak{S}_d$ on $V^{\otimes d}$ commute with each other.

2.7. Polynomials on the space of matrices

Consider homogeneous polynomials on the space of $\mathbf{a} \times \mathbf{b}$ matrices. We will be interested in how the polynomials change under changes of bases in $\mathbb{C}^{\mathbf{a}}$ and $\mathbb{C}^{\mathbf{b}}$.

More invariantly, we will study our polynomials as $GL(A) \times GL(B)$-modules. Let $V = A{\otimes}B$. We study the degree d homogeneous polynomials on $A^*{\otimes}B^*$, S^dV as a $G := GL(A) \times GL(B)$-module.

Warning to the reader: I am identifying our vector space of matrices $\mathbb{C}^{\mathbf{a}}{\otimes}\mathbb{C}^{\mathbf{b}}$ with $A^*{\otimes}B^*$, in order to minimize the use of $*$'s below.

Remark. In the examples that follow, the reader may wonder why I am belaboring such familiar polynomials. The reason is that later on we will encounter natural polynomials that are unfamiliar and will need ways of writing them down explicitly from an invariant description.

2.7.1. Quadratic polynomials. While S^2V is irreducible as a $GL(V)$-module, if $V = A{\otimes}B$, one expects to be able to decompose it further as a $G = GL(A) \times GL(B)$-module. One way to get an element of $S^2(A{\otimes}B)$ is simply to multiply an element of S^2A with an element of S^2B, i.e., given $\alpha \in S^2A$, $\beta \in S^2B$, $\alpha{\otimes}\beta$ is defined by $\alpha{\otimes}\beta(x_1{\otimes}y_1, x_2{\otimes}y_2) = \alpha(x_1, x_2)\beta(y_1, y_2)$, where $x_j \in A^*$, $y_j \in B^*$. Clearly $\alpha{\otimes}\beta(x_1{\otimes}y_1, x_2{\otimes}y_2) = \alpha{\otimes}\beta(x_2{\otimes}y_2, x_1{\otimes}y_1)$, so $\alpha{\otimes}\beta$ is indeed an element of $S^2(A{\otimes}B)$.

The space $S^2A{\otimes}S^2B$ is a G-invariant subspace of $S^2(A{\otimes}B)$, i.e., if $T \in S^2A{\otimes}S^2B$, then $g \cdot T \in S^2A{\otimes}S^2B$ for all $g \in G$. One may think of the embedding $S^2A{\otimes}S^2B \to S^2(A{\otimes}B)$ as the result of the composition of the inclusion

$$S^2A{\otimes}S^2B \to A{\otimes}A{\otimes}B{\otimes}B = (A{\otimes}B)^{\otimes 2}$$

with the projection π_S (see §2.6.3),

$$(A{\otimes}B)^{\otimes 2} \to S^2(A{\otimes}B).$$

Since S^2A is an irreducible $GL(A)$-module and S^2B is an irreducible $GL(B)$-module, $S^2A{\otimes}S^2B$ is an irreducible G-module.

On the other hand, $\dim S^2V = \binom{ab+1}{2} = (ab+1)ab/2$ and $\dim(S^2A\otimes S^2B) = \binom{a+1}{2}\binom{b+1}{2} = (ab+a+b+1)ab/4$. So we have not found all possible elements of S^2V.

To find the missing polynomials, consider $\alpha \in \Lambda^2 A$, $\beta \in \Lambda^2 B$, and define $\alpha\otimes\beta(x_1\otimes y_1, x_2\otimes y_2) = \alpha(x_1,x_2)\beta(y_1,y_2)$. Observe that

$$\begin{aligned}\alpha\otimes\beta(x_2\otimes y_2, x_1\otimes y_1) &= \alpha(x_2,x_1)\beta(y_2,y_1)\\ &= (-\alpha(x_1,x_2))(-\beta(y_1,y_2))\\ &= \alpha\otimes\beta(x_1\otimes y_1, x_2\otimes y_2).\end{aligned}$$

Thus $\alpha\otimes\beta \in S^2(A\otimes B)$, and extending the map linearly yields an inclusion $\Lambda^2A\otimes\Lambda^2B \subset S^2(A\otimes B)$.

Now $\dim(\Lambda^2A\otimes\Lambda^2B) = (ab-a-b+1)ab/4$ so $\dim(\Lambda^2A\otimes\Lambda^2B) + \dim(S^2A\otimes S^2B) = S^2(A\otimes B)$, and thus

$$S^2(A\otimes B) = (\Lambda^2A\otimes\Lambda^2B) \oplus (S^2A\otimes S^2B)$$

is a decomposition of $S^2(A\otimes B)$ into $GL(A)\times GL(B)$-submodules, in fact into $GL(A)\times GL(B)$-irreducible submodules.

Exercise 2.7.1.1: Verify that the above decomposition is really a direct sum, i.e., that $(\Lambda^2A\otimes\Lambda^2B)$ and $(S^2A\otimes S^2B)$ are disjoint.

Exercise 2.7.1.2: Decompose $\Lambda^2(A\otimes B)$ as a $GL(A)\times GL(B)$-module.

2.7.2. Two by two minors. I now describe the inclusion $\Lambda^2A\otimes\Lambda^2B \to S^2(A\otimes B)$ in bases. Let (a_i), (b_s) respectively be bases of A, B, and let $(a_i\otimes b_s)$ denote the induced basis of $A\otimes B$ and $(\alpha^i\otimes\beta^s)$ the induced dual basis for $A^*\otimes B^*$. Identify $\alpha^i\otimes\beta^s$ with the matrix having a one in the (i,s)-entry and zeros elsewhere. Consider the following quadratic polynomial on $A^*\otimes B^*$, viewed as the space of $\mathbf{a}\times\mathbf{b}$ matrices with coordinates $x^{i,s}$; i.e., $X = \sum_{i,s} x^{i,s} a_i\otimes b_s$ corresponds to the matrix whose (i,s)-th entry is $x^{i,s}$:

$$P_{jk|tu}(X) := x^{j,t}x^{k,u} - x^{k,t}x^{j,u},$$

which is the two by two minor $(jk|tu)$. As a tensor,

$$\begin{aligned}P_{jk|tu} &= (a_j\otimes b_t)(a_k\otimes b_u) - (a_k\otimes b_t)(a_j\otimes b_u)\\ &= \frac{1}{2}[a_j\otimes b_t\otimes a_k\otimes b_u + a_k\otimes b_u\otimes a_j\otimes b_t - a_k\otimes b_t\otimes a_j\otimes b_u - a_j\otimes b_u\otimes a_k\otimes b_t].\end{aligned}$$

Exercise 2.7.2.1: Show that $A\otimes B$ is canonically isomorphic to $B\otimes A$ as a $GL(A)\times GL(B)$-module. More generally, for all $\sigma\in\mathfrak{S}_k$, $V_{\sigma(1)}\otimes\cdots\otimes V_{\sigma(k)}$ is isomorphic to $V_1\otimes\cdots\otimes V_k$ as a $GL(V_1)\times\cdots\times GL(V_k)$-module.

Now use the canonical isomorphism of Exercise 2.7.2.1, $A{\otimes}B{\otimes}A{\otimes}B \simeq A{\otimes}A{\otimes}B{\otimes}B$:

$$\begin{aligned}
P_{jk|tu} &= \frac{1}{2}[a_j{\otimes}a_k{\otimes}b_t{\otimes}b_u + a_k{\otimes}a_j{\otimes}b_u{\otimes}b_t - a_k{\otimes}a_j{\otimes}b_t{\otimes}b_u - a_j{\otimes}a_k{\otimes}b_u{\otimes}b_t] \\
&= \frac{1}{2}[(a_j{\otimes}a_k - a_k{\otimes}a_j){\otimes}b_t{\otimes}b_u + (a_k{\otimes}a_j - a_j{\otimes}a_k){\otimes}b_u{\otimes}b_t] \\
&= (a_j \wedge a_k){\otimes}b_t{\otimes}b_u + (a_k \wedge a_j){\otimes}b_u{\otimes}b_t \\
&= 2(a_j \wedge a_k){\otimes}(b_t \wedge b_u).
\end{aligned}$$

Thus

A two by two minor of a matrix, expressed as a tensor in $S^2(A{\otimes}B)$, corresponds to an element of the subspace $\Lambda^2A{\otimes}\Lambda^2B \subset S^2(A{\otimes}B)$.

Compare this remark with (2.6.11).

Another perspective on the space of two by two minors is as follows: a linear map $x : A \to B^*$ has rank (at most) one if and only if the induced linear map

$$x^{\wedge 2} : \Lambda^2A \to \Lambda^2B^*$$

is zero. Now $x^{\wedge 2} \in \Lambda^2A^*{\otimes}\Lambda^2B^*$, and for any vector space U and its dual U^* there is a perfect pairing $U \times U^* \to \mathbb{C}$. Thus a way to test if x has rank one is to check if $x^{\wedge 2}$ pairs with each element of $\Lambda^2A{\otimes}\Lambda^2B$ to be zero.

In contrast, consider the induced map $x^2 : S^2A \to S^2B^*$. This map is never identically zero if x is nonzero, so the set $\{x \in A^*{\otimes}B^* \mid \phi(x) = 0 \ \forall\phi \in S^2A{\otimes}S^2B\}$ is just the zero vector $0 \in A^*{\otimes}B^*$.

2.7.3. Exercises on equations for the set of rank at most $k-1$ matrices. Let $\alpha^1, \dots, \alpha^k \in A^*$, $\beta^1, \dots, \beta^k \in B^*$ and consider $P := (\alpha^1 \wedge \cdots \wedge \alpha^k){\otimes}(\beta^1 \wedge \cdots \wedge \beta^k) \in \Lambda^kA^*{\otimes}\Lambda^kB^*$. By Exercise 2.7.3(2) one may consider $P \in S^k(A{\otimes}B)^*$, i.e., as a homogeneous polynomial of degree k on $A{\otimes}B$.

(1) Show that $\Lambda^2A{\otimes}\Lambda^2B \subset S^2(A{\otimes}B)$ is exactly the span of the collection of two by two minors (with respect to any choice of bases) considered as quadratic polynomials on $A^*{\otimes}B^*$.

(2) Show that $\Lambda^kA{\otimes}\Lambda^kB \subset S^k(A{\otimes}B)$, and that it corresponds to the span of the collection of $k \times k$ minors.

(3) Show that if $T \in A{\otimes}B$ is of the form $T = a_1{\otimes}b_1 + \cdots + a_k{\otimes}b_k$, where $(a_1, \dots, a_k)$ and $(b_1, \dots, b_k)$ are linearly independent sets of vectors, then there exists $P \in \Lambda^kA^*{\otimes}\Lambda^kB^*$ such that $P(T) \neq 0$. Conclude that the set of rank at most $k-1$ matrices is the common zero locus of the polynomials in $\Lambda^kA^*{\otimes}\Lambda^kB^*$.

(4) Given $T \in A \otimes B$, consider $T : A^* \to B$ and the induced linear map $T^{\wedge k} : \Lambda^k A^* \to \Lambda^k B$, i.e. $T^{\wedge k} \in \Lambda^k A \otimes \Lambda^k B$. Show that the rank of T is less than k if and only if $T^{\wedge k}$ is zero.

The perspectives of the last two exercises are related by noting the perfect pairing

$$(\Lambda^k A \otimes \Lambda^k B) \times (\Lambda^k A^* \otimes \Lambda^k B^*) \to \mathbb{C}.$$

2.7.4. The Pfaffian. Let E be a vector space of dimension $n = 2m$ and let $\Omega_E \in \Lambda^n E$ be a volume form. Let $x \in \Lambda^2 E$ and consider $x^{\wedge m} \in \Lambda^n E$. Since $\dim(\Lambda^n E) = 1$, there exists $c_x \in \mathbb{C}$ such that $x^{\wedge m} = c_x \Omega_E$. Define $\mathrm{Pf} \in S^m(\Lambda^2 E)$ by $\mathrm{Pf}(x) = \frac{1}{m!} c_x$. By its definition, Pf depends only on a choice of volume form and thus is invariant under $SL(E) = SL(E, \Omega_E)$ (which, by definition, is the group preserving Ω_E).

The Pfaffian is often used in connection with the orthogonal group because endomorphisms $E \to E$ that are "skew-symmetric" arise often in practice, where, in order to make sense of "skew-symmetric" one needs to choose an isomorphism $E \to E^*$, which can be accomplished, e.g., with a nondegenerate quadratic form $Q \in S^2 E^*$. It plays an important role in differential geometry as it is the key to Chern's generalization of the Gauss-Bonnet theorem, see, e.g., [**296**, Chap. 13].

For example, if $n = 4$ and $x = \sum_{i,j} x^{ij} e_i \wedge e_j$, then

$$x \wedge x = \sum_{ijkl} x^{ij} x^{kl} e_i \wedge e_j \wedge e_k \wedge e_l = \sum_{\sigma \in \mathfrak{S}_4} x^{\sigma(1)\sigma(2)} x^{\sigma(3)\sigma(4)} e_1 \wedge e_2 \wedge e_3 \wedge e_4.$$

In particular $\mathrm{Pf}(x) = x^{12}x^{34} - x^{13}x^{24} + x^{14}x^{23}$.

2.7.5. Exercises on the Pfaffian.

(1) Let x be a skew-symmetric matrix. If $x^{ij} = 0$ for $i > j+1$, show that $\mathrm{Pf}(x) = x^{12}x^{34} \cdots x^{2m-1,2m}$. Note that the eigenvalues of x are $\pm\sqrt{-1}x^{j,j+1}$.

(2) Using the previous exercise, show that if x is skew-symmetric, $\mathrm{Pf}(x)^2 = \det(x)$.

(3) Fix a basis $e_1, \ldots, e_n$ of E so $x \in \Lambda^2 E$ may be written as a skew-symmetric matrix $x = (x^i_j)$. Show that in this basis

$$\begin{aligned} \mathrm{Pf}(x^i_j) &= \frac{1}{2^n n!} \sum_{\sigma \in \mathfrak{S}_{2n}} \mathrm{sgn}(\sigma) x^{\sigma(1)}_{\sigma(2)} \cdots x^{\sigma(2m-1)}_{\sigma(2m)} \\ &= \frac{1}{2^n} \sum_{\sigma \in \mathcal{P}} \mathrm{sgn}(\sigma) x^{\sigma(1)}_{\sigma(2)} \cdots x^{\sigma(2m-1)}_{\sigma(2m)}, \end{aligned} \tag{2.7.1}$$

where $\mathcal{P} \subset \mathfrak{S}_{2m}$ consists of the permutations such that $\sigma(2i-1) < \sigma(2i)$ for all $0 \leq i \leq m$. Note that these expressions are defined for arbitrary matrices and the $SL(V)$-invariance still holds as $S^n(\Lambda^2 V^*)$ is an $SL(V)$-submodule of $S^n(V^* \otimes V^*)$.

2.8. Decomposition of $V^{\otimes 3}$

When $d > 2$, there are subspaces of $V^{\otimes d}$ other than the completely symmetric and skew-symmetric tensors that are invariant under changes of bases. These spaces will be studied in detail in Chapter 6. In this section, as a preview, I consider $V^{\otimes 3}$.

Change the previous notation of $\pi_S : V^{\otimes 3} \to V^{\otimes 3}$ and $\pi_\Lambda : V^{\otimes 3} \to V^{\otimes 3}$ to $\rho_{\ytableaushort{123}}$ and $\rho_{\ytableaushort{1,2,3}}$ respectively.

Define the projections

$$\rho_{\ytableaushort{1,2}} : V \otimes V \otimes V \to \Lambda^2 V \otimes V,$$

$$v_1 \otimes v_2 \otimes v_3 \mapsto \frac{1}{2}(v_1 \otimes v_2 \otimes v_3 - v_2 \otimes v_1 \otimes v_3),$$

$$\rho_{\ytableaushort{13}} : V \otimes V \otimes V \to V \otimes S^2 V,$$

$$v_1 \otimes v_2 \otimes v_3 \mapsto \frac{1}{2}(v_1 \otimes v_2 \otimes v_3 + v_3 \otimes v_2 \otimes v_1),$$

which are also endomorphisms of $V^{\otimes 3}$. Composing them gives

$$\rho_{\ytableaushort{13,2}} = \rho_{\ytableaushort{13}} \circ \rho_{\ytableaushort{1,2}} : V \otimes V \otimes V \to S_{\ytableaushort{13,2}} V,$$

where $S_{\ytableaushort{13,2}} V$ is defined as the image of $\rho_{\ytableaushort{13,2}}$. The essential point here is:

The maps $\rho_{\ytableaushort{13,2}}, \rho_{\ytableaushort{13}}, \rho_{\ytableaushort{1,2}}$ all commute with the action of $GL(V)$ on $V^{\otimes 3}$. Therefore the image of $\rho_{\ytableaushort{13,2}}$ is a $GL(V)$-submodule of $V^{\otimes 3}$. Similarly define $S_{\ytableaushort{12,3}} V$ as the image of $\rho_{\ytableaushort{12,3}}$.

Warning: The image of $\rho_{\ytableaushort{13}} : \Lambda^2 V \otimes V \to V^{\otimes 3}$ is no longer skew-symmetric in its first two arguments. Similarly the image of $\rho_{\ytableaushort{1,2}} : V \otimes S^2 V \to V^{\otimes 3}$ is not symmetric in its second and third argument.

2.8.1. Exercises.

(1) Show that the sequence
$$S_{\begin{smallmatrix}1&2\\3&\end{smallmatrix}}V \to V{\otimes}\Lambda^2 V \to \Lambda^3 V$$
is exact.

(2) Show that $S_{\begin{smallmatrix}1&2\\3&\end{smallmatrix}}V \cap S_{\begin{smallmatrix}1&3\\2&\end{smallmatrix}}V = (0)$. ⊚

(3) Show that there is a direct sum decomposition:
$$\begin{aligned} V^{\otimes 3} &= S^3V \oplus S_{\begin{smallmatrix}1&3\\2&\end{smallmatrix}}V \oplus \Lambda^3 V \oplus S_{\begin{smallmatrix}1&2\\3&\end{smallmatrix}}V \\ &= S_{\begin{smallmatrix}1&2&3\end{smallmatrix}}V \oplus S_{\begin{smallmatrix}1&2\\3&\end{smallmatrix}}V \oplus S_{\begin{smallmatrix}1\\2\\3\end{smallmatrix}}V \oplus S_{\begin{smallmatrix}1&3\\2&\end{smallmatrix}}V. \end{aligned} \tag{2.8.1}$$

(4) Let $\dim V = 2$ with basis e_1, e_2. Calculate the images $S_{\begin{smallmatrix}1&3\\2&\end{smallmatrix}}V$ and $S_{\begin{smallmatrix}1&2\\3&\end{smallmatrix}}V$ explicitly.

(5) Now consider $S_{\begin{smallmatrix}3&1\\2&\end{smallmatrix}}V$ to be the image of $\rho_{\begin{smallmatrix}3&1\\2&\end{smallmatrix}}$. Show it is also a $GL(V)$-invariant subspace. Using the previous exercise, show that when $\dim V = 2$, $S_{\begin{smallmatrix}3&1\\2&\end{smallmatrix}}V \subset S_{\begin{smallmatrix}1&3\\2&\end{smallmatrix}}V \oplus S_{\begin{smallmatrix}1&2\\3&\end{smallmatrix}}V$.

(6) Now let $\dim V = 3$ with basis e_1, e_2, e_3 and calculate the images $S_{\begin{smallmatrix}1&3\\2&\end{smallmatrix}}V$ and $S_{\begin{smallmatrix}1&2\\3&\end{smallmatrix}}V$ explicitly. Show that $S_{\begin{smallmatrix}3&1\\2&\end{smallmatrix}}V \subset S_{\begin{smallmatrix}1&3\\2&\end{smallmatrix}}V \oplus S_{\begin{smallmatrix}1&2\\3&\end{smallmatrix}}V$. More generally, show that each of these spaces is contained in the span of the two others.

(7) Explain why the previous exercise implies $S_{\begin{smallmatrix}3&1\\2&\end{smallmatrix}}V \subset S_{\begin{smallmatrix}1&3\\2&\end{smallmatrix}}V \oplus S_{\begin{smallmatrix}1&2\\3&\end{smallmatrix}}V$ for V of arbitrary dimension.

(8) Consider the sequence of maps $d : S^pV{\otimes}\Lambda^q V \to S^{p-1}V{\otimes}\Lambda^{q+1}V$ given in coordinates by $f{\otimes}u \mapsto \sum \frac{\partial f}{\partial x^i}{\otimes}x_i \wedge u$. Show that $d^2 = 0$ so we have an exact sequence of $GL(V)$-modules: ⊚

$$0 \to S^dV \to S^{d-1}V{\otimes}V \to S^{d-2}V{\otimes}\Lambda^2 V \to \cdots \to V{\otimes}\Lambda^{d-1}V \to \Lambda^d V \to 0. \tag{2.8.2}$$

2.8.2. Isotypic decompositions. Recall that if a linear map $f : V \to V$ has distinct eigenvalues, there is a canonical decomposition of V into a direct sum of one-dimensional eigenspaces. But if there are eigenvalues that occur with multiplicity, even if f is diagonalizable, there is no canonical splitting into eigenlines. The same phenomenon is at work here. Although our decomposition (2.8.1) is *invariant* under the action of $GL(V)$, it is not *canonical*, i.e., independent of choices. The space $S_{\begin{smallmatrix}1&3\\2&\end{smallmatrix}}V \oplus S_{\begin{smallmatrix}1&2\\3&\end{smallmatrix}}V$ is the analog of an eigenspace for an eigenvalue with multiplicity two. In fact, I claim that $S_{\young(31,2)}V, S_{\young(12,3)}V, S_{\young(13,2)}V$ are all isomorphic $GL(V)$-modules!

It is exactly that these modules are isomorphic which causes the decomposition to be noncanonical.

Exercise 2.8.2.1: Prove the claim. ⊚

Exercise 2.8.2.2: Calculate $\dim S_{\young(12,3)}V$ in terms of $n = \dim V$ by using your knowledge of $\dim V^{\otimes 3}, \dim S^3V, \dim \Lambda^3 V$ and the fact that $S_{\young(31,2)}V, S_{\young(12,3)}V$ are isomorphic vector spaces.

Definition 2.8.2.3. Let G be a group. G is said to be *reductive* if every G-module V admits a decomposition into a direct sum of irreducible G-modules.

For example, $GL(V)$ is reductive but the group

$$N := \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \mid t \in \mathbb{C} \right\}$$

is not.

Exercise 2.8.2.4: Show that N is not reductive by showing that there is a line L in $\mathbb{C}^2$ preserved by N, but no N-invariant complement to L exists.

Remark 2.8.2.5. If one is working over finite fields, there are several different notions of reductive which coincide over $\mathbb{C}$, see e.g. [**116**].

Definition 2.8.2.6. Let G be reductive and let V be a G-module. Write the decomposition of V into irreducible G-modules as

$$V = W_{1,1} \oplus \cdots \oplus W_{1,m_1} \oplus W_{2,1} \oplus \cdots \oplus W_{2,m_2} \oplus \cdots \oplus W_{r,1} \oplus \cdots \oplus W_{r,m_r},$$

where the $W_{i,1}, \dots, W_{i,m_i}$ are isomorphic G-modules and no $W_{i,j}$ is isomorphic to a $W_{k,l}$ for $k \neq i$. While this decomposition is invariant under the action of G, it is not canonical. If $U_i = W_{i,1} \oplus \cdots \oplus W_{i,m_i}$, then the decomposition $V = \bigoplus_i U_i$ is canonical and is called the *isotypic* decomposition of V as a G-module. The U_i are called *isotypic components*. We say m_j is the *multiplicity* of the irreducible module $W_{j,1}$ in V.

Definition 2.8.2.7. Let $S_{21}V$ denote the irreducible $GL(V)$ module isomorphic to each of $S_{\young(12,3)}V, S_{\young(13,2)}V, S_{\young(23,1)}V$.

Exercise 2.8.2.8: Show that for $d > 3$ the kernel of the last nonzero map in Exercise 2.8.1(8) and the kernel of the second map give rise to different generalizations of $S_{21}V$. ⊚

In Chapter 6, the $GL(V)$-modules in $V^{\otimes d}$ will be studied for all d. They will be indexed by partitions of d. For example, the partitions of 3 give rise to the three modules $S_3V = S^3V$, $S_{111}V = \Lambda^3 V$, and our new friend $S_{21}V$.

2.8.3. Exercises.

(1) Show that there is an invariant decomposition

$$S^3(A\otimes B) = (S^3A\otimes S^3B) \oplus \pi_S(S_{\begin{array}{|c|c|}\hline 3&1\\\hline 2\\\cline{1-1}\end{array}}A\otimes S_{\begin{array}{|c|c|}\hline 3&1\\\hline 2\\\cline{1-1}\end{array}}B) \oplus \Lambda^3A\otimes\Lambda^3B$$

as a $GL(A)\times GL(B)$-module.

(2) Decompose $\Lambda^3(A\otimes B)$ as a $GL(A)\times GL(B)$-module.

(3) Let $\tilde{S}_{\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}}A$ denote the kernel of the map $S^2A\otimes A \to S^3A$, so it represents a copy of the module $S_{21}A$ in $A^{\otimes 3}$.

Show that if $R \in \tilde{S}_{\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}}A$, then $R(u,v,w) = R(v,u,w)$ for all $u,v,w \in A^*$ and

$$R(u,v,u) = -\frac{1}{2}R(u,u,v) \quad \forall u,v,\in A^*. \tag{2.8.3}$$

2.9. Appendix: Basic definitions from algebra

2.9.1. Linear algebra definitions. Vector spaces, dual spaces, linear maps and bilinear maps are defined in §1.2.1.

Definition 2.9.1.1. The *dimension* of a vector space V is the smallest number $\mathbf{v}$ such that there exist $e_1,\dots,e_{\mathbf{v}} \in V$ such that any $x\in V$ may be written $x = \sum c^ie_i$ for constants c^i. Such a set of vectors $\{e_j\}$ is called a *basis* of V.

Definition 2.9.1.2. The *rank* of a linear map $f : V \to W$ is $\dim(f(V))$.

Definition 2.9.1.3. Given a linear map $f: V\to V$, a nonzero vector $v\in V$ such that $f(v) = \lambda v$ for some $\lambda\in\mathbb{C}$ is called an *eigenvector.*

If V has basis $v_1,\dots,v_{\mathbf{v}}$, let $\alpha^1,\dots,\alpha^{\mathbf{v}}$ be the basis of V^* such that $\alpha^i(v_j) = \delta^i_j$. It is called the *dual basis* to $(v_1,\dots,v_{\mathbf{v}})$. Let W have basis $w_1,\dots,w_{\mathbf{w}}$. A linear map $f : V\to W$ is determined by its action on a basis. Write $f(v_j) = f^s_jw_s$. Then the matrix representing f with respect to these bases is (f^s_j).

Definition 2.9.1.4. Given vector spaces U, V, and W, and linear maps $f : U\to V$ and $g: V\to W$, one says that the sequence

$$0\to U\xrightarrow{f}V\xrightarrow{g}W\to 0$$

is an *exact sequence of vector spaces* if f is injective, g is surjective, and $\ker(g) = \mathrm{image}(f)$.

Remark 2.9.1.5. Regarding dual spaces, the spaces V and V^* can be thought of as covariant and contravariant. There is no canonical isomorphism $V \cong V^*$. If V is endowed with a Hermitian inner product, there is then a canonical identification of V^* with $\overline{V}$, the complex conjugate. However in this book I will work with projectively invariant properties so Hermitian inner products will not play a role.

Definition 2.9.1.6. Given $f \in \mathrm{Hom}(V, W)$, define $f^T \in \mathrm{Hom}(W^*, V^*)$, called the *transpose* or *adjoint* of f, by $f^T(\beta)(v) = \beta(f(v))$.

2.9.2. Definitions regarding groups and rings.

Definition 2.9.2.1. A *group* is a set G, with a pairing $G \times G \to G$, $(a, b) \mapsto ab$, a preferred element (the identity) $\mathrm{Id} \in G$, such that $a(\mathrm{Id}) = (\mathrm{Id})a = a$ for all $a \in G$ and such that for each $a \in G$ there is an inverse element which is both a left and right inverse.

For example, a vector space is a group with the operation $+$ and the identity element 0.

One of the most important groups is the permutation group:

Definition 2.9.2.2 (The group $\mathfrak{S}_n$)**.** Given a collection of n ordered objects, the set of permutations of the objects forms a group, called the *symmetric group on n elements* or the *permutation group*, and is denoted $\mathfrak{S}_n$.

An important fact about elements $\sigma \in \mathfrak{S}_n$ is that they may be written as a product of transpositions, e.g. $(1, 2, 3) = (2, 3)(1, 2)$. This decomposition is not unique, nor is the number of transpositions used in the expression unique, but the parity is.

Definition 2.9.2.3. For a permutation σ, define the *sign* of σ, $\mathrm{sgn}(\sigma)$, to be $+1$ if an even number of transpositions are used and -1 if an odd number are used.

Definition 2.9.2.4. Let G, H be groups. A map $f : G \to H$ is called a *group homomorphism* if $f(g_1 g_2) = f(g_1)f(g_2)$ for all $g_1, g_2 \in G$.

Definition 2.9.2.5. A *ring* is a set equipped with an additive group structure and a multiplicative operation such that the two operations are compatible.

An *ideal* $I \subset R$ is a subset that is a group under addition, and strongly closed under multiplication in the sense that if $P \in I$ and $Q \in R$, then $PQ \in I$.

Definition 2.9.2.6. An *algebra* is a vector space V equipped with a multiplication compatible with the vector space structure, in other words, a ring that is also a vector space.

2.10. Appendix: Jordan and rational canonical form

Most linear algebra texts cover Jordan canonical form of a linear map $f : V \to V$ so I just state the result.

Let $f : V \to V$ be a linear map. If it has distinct eigenvalues, it is diagonalizable under the action of $GL(V)$. If not, one can decompose V into *generalized eigenspaces* for f, i.e., say there are k distinct eigenvalues $\lambda_1, \ldots, \lambda_k$. One may write

$$V = V_1 \oplus \cdots \oplus V_k,$$

where, if the minimal polynomial of f is denoted ϕ_f, one can write $\phi_f = \phi_1^{a_1} \cdots \phi_k^{a_k}$, where $\phi_j(\lambda) = (\lambda - \lambda_j)$. Then $g_j := f\,|_{V_j} - \lambda_j \operatorname{Id}_{V_j}$ is nilpotent on V_j, and choosing bases with respect to generating vectors one obtains (noncanonically) blocks of sizes $s_1^j = a_j \geq s_2^j \geq \cdots \geq s_{m_j}^j$, where each block in matrices looks like

$$\begin{pmatrix} \lambda_j & 1 & 0 & \cdots & 0 \\ 0 & \lambda_j & 1 & & \vdots \\ & \ddots & \ddots & \ddots & 0 \\ & & & \lambda_j & 1 \\ & & & 0 & \lambda_j \end{pmatrix}.$$

In the end one obtains a matrix representing f whose entries are zero except on the diagonal, which have the eigenvalues, and some of the entries above the diagonal, which are ones.

To obtain the rational canonical form from the Jordan form, in each V_j, first take a vector $v_j = v_{j,1}$ such that $g_j^{\alpha_j - 1}(v_j) \neq 0$ and let $w_1 = \sum v_{j,1}$. Let $W_1 \subset V$ be the subspace generated by successive images of w_1 under f. Note that the minimal polynomial of f restricted to W_1, call it ψ_1, is ϕ_f. From this one deduces that f restricted to W_1 has the form

$$\begin{pmatrix} 0 & & & -p_0 \\ 1 & \ddots & & \vdots \\ & \ddots & 0 & -p_{d-2} \\ & & 1 & -p_{d-1} \end{pmatrix},$$

where $\psi_1 = \phi_f = p_0 + p_1\lambda + \cdots + p_{d-1}\lambda^{d-1} + \lambda^d$.

Then in each V_j, in the complement of the subspace generated by the images of $v_{j,1}$ under g_j, take a vector $v_{j,2}$ such that the space $\langle g_j^s v_{j,2} \mid s \in \mathbb{N} \rangle$ has maximal dimension. Now let $w_2 = \sum v_{j,2}$ and consider the corresponding space W_2. The minimal polynomial of f restricted to W_2, call it ψ_2, divides ψ_1, and one obtains a matrix of the same form as above with respect to ψ_2 representing f restricted to W_2. One continues in this fashion. (For $u > m_j$, the contribution of V_j to the vector generating w_u is zero.) The polynomials ψ_u are called the *invariant divisors* of f and are independent of choices.

Note that ψ_{u+1} divides ψ_u and that, ignoring 1×1 blocks, the maximum number of Jordan blocks of size at least two associated to any eigenvalue of f is the number of invariant divisors.

Rational canonical form is described in [**136**, VI.6], also see, e.g., [**128**, §7.4]. For a terse description of rational canonical form via the Jordan form, see [**251**, Ex. 7.4.8].

2.11. Appendix: Wiring diagrams

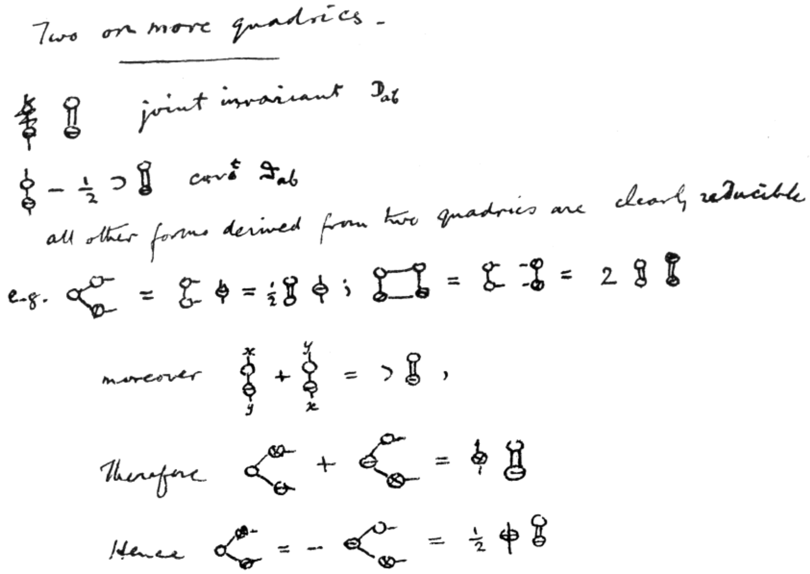


Figure 2.11.1. Wiring diagrams from unpublished notes of Clifford 1881.

Wiring diagrams may be used to represent many tensors and tensor operations including contractions. Such diagrams date at least back to Clifford and were used by Feynman and Penrose [**264**] in physics, Cvitanović [**105**] (who calls them *birdtracks*) to study representation theory via invariant tensors, Kuperberg [**198**], Bar-Natan and Kontsevich [**15**], and Reshetikhin and Turaev [**275**] in knot theory/quantum groups, Deligne [**111**] and Vogel [**327, 326**] in their proposed categorical generalizations of Lie algebras, and many others.

A *wiring diagram* is a diagram that encodes a tensor as described below. A *natural wiring diagram*, or more precisely a *G-natural wiring diagram* is a diagram that encodes a G-invariant tensor $T \in A_1^{\otimes d_1} \otimes A_1^{*\otimes \delta_1} \otimes \cdots \otimes A_n^{\otimes d_n} \otimes A_n^{*\otimes \delta_n}$, where $G = GL(A_1) \otimes \cdots \otimes GL(A_n)$. Often such invariant tensors will be viewed as natural operations on other tensors. For example, $\mathrm{tr} = \mathrm{Id}_V \in V \otimes V^*$ is a $GL(V)$-invariant tensor.

When a diagram is viewed as an operation, it is to be read top to bottom. Strands going in and coming out represent vector spaces, and a group of strands represents the tensor product of the vector spaces in the group. If there are no strands going in (resp. coming out) we view this as inputing a scalar (resp. outputing a scalar).

For example, encode a linear map $f : A \to A$ by the diagram in Figure 2.11.2 (a). The space of tensors designated in a box at the top and bottom of the figure indicates the input and output of the operator. More generally, if a diagram contains a T with a circle around it with k strands with arrows going out, and ℓ strands with arrows going in, then $T \in V^{\otimes k} \otimes V^{*\otimes \ell}$.

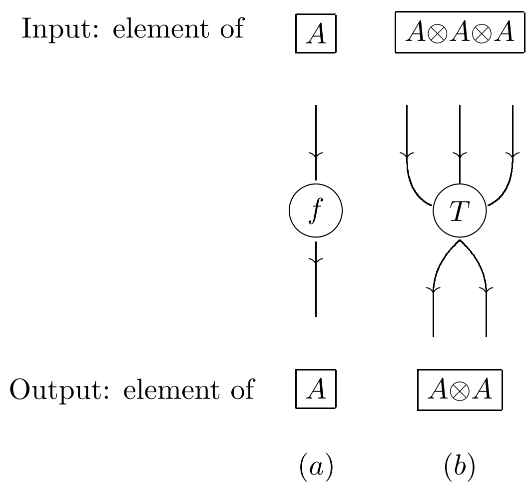


Figure 2.11.2. Wiring diagram of (a) a map $f : A \to A$, i.e., of $f \in A^* \otimes A$ and (b) a tensor $T \in (A^*)^{\otimes 3} \otimes A^{\otimes 2}$.

Recalling that one may also view $f \in A^* \otimes A$ as a map $A^* \to A^*$, or as a bilinear map $A \times A^* \to \mathbb{C}$, one may write the diagram in the three ways depicted in Figure 2.11.3. Note that the *tensor* is the same in all four cases; one just changes how one views it as an operator.

Of particular importance is the identity map $\mathrm{Id} : A \to A$, the first of the four diagrams in Figure 2.11.4. In these diagrams I indicate the identity by omitting the letter f.

If an arrow is present, it points from a vector space to its dual. The identity and dualizing operators are depicted in Figure 2.11.4.

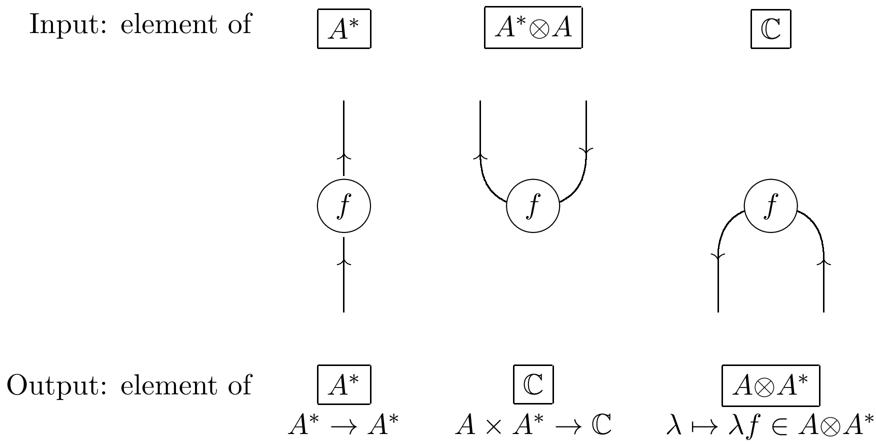


Figure 2.11.3. Three views of the same tensor as maps.

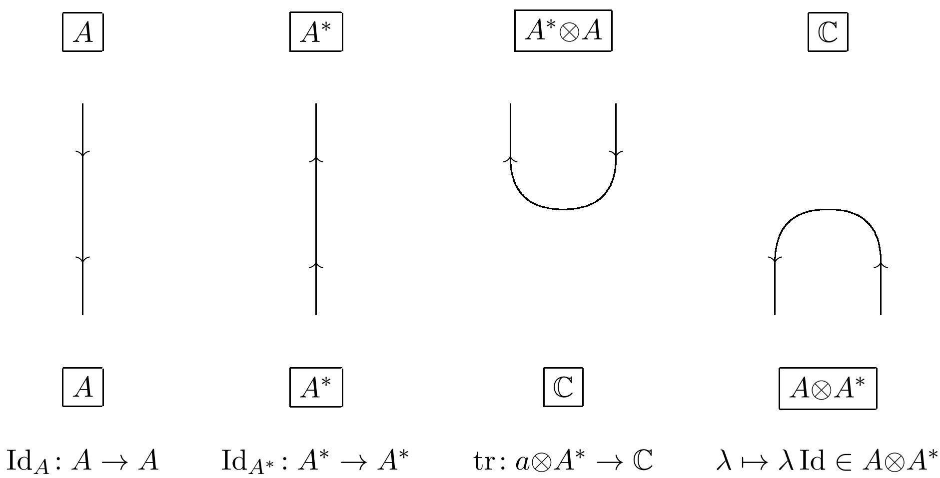
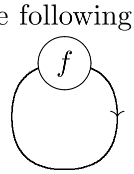


Figure 2.11.4. $\mathrm{Id} \in A^*{\otimes}A$ viewed as: a map $A \to A$, a map $A^* \to A^*$, a map $A^*{\otimes}A \to \mathbb{C}$ (i.e., a linear map maps to its trace), and a map $\mu_A : \mathbb{C} \to A^*{\otimes}A$ ($\lambda \mapsto \lambda\,\mathrm{Id}_A$).

Exercise 2.11.0.7: Show that the following diagram represents the scalar $\mathrm{tr}(f)$, for a linear map $f : V \to V$:



In particular, if $f : V \to V$ is a projection, then the diagram represents $\dim \mathrm{image}(f)$. (Compare with Exercise 2.3.2(9).)

2.11.1. A wiring diagram for the dimension of V. If we compose the maps $\mu_V : \mathbb{C} \to V^*{\otimes}V$ of Figure 2.11.4 and $\mathrm{tr}_V : V^*{\otimes}V \to \mathbb{C}$, we obtain a map $\mathrm{tr}_V \circ \mu_V : \mathbb{C} \to \mathbb{C}$ that is multiplication by some scalar.

Exercise 2.11.1.1: Show that the scalar is $\dim V$.

Thus the picture

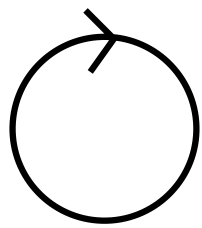


Figure 2.11.5. Symbolic representation of $\dim V$.

should be interpreted as the scalar $\dim V$. Similarly, since $\mathrm{Id}_{V^{\otimes d}}$ has trace equal to $(\dim V)^d$, the union of d disjoint circles should be interpreted as the scalar $(\dim V)^d$.

Below I will take formal sums of diagrams, and under such a formal sum one adds scalars and tensors.

Recall the linear map

$$\begin{aligned} \sigma : V{\otimes}V &\to V{\otimes}V, \\ a{\otimes}b &\mapsto b{\otimes}a. \end{aligned}$$

It has the wiring diagram shown in Figure 2.11.6.

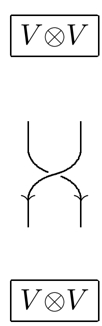


Figure 2.11.6. A wiring diagram for the map $a{\otimes}b \mapsto b{\otimes}a$.

Exercise 2.11.1.2: Show pictorially that $\mathrm{tr}\,\sigma = \dim V$ by composing the picture with σ with the picture for $\mathrm{Id}_{V\otimes V}$. (Of course, one can also obtain the result by considering the matrix of σ with respect to the basis $e_i{\otimes}e_j$.)

In Chapter 6, I show that *all* $GL(V)$-natural wiring diagrams in $V^{\otimes d} \otimes V^{*\otimes \delta}$ are built from Id_V and σ.

2.11.2. A wiring diagram for matrix multiplication. Matrix multiplication is depicted as a wiring diagram in Figure 2.11.7.

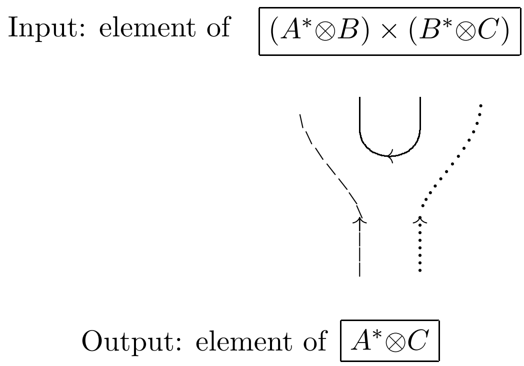


Figure 2.11.7. Matrix multiplication as an operator from $(A^*\otimes B) \times (B^*\otimes C)$ to $A^*\otimes C$.

Exercise 2.11.2.1: Show that the diagram in Figure 2.11.7 agrees with the matrix multiplication you know and love.

Encode the tensor π_S of §2.6.2 with the white box shorthand on the left hand side. It is one half the formal sum of the wiring diagrams for σ and $\mathrm{Id}_{V^{\otimes 2}}$, as on the right hand side of Figure 2.11.8.

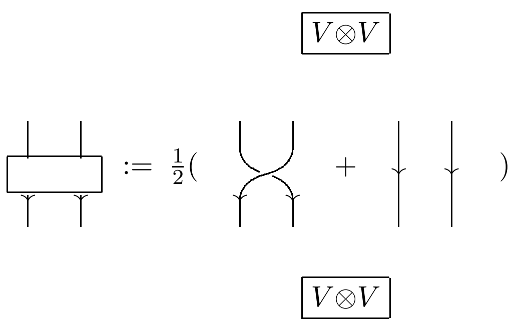


Figure 2.11.8. The wiring diagram $\frac{1}{2}(\sigma + \mathrm{Id}_{V^{\otimes 2}})$.

Encode $\pi_S : V^{\otimes d} \to V^{\otimes d}$ by a diagram as in Figure 2.11.10 with d strands. This is $\frac{1}{d!}$ times the formal sum of diagrams corresponding to all permutations. Recall that since each permutation is a product of transpositions, a wiring diagram for any permutation can be written as a succession of σ's acting on different pairs of factors; for example, one can write the diagram for the cyclic permutation on three factors as in Figure 2.11.11.

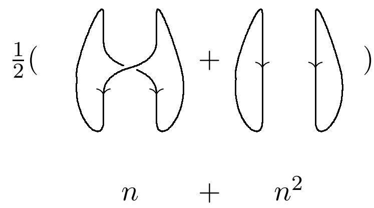


Figure 2.11.9. Symbolic representation of $\dim(S^2V)$.

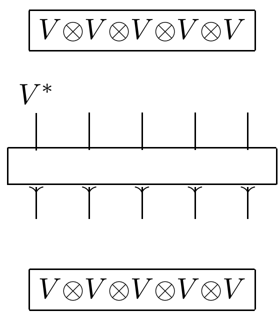


Figure 2.11.10. The symmetrizing wiring diagram π_S for $d = 5$.

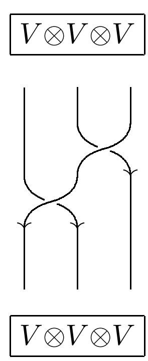


Figure 2.11.11. The cyclic permutation on three factors.

2.11.3. Exercises.

(1) Reprove that $\pi_S : V^{\otimes 2} \to V^{\otimes 2}$ is a projection operator by showing that the concatenation of two wiring diagrams for π_S yields the diagram of π_S.

(2) Show that $\dim(S^2V) = \binom{n+1}{2}$ pictorially by using Figure 2.11.9.

(3) Define and give a wiring diagram for $\pi_\Lambda : V^{\otimes 2} \to V^{\otimes 2}$, the projection operator to $\Lambda^2 V$, and use the diagram to compute the dimension of $\Lambda^2 V$.

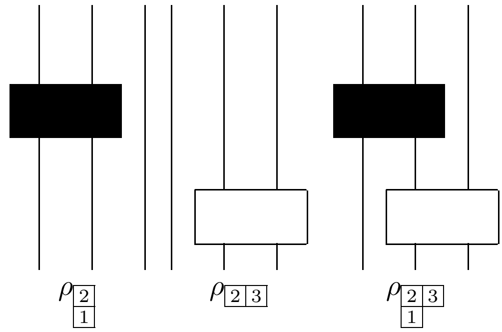


Figure 2.11.12. The wiring diagrams for three projection operators.

(4) Show that $V^{\otimes 2} = S^2V \oplus \Lambda^2 V$ by showing that the diagram of $\mathrm{Id}_{V^{\otimes 2}}$ is the sum of the diagrams for S^2V and $\Lambda^2 V$.

(5) Calculate the trace of the cyclic permutation on three factors $V\otimes V \otimes V \to V\otimes V\otimes V$ using wiring diagrams. Verify your answer by using bases.

(6) Show diagrammatically that $V^{\otimes 3} \neq \Lambda^3 V \oplus S^3 V$ by showing that the diagram for $\mathrm{Id}_{V^{\otimes 3}}$ is not the sum of the diagrams for π_S and π_Λ.

(7) Use wiring diagrams to calculate $\dim S^3V$ and $\dim \Lambda^3 V$.

The three maps $\rho_{\boxed{1}\boxed{3}}$, $\rho_{\substack{\boxed{1}\\ \boxed{2}}}$, and $\rho_{\substack{\boxed{1}\boxed{3}\\ \boxed{2}\phantom{\boxed{3}}}}$ are depicted visually in the three wiring diagrams of Figure 2.11.12, which add new components represented by the black and white boxes. Each of these diagrams, just as in the definitions above, shows what to do with a rank one element of $V\otimes V\otimes V$, which is extended linearly. The three vertical lines in each diagram correspond to the three tensor factors of a rank one element $v_1\otimes v_2\otimes v_3$. A black box indicates skew-symmetrization, and a white box symmetrization. The factor of $\frac{1}{b!}$, where b is the number of wires passing through a box, is implicit.

$\rho_{\substack{\boxed{1}\boxed{2}\\ \boxed{3}\phantom{\boxed{2}}}}$ may be described by the wiring diagram in Figure 2.11.13.

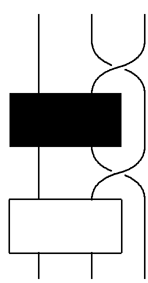
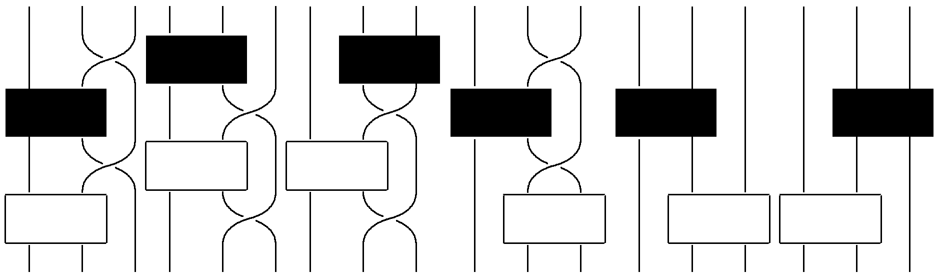


Figure 2.11.13. A wiring diagram for the projection map $\rho_{12,3}$.



$\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}$ $\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}$ $\begin{array}{|c|c|}\hline 3&1\\\hline 2\\\cline{1-1}\end{array}$ $\begin{array}{|c|c|}\hline 3&2\\\hline 1\\\cline{1-1}\end{array}$ $\begin{array}{|c|c|}\hline 2&3\\\hline 1\\\cline{1-1}\end{array}$ $\begin{array}{|c|c|}\hline 2&1\\\hline 3\\\cline{1-1}\end{array}$

(a) **(b)** **(c)** **(d)** **(e)** **(f)**

Figure 2.11.14. 6 natural realizations of $S_{21}V$.

Exercise 2.11.3.1: Prove the direct sum decomposition (2.8.1) diagrammatically.

Exercise 2.11.3.2: Calculate $\dim S_{\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}}V$ in terms of $n = \dim V$ by using wiring diagrams. ⊚

Chapter 3

Elementary results on rank and border rank

This chapter presents results on rank and border rank that can be stated without the language of algebraic geometry and representation theory. When the proofs do not need such language, they are presented as well. It will give practitioners outside of geometry and representation theory a quick introduction to the state of the art.

It begins with elementary results on rank and symmetric rank in §3.1 and §3.2 respectively. The sections include remarks on the maximal possible rank, typical rank, efficient presentation of tensors, uniqueness of expressions, symmetric tensor rank of monomials, and the notion of rank for partially symmetric tensors. In §3.3, uniqueness of CP decompositions is discussed. There are three topics: (i) unique up to finite decompositions (sometimes called "partially unique" decompositions), (ii) Kruskal's test, and (iii) the property NWD (non-weak-defectivity) which assures that with probability one, a tensor of rank r will have a unique CP decomposition. The bulk of this chapter, §§3.4–3.10, concerns tests for border rank and symmetric border rank in the form of equations. The property of having border rank at most r is an algebraic property; in particular, it can in principle be precisely tested by the vanishing of polynomial equations. The state of the art regarding what is known for both border rank and symmetric border rank is presented. Included in §3.5 is a discussion of the Comas-Seguir theorem that characterizes ranks of tensors in $S^d\mathbb{C}^2$, as well as an improved version of Sylvester's algorithm for expressing an element of $S^d\mathbb{C}^2$ as a sum of d-th powers. The chapter concludes with a discussion of Kronecker's normal form for pencils of matrices and the complete determination of the rank of a tensor in $\mathbb{C}^2{\otimes}\mathbb{C}^{\mathbf{b}}{\otimes}\mathbb{C}^{\mathbf{c}}$ with a given normal form in §3.11.

3.1. Ranks of tensors

3.1.1. A characterization of rank in terms of spaces of matrices. Since spaces of matrices have been studied for a long time, it is common practice to convert questions about tensors to questions about spaces of matrices. Here is such an example, which appears in [**54**, Prop. 14.45] as an inequality, where it is said to be classical, and as an equality in [**129**, Thm. 2.4]. It is also used, e.g., in [**181**]:

Theorem 3.1.1.1. *Let $T \in A{\otimes}B{\otimes}C$. Then $\mathbf{R}(T)$ equals the number of rank one matrices needed to span (a space containing) $T(A^*) \subset B{\otimes}C$ (and similarly for the permuted statements).*

Proof. Let T have rank r so there is an expression $T = \sum_{i=1}^{r} a_i{\otimes}b_i{\otimes}c_i$. (I remind the reader that the vectors a_i need not be linearly independent, and similarly for the b_i and c_i.) Then $T(A^*) \subseteq \langle b_1{\otimes}c_1, \dots, b_r{\otimes}c_r\rangle$ shows that the number of rank one matrices needed to span $T(A^*) \subset B{\otimes}C$ is at most $\mathbf{R}(T)$.

On the other hand, say $T(A^*)$ is spanned by rank one elements $b_1{\otimes}c_1, \dots, b_r{\otimes}c_r$. Let $a^1, \dots, a^{\mathbf{a}}$ be a basis of A^*, with dual basis $a_1, \dots, a_{\mathbf{a}}$ of A. Then $T(a^i) = \sum_{s=1}^{r} x_s^i b_s{\otimes}c_s$ for some constants x_s^i. But then $T = \sum_{s,i} a_i{\otimes}(x_s^i b_s{\otimes}c_s) = \sum_{s=1}^{r} (\sum_i x_s^i a_i){\otimes}b_s{\otimes}c_s$ proving $\mathbf{R}(T)$ is at most the number of rank one matrices needed to span $T(A^*) \subset B{\otimes}C$. $\square$

Exercise 3.1.1.2: State and prove an analog of Theorem 3.1.1.1 for $T \in A_1{\otimes}\cdots{\otimes}A_n$.

3.1.2. Maximal possible rank. The following bound on rank is a corollary of Theorem 3.1.1.1:

Corollary 3.1.2.1. *Let $\mathbf{a} \leq \mathbf{b} \leq \mathbf{c}$. Given $T \in A{\otimes}B{\otimes}C$, $\mathbf{R}(T) \leq \mathbf{ab}$.*

Exercise 3.1.2.2: Prove Corollary 3.1.2.1.

Corollary 3.1.2.1 is not sharp in general. For example:

Theorem 3.1.2.3 ([**149**])**.** *The maximum rank of a tensor in $\mathbb{C}^2{\otimes}\mathbb{C}^{\mathbf{b}}{\otimes}\mathbb{C}^{\mathbf{b}}$ is $\lfloor \frac{3\mathbf{b}}{2} \rfloor$.*

Theorem 3.1.2.3 is proved in §10.3.

Exercise 3.1.2.4: Use Exercise 3.1.1.2 to deduce an upper bound for the maximal rank of an element in $A_1{\otimes}\cdots{\otimes}A_n$.

3.1.3. Concision of tensors. The following proposition is evidently classical:

Proposition 3.1.3.1. *Let $n > 2$. Let $T \in A_1{\otimes}\cdots{\otimes}A_n$ have rank r. Say $T \in A_1'{\otimes}\cdots{\otimes}A_n'$, where $A_j' \subseteq A_j$, with at least one inclusion proper. Then any expression $T = \sum_{i=1}^{\rho} u_i^1{\otimes}\cdots{\otimes}u_i^n$ with some $u_j^s \notin A_s'$ has $\rho > r$.*

Proof. Choose complements A_t'' so $A_t = A_t' \oplus A_t''$. Write $u_j^t = u_j^{t'} + u_j^{t''}$ with $u_j^{t'} \in A_t'$, $u_j^{t''} \in A_t''$. Then $T = \sum_{i=1}^{\rho} u_i^{1'} \otimes \cdots \otimes u_i^{n'}$ so all the other terms must cancel. Assume $\rho = r$, and say, e.g., some $u_{j_0}^{1''} \neq 0$. Then $\sum_{j=1}^{r} u_j^{1''} \otimes (u_j^{2'} \otimes \cdots \otimes u_j^{n'}) = 0$, but all the terms $(u_j^{2'} \otimes \cdots \otimes u_j^{n'})$ must be linearly independent in $A_2' \otimes \cdots \otimes A_n'$ (otherwise r would not be minimal), thus the $u_j^{1''}$ must all be zero, a contradiction. $\square$

Definition 3.1.3.2. $T \in A_1 \otimes \cdots \otimes A_n$ is called *concise* if each map $T : A_i^* \to A_1 \otimes \cdots \otimes A_{i-1} \otimes A_{i+1} \otimes \cdots \otimes A_n$ is injective. Similarly $\phi \in S^d V$ is concise if there does not exist a proper linear subspace $V' \subset V$ such that $\phi \in S^d V'$.

Exercise 3.1.3.3: Show that if $T \in A_1 \otimes \cdots \otimes A_n$ is concise, then $\underline{\mathbf{R}}(T) \geq \max\{\mathbf{a}_i\}$.⊚

Definition 3.1.3.4. For integers $r_j \leq \mathbf{a}_j$, define $\hat{Sub}_{r_1,\dots,r_n}(A_1 \otimes \cdots \otimes A_n) \subset A_1 \otimes \cdots \otimes A_n$ to be the set of tensors $T \in A_1 \otimes \cdots \otimes A_n$ such that there exists $A_j' \subset A_j$, $\dim A_j' = r_j$, with $T \in A_1' \otimes \cdots \otimes A_n'$. When all the r_j are equal, write $\hat{Sub}_r(A_1 \otimes \cdots \otimes A_n)$. Define $\hat{Sub}_r(S^d V)$ similarly.

3.1.4. Typical rank. A geometric definition of typical rank is given in §5.2.1. For now, I use the following provisional definition:

Definition 3.1.4.1. Put any Euclidean structure on the (real or complex) vector spaces A_j and V, inducing Euclidean structures and measures on $A_1 \otimes \cdots \otimes A_n$ and $S^d V$. A *typical rank* for $A_1 \otimes \cdots \otimes A_n$ is any r such that the set of tensors having rank r has positive measure. Similarly, define a *typical symmetric rank* as any r such that the set of symmetric tensors in $S^d V$ having symmetric tensor rank r has positive measure. Alternatively, the typical ranks (resp. symmetric ranks) may be defined to be the numbers r such that the set of tensors (resp. symmetric tensors) of rank (resp. symmetric rank) r has nonempty interior in the topology induced from the linear structure. In particular the set of typical ranks is independent of the choice of Euclidean structure.

Definition 3.1.4.2. Over $\mathbb{C}$ the typical rank is unique (see §5.2.1) and also called the *generic rank*. Unless otherwise stated, I will work exclusively over $\mathbb{C}$.

This definition is undesirable when working over the complex numbers, as the uniqueness is not transparent from the definition.

The set of rank one tensors in $A_1 \otimes \cdots \otimes A_n$ is of dimension $\mathbf{a}_1 + \cdots + \mathbf{a}_n - (n-1)$, in the sense that they are parametrized by elements $(v_1, \dots, v_n) \in A_1 \times \cdots \times A_n$ up to scale (plus one scalar). One needs the notion of a *manifold* or *algebraic variety* to speak precisely about dimension of sets more complicated than vector spaces, but for the moment, one can interpret it as how many independent parameters are needed to describe a point in the set. See Definition 4.6.1.8 for a precise definition.

Similarly, one expects $r(\mathbf{a}_1 + \cdots + \mathbf{a}_n - (n-1)) + r - 1$ parameters for the tensors of rank r. Since the ambient space has dimension $\mathbf{a}_1 \cdots \mathbf{a}_n$, we conclude:

The expected generic rank of an element of $\mathbb{C}^{\mathbf{a}_1} \otimes \cdots \otimes \mathbb{C}^{\mathbf{a}_n}$ *is*

$$\lceil \frac{\mathbf{a}_1 \cdots \mathbf{a}_n}{\mathbf{a}_1 + \cdots + \mathbf{a}_n - n + 1} \rceil.$$

In particular, if $\mathbf{a}_1 = \cdots = \mathbf{a}_n =: \mathbf{a}$, one expects $\frac{\mathbf{a}^n}{n(\mathbf{a}-1)+1} \sim \frac{\mathbf{a}^{n-1}}{n}$, compared with the known maximal rank bound of $\mathbf{a}^{n-1}$.

Here are some cases:

Theorem 3.1.4.3. (1) *(Strassen) The generic rank of an element of* $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ *is five.*

(2) *(Strassen-Lickteig) For all* $\mathbf{v} \neq 3$, *the generic rank of an element of* $\mathbb{C}^{\mathbf{v}} \otimes \mathbb{C}^{\mathbf{v}} \otimes \mathbb{C}^{\mathbf{v}}$ *is the expected* $\lceil \frac{\mathbf{v}^3 - 1}{3\mathbf{v} - 2} \rceil$.

(3) *The typical rank of elements of* $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^3$ *and* $\mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ *are the expected* 3.

See Theorems 5.4.1.1 and 5.5.1.1 for more detailed statements.

3.1.5. Possible ranks given the border rank. If one knows the border rank of a tensor, what are the restrictions on its rank? For tensors in $A_1 \otimes \cdots \otimes A_n$ of border rank two, the rank can be anywhere from 2 to n inclusive. Very little beyond this is known. For example, for tensors in $A_1 \otimes A_2 \otimes A_3$ of border rank 3, the rank can be $3, 4$ or 5. See §10.10 for a further discussion. In the case of symmetric tensors, discussed below in §3.5.2, much more is known.

3.2. Symmetric rank

3.2.1. Symmetric tensor rank and polarization. For $\phi \in S^d V$, let $\phi_{s,d-s} \in S^s V \otimes S^{d-s} V$ denote its partial polarization. We may think of $\phi_{s,d-s}$ as a linear map $\phi_{s,d-s} : S^{d-s} V^* \to S^s V$. Here is a symmetric analog of one direction of Theorem 3.1.1.1.

Proposition 3.2.1.1. *Given* $\phi \in S^d W$ *and* $1 \leq s \leq d-1$, $\mathbf{R}_S(\phi)$ *is at least the minimal number of rank one elements of* $S^s W$ *needed to span (a space containing)* $\phi_{s,d-s}(S^{d-s} W^*)$.

Proof. If $\phi = \eta_1^d + \cdots + \eta_r^d$, then $\phi_{s,d-s}(S^{d-s} W^*) \subseteq \langle \eta_1^s, \ldots, \eta_r^s \rangle$. □

One has the bound $\underline{\mathbf{R}}_S(\phi) \geq \operatorname{rank}(\phi_{s,d-s})$, see §3.5, and there is a stronger bound on symmetric rank that takes into account the nature of the kernel of $\phi_{s,d-s}$, see Theorem 9.2.1.4.

3.2.2. Maximum, typical and generic symmetric rank. The best general upper bound for symmetric rank is

Theorem 3.2.2.1 ([**208**])**.** *For $\phi \in S^d\mathbb{C}^{\mathbf{v}}$, $\mathbf{R}_S(\phi) \leq \binom{\mathbf{v}+d-1}{d} + 1 - \mathbf{v}$.*

Theorem 3.2.2.1 is sharp when $\mathbf{v} = 2$. It is not known if it is sharp when $\mathbf{v} > 2$. The result is a very special case of a much more general result, Theorem 9.1.3.1, where the proof is also presented.

Exercise 3.2.2.2: State and prove an analog of Proposition 3.1.3.1 for symmetric tensors.

Example 3.2.2.3. In $S^d\mathbb{R}^2$ all polynomials with distinct roots have a typical rank. Let $\Delta \subset S^d\mathbb{R}^2$ denote the subset of polynomials with a repeated root (the zero set of the classical discriminant). The set $S^d\mathbb{R}^2\backslash\Delta$ has $\lfloor \frac{d}{2} \rfloor + 1$ components, so there are at most $\lfloor \frac{d}{2} \rfloor + 1$ possible typical ranks because rank is semi-continuous. Sylvester showed that both d and $\lfloor \frac{d}{2} \rfloor + 1$ occur as typical ranks. Comon and Ottaviani [**100**] showed that this possible maximum occurs for $d \leq 5$ and conjectured that it always occurs.

The expected generic symmetric tensor rank in $S^d\mathbb{C}^{\mathbf{v}}$ is

$$\left\lceil \frac{\binom{\mathbf{v}+d-1}{d}}{\mathbf{v}} \right\rceil. \tag{3.2.1}$$

Theorem 3.2.2.4 (Alexander-Hirschowitz [**7**])**.** *The generic symmetric rank of an element of $S^d\mathbb{C}^{\mathbf{v}}$ is the expected $\lceil \frac{\binom{\mathbf{v}+d-1}{d}}{\mathbf{v}} \rceil$ with the following exceptions: $d = 2$ where it is $\mathbf{v}$, and the pairs $(d, \mathbf{v}) = (3,5), (4,3), (4,4), (4,5)$ where the typical symmetric rank is the expected plus one.*

3.2.3. Symmetric ranks and border ranks of monomials. The ranks and border ranks of some monomials are known. Here is a list of the state of the art. Proofs are given in Chapter 9.

Write $\mathbf{b} = (b_1, \dots, b_m)$. Define

$$S_{\mathbf{b},\delta} := \sum_{k=0}^{m} (-1)^k \left[\sum_{|I|=k} \binom{\delta + m - k - (b_{i_1} + \cdots + b_{i_k})}{m} \right],$$

$$T_{\mathbf{b}} := \prod_{i=1}^{m} (1 + b_i).$$

Theorem 3.2.3.1 ([**208**])**.** *Let $b_0 \geq b_1 \geq \cdots \geq b_n$ and write $d = b_0 + \cdots + b_n$. Then*

(1) $S_{(b_0,b_1,\dots,b_n),\lfloor\frac{d}{2}\rfloor} \leq \underline{\mathbf{R}}_S(x_0^{b_0}x_1^{b_1}\cdots x_n^{b_n}) \leq T_{(b_1,\dots,b_n)}$.

(2) *If* $b_0 \geq b_1+\cdots+b_n$, *then* $S_{\mathbf{b},\lfloor\frac{d}{2}\rfloor} = T_{\mathbf{b}}$ *so* $\underline{\mathbf{R}}_S(x_0^{b_0}x_1^{b_1}\cdots x_n^{b_n}) = T_{(b_1,\dots,b_n)}$.

(3) *In particular, if* $a \geq n$, *then* $\underline{\mathbf{R}}_S(x_0^a x_1\cdots x_n) = 2^n$. *Otherwise,*

$$\binom{n}{\lfloor\frac{d}{2}\rfloor - a} + \binom{n}{\lfloor\frac{d}{2}\rfloor - a+1} + \cdots + \binom{n}{\lfloor\frac{d}{2}\rfloor} \leq \underline{\mathbf{R}}_S(x_0^a x_1\cdots x_n) \leq 2^n.$$

(4) $\binom{n}{\lfloor n/2\rfloor} \leq \underline{\mathbf{R}}_S(x_1\cdots x_n) \leq 2^{n-1}$.

Proofs and a discussion are in §9.3.3.

Theorem 3.2.3.2 ([**272**]). $\mathbf{R}_S(x_1\cdots x_n) = 2^{n-1}$.

Just before this book went to press, the following result appeared:

Theorem 3.2.3.3 ([**69**]). *Let* $b_1 \geq \cdots \geq b_n$; *then*

$$\mathbf{R}_S(x_1^{b_1}x_2^{b_2}\cdots x_n^{b_n}) = \prod_{i=1}^{n-1}(1+b_i).$$

3.3. Uniqueness of CP decompositions

There are several types of "uniqueness" that one can discuss regarding tensor decomposition. The first is true uniqueness (up to trivialities like reordering terms). The second is called *unique up to finite* or *partially unique*, which means that a given tensor has at most a finite number of CP decompositions. The third (respectively fourth) is the property that a general tensor of rank r has a unique (respectively unique up to finite) CP decomposition. I.e., if one draws a tensor of rank r at random, with probability one, there will be a unique (resp. unique up to finite) decomposition. I discuss each of these uniqueness notions in this section.

3.3.1. Unique up to to finite decomposition. It is relatively easy to determine cases where tensors have at most a finite number of decompositions. Sometimes this situation is called *partial uniqueness*.

Proposition 3.3.1.1. *Assume the rank of a tensor (resp. symmetric rank of a symmetric tensor* $T \in S^dV$*) is* r*, which is less than the generic (see Theorems* 3.1.4.3 *and* 3.2.2.4 *for a list of generic ranks).*

(1) *If* T *is symmetric, then with probability one there are finitely many decompositions of* T *into a sum of* r *(symmetric) rank one tensors unless* $d = 2$*, where there are always infinitely many decompositions when* $r > 1$.

(2) *If $T \in \mathbb{C}^{\mathbf{v}} \otimes \mathbb{C}^{\mathbf{v}} \otimes \mathbb{C}^{\mathbf{v}}$, then as long as $\mathbf{v} \neq 3$, with probability one there are finitely many decompositions of T into a sum of r rank one tensors.*

Proposition 3.3.1.1 is a consequence of Corollary 5.3.1.3.

Proposition 3.3.1.2. *Say r is the generic tensor rank in $A \otimes B \otimes C$. Let $T \in A \otimes B \otimes C$ be generic. Then T will have parameters worth of expressions as a sum of r decomposable tensors unless $\frac{\mathbf{abc}}{\mathbf{a}+\mathbf{b}+\mathbf{c}-2}$ is an integer. If $\frac{\mathbf{abc}}{\mathbf{a}+\mathbf{b}+\mathbf{c}-2}$ is an integer and the generic tensor rank is the expected one, there will be a finite number of expressions for T as a sum of r decomposable elements.*

See §5.3 for a proof. This finite number of decompositions for generic tensors is almost never one. For example, when $A \otimes B \otimes C = \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^5$, there are exactly six decompositions of T as a sum of five decomposable elements, see §12.3.1. It is one, however, when $A \otimes B \otimes C = \mathbb{C}^2 \otimes \mathbb{C}^{\mathbf{b}} \otimes \mathbb{C}^{\mathbf{b}}$, see §12.3.1.

3.3.2. Kruskal's theorem.

Definition 3.3.2.1. Let $\mathcal{S} = \{w_1, \dots, w_p\} \subset W$ be a set of vectors. One says the points of $\mathcal{S}$ are in 2-general linear position if no two points are colinear; they are in 3-general linear position if no three lie on a plane; and more generally they are in *r-general linear position* if no r of them lie in a $\mathbb{C}^{r-1}$. The *Kruskal rank* of $\mathcal{S}$, $k_{\mathcal{S}}$, is defined to be the maximum number r such that the points of $\mathcal{S}$ are in r-general linear position.

Remark on the tensor literature. If one chooses a basis for W so that the points of $\mathcal{S}$ can be written as columns of a matrix (well defined up to rescaling the columns), then $k_{\mathcal{S}}$ will be the maximum number r such that all subsets of r column vectors of the corresponding matrix are linearly independent. This was Kruskal's original definition.

Theorem 3.3.2.2 (Kruskal [**195**]). *Let $T \in A \otimes B \otimes C$. Say T admits an expression $T = \sum_{i=1}^{r} u_i \otimes v_i \otimes w_i$. Let $\mathcal{S}_A = \{u_i\}$, $\mathcal{S}_B = \{v_i\}$, $\mathcal{S}_C = \{w_i\}$. If*

$$r \leq \frac{1}{2}(k_{\mathcal{S}_A} + k_{\mathcal{S}_B} + k_{\mathcal{S}_C}) - 1, \tag{3.3.1}$$

then T has rank r and its expression as a rank r tensor is essentially unique.

Note that Kruskal's theorem studies the sets of vectors appearing in T from each vector space A, B, C in isolation, not paying attention to the vectors from the other spaces that they are paired with. This indicates how one might improve upon the theorem. A proof of Kruskal's theorem is given in §12.5.

3.3.3. NWD: non-weak-defectivity. There is a test coming from algebraic geometry which assures that a general tensor of rank r has a unique decomposition as a sum of r rank one tensors, and it can also be used if one is handed a particular tensor written as a sum of r rank one tensors. Call this property r-*NWD*, for r-*not weakly defective*.

Roughly speaking, one takes the span of the tangent spaces to the set of rank one elements at each of the r points, and sees if a general hyperplane containing this span also contains the tangent space to the set of rank one elements at another point.

The symmetric tensors which are weakly defective have been classified.

Theorem 3.3.3.1 ([**86, 233, 14**])**.** *The spaces $S^d\mathbb{C}^{\mathbf{v}}$ where a general element of border rank k which is less than generic fails to have a unique decomposition as a sum of rank one symmetric tensors $(k, d, \mathbf{v})$ are:*

(i) $(k, 2, \mathbf{v})$, $k = 2, \ldots, \binom{\mathbf{v}+1}{2}$, $(5, 4, 3)$, $(9, 4, 4)$, $(14, 4, 5)$, $(7, 3, 5)$, *where there are infinitely many decompositions,*

and

(ii) $(9, 6, 3)$, $(8, 4, 4)$, *where there are finitely many decompositions.*

Theorem 3.3.3.2 ([**88**])**.** *Let $\mathbf{a} \leq \mathbf{b} \leq \mathbf{c}$. Let $T \in A \otimes B \otimes C$. If $\underline{\mathbf{R}}(T) \leq \frac{\mathbf{ab}}{16}$, then, with probability one, T has a unique CP decomposition into a sum of $\underline{\mathbf{R}}(T)$ rank one terms.*

More precisely, let $2^\alpha \leq \mathbf{a} < 2^{\alpha+1}$ and $2^\beta \leq \mathbf{b} < 2^{\beta+1}$ and assume $\underline{\mathbf{R}}(T) \leq 2^{\alpha+\beta-2}$. Then, with probability one, T has a unique CP decomposition into a sum of $\underline{\mathbf{R}}(T)$ rank one terms.

Compare this with Kruskal's theorem, which only assures a unique CP decomposition up to roughly $\frac{\mathbf{a}+\mathbf{b}+\mathbf{c}}{2}$. Also note that if $\mathbf{a} = \mathbf{b} = \mathbf{c} > 3$, then $\underline{\mathbf{R}}(T) \leq \lceil \frac{\mathbf{a}^3-1}{3\mathbf{a}-2} \rceil$, and if $\underline{\mathbf{R}}(T) \leq \lceil \frac{\mathbf{a}^3-1}{3\mathbf{a}-2} \rceil - 1$, then with probability one T has at most a finite number of decompositions into a sum of $\underline{\mathbf{R}}(T)$ rank one tensors.

3.4. First tests of border rank: flattenings

As mentioned earlier, although border rank is defined in terms of limits, it turns out to be an algebraic property. Thus we may test if a tensor has border rank at most r using polynomials. More precisely, by Definition 4.2.1.7, if a polynomial vanishes on on all tensors T of rank r, it will also vanish on a tensor that is a limit of tensors of rank r.

This section, and the next five sections, discuss tests for border rank and symmetric border rank in terms of polynomial equations. Essentially by definition, the tensors of border rank at most r are exactly the zero set of a collection of polynomials. The explicit polynomials are known in just a few cases.

The set of all homogeneous polynomials vanishing on $\hat{\sigma}_{r,A_1\otimes\cdots\otimes A_n}$, the set of tensors of border rank at most r in $A_1\otimes\cdots\otimes A_n$, forms an *ideal* in $S^\bullet(A_1\otimes\cdots\otimes A_n)^*$, that is, if $P_1, P_2 \in S^\bullet(A_1\otimes\cdots\otimes A_n)^*$ both vanish on $\hat{\sigma}_{r,A_1\otimes\cdots\otimes A_n}$, then $P_1Q_1 + P_2Q_2$ vanish on $\hat{\sigma}_{r,A_1\otimes\cdots\otimes A_n}$ for all $Q_1, Q_2 \in S^\bullet(A_1\otimes\cdots\otimes A_n)^*$. See Chapter 4 for more on this essential property. In this chapter, when I discuss polynomials, I just mean enough polynomials in the ideal such that their common zero locus is $\hat{\sigma}_{r,A_1\otimes\cdots\otimes A_n}$.

3.4.1. The subspace variety and flattenings. We reconsider the subspace variety from Definition 3.1.3.4:

$$\begin{aligned}
&\hat{S}ub_{\mathbf{b}_1,\ldots,\mathbf{b}_n}(A_1\otimes\cdots\otimes A_n)\\
&\qquad = \{T\in A_1\otimes\cdots\otimes A_n \mid \mathbf{R}_{\mathrm{multlin}}(T)\le(\mathbf{b}_1,\ldots,\mathbf{b}_n)\}\\
&\qquad = \{T\in A_1\otimes\cdots\otimes A_n \mid \dim(T(A_j^*))\le \mathbf{b}_j\ \forall 1\le j\le n\}\\
&\qquad = \{T\in A_1\otimes\cdots\otimes A_n \mid P(T)=0\\
&\qquad\qquad\qquad\qquad\qquad \forall P\in\Lambda^{\mathbf{b}_j+1}A_j^*\otimes\Lambda^{\mathbf{b}_j+1}A_{\hat{j}}^*\ \forall 1\le j\le n\}.
\end{aligned}$$

Here $(\mathbf{c}_1,\ldots,\mathbf{c}_n)\le(\mathbf{b}_1,\ldots,\mathbf{b}_n)$ if $\mathbf{c}_j\le\mathbf{b}_j$ for all j. To connect the second and third description with the original one (the first line), for the second, one variously thinks of T as a linear map $A_1^*\to A_2\otimes\cdots\otimes A_n,\ldots,A_n^*\to A_1\otimes\cdots\otimes A_{n-1}$ and requires the rank of the j-th map to be at most $\mathbf{b}_j$. In §2.7.3, I discussed the $(r+1)\times(r+1)$ minors of a linear map extensively. As a space of equations, they are

$$\Lambda^{r+1}A_j^*\otimes\Lambda^{r+1}(A_1\otimes\cdots\otimes A_{j-1}\otimes A_{j+1}\otimes\cdots\otimes A_n)^* =: \Lambda^{r+1}A_j^*\otimes\Lambda^{r+1}A_{\hat{j}}^*, \tag{3.4.1}$$

and the zero set of these equations is exactly the set of linear maps of rank at most r, yielding the third description.

Remark 3.4.1.1. The sum of the vector spaces of minors in the third description is not direct. With the aid of representation theory one can determine the exact redundancy among these equations. See Theorem 7.1.1.2.

We saw that $\hat{\sigma}_r\subset A_1\otimes\cdots\otimes A_n$, the set of tensors of border rank at most r, is contained in $\hat{S}ub_r(A_1\otimes\cdots\otimes A_n)$, so polynomials in the ideal of $\hat{S}ub_r(A_1\otimes\cdots\otimes A_n)$ furnish tests for membership in $\hat{\sigma}_r$. In general these will only be necessary conditions and further polynomials will be needed to get an exact test. They are sufficient only when $r=1$.

Exercise 3.4.1.2: Show that they are sufficient when $r=1$. ⊚

Consider $A_1\otimes\cdots\otimes A_n=(A_{i_1}\otimes\cdots\otimes A_{i_k})\otimes(A_{j_1}\otimes\cdots\otimes A_{j_{n-k}})=:A_I\otimes A_J$ where $I\cup J=\{1,\ldots,n\}$. Write $J=I^c$, as it is a complementary index set. We may consider $T\in A_1\otimes\cdots\otimes A_n$ as $T\in A_I\otimes A_{I^c}$, i.e., as a linear map

$A_I^* \to A_{I^c}$, and if $\underline{\mathbf{R}}(T) \leq r$, the rank of this linear map is at most r; i.e., the $(r+1)\times(r+1)$ minors of this map must vanish, i.e.,

$\Lambda^{r+1}A_I^* \otimes \Lambda^{r+1}A_{I^c}^* \subset S^{r+1}(A_1\otimes\cdots\otimes A_n)^*$ *furnish equations for* $\hat{\sigma}_r$.

These equations are called *equations of flattenings.*

Flattenings are of use for relatively small values of r. Consider the case of $\hat{\sigma}_{3,\mathbb{C}^3\otimes\mathbb{C}^3\otimes\mathbb{C}^3}$. This is not the ambient space by Theorem 3.1.4.3, so there are equations to be found. However there are no equations to be obtained from flattenings.

3.4.2. Flattenings are enough for "unbalanced" cases. If the dimensions of the vector spaces involved are such that the last is much greater than the rest, $A_1\otimes\cdots\otimes A_n$ will "look like" $V\otimes A_n$ with $V = A_1\otimes\cdots\otimes A_{n-1}$ from the perspective of rank and border rank. Here is a precise statement; a more general statement and proof are given in §7.3.1:

Theorem 3.4.2.1 ([**81**, Thm 2.4.2]). *Consider tensors in* $A_1\otimes\cdots\otimes A_n$, *where* $\dim A_s = \mathbf{a}_s$, $1 \leq s \leq n$. *Assume* $\mathbf{a}_n \geq \prod_{i=1}^{n-1}\mathbf{a}_i - \sum_{i=1}^{n-1}\mathbf{a}_i - n + 1$.

(1) *If* $\mathbf{a}_n > r \geq \prod_{i=1}^{n-1}\mathbf{a}_i - \sum_{i=1}^{n-1}\mathbf{a}_i - n + 1$, *then* $\hat{\sigma}_r$ *is the zero set of* $\Lambda^{r+1}(A_1\otimes\cdots\otimes A_{n-1})^*\otimes\Lambda^{r+1}A_n^*$.
(2) *If* $r \geq \mathbf{a}_1\cdots\mathbf{a}_{n-1}$, *then* $\hat{\sigma}_r = A_1\otimes\cdots\otimes A_n$.

3.5. Symmetric border rank

3.5.1. Symmetric flattenings. For $\phi \in S^dV$, define the partial polarizations

$$\phi_{s,d-s} \in S^sV\otimes S^{d-s}V. \tag{3.5.1}$$

Identify $\phi_{s,d-s}$ with the associated map $S^sV^* \to S^{d-s}V$. I generally restrict attention to $\phi_{s,d-s}$ for $1 \leq s \leq \lfloor d/2 \rfloor$ to avoid redundancies. If $\phi = x_1^d + \cdots + x_r^d$, then $\phi_{s,d-s} = x_1^s\otimes x_1^{d-s} + \cdots + x_1^s\otimes x_1^{d-s}$. One concludes:

Proposition 3.5.1.1. *For all* $1 \leq s \leq \lfloor d/2 \rfloor$, *the size* $r+1$ *minors of* $\phi_{s,d-s}$,

$$\Lambda^{r+1}(S^sV^*)\otimes\Lambda^{r+1}(S^{d-s}V^*),$$

furnish equations of degree $r+1$ *for* $\hat{\sigma}_{r,S^dV^*}$.

Exercise 3.5.1.2: Prove Proposition 3.5.1.1.

Such equations are called *symmetric flattenings.* The symmetric flattenings, also known as *minors of catalecticant matrices*, or simply *catalecticant minors*, date back to Macaulay [**224**].

Theorem 3.5.1.3 (Gundelfinger [**172**]). $\hat{\sigma}_{r,S^d\mathbb{C}^2}$ *is the zero set of any of the nontrivial size* $r+1$ *minors of the symmetric flattenings.*

Proposition 3.5.1.4. *Let $\delta = \lfloor \frac{d}{2} \rfloor$. Symmetric flattenings furnish nontrivial equations for $\hat{\sigma}_{r,S^dV}$ when*

$$r < \binom{\delta + \mathbf{v} - 1}{\delta}$$

and $r < \frac{1}{\mathbf{v}}\binom{\mathbf{v}-1+d}{d}$ (so that $\hat{\sigma}_{r,S^dV} \neq S^dV$). If r fails to satisfy the bounds, then symmetric flattenings do not furnish equations except when $d = 2$ (where there are flattenings for $r < \mathbf{v}$), and the pairs $(d, \mathbf{v})$ $(4, \mathbf{v})$, $3 \leq \mathbf{v} \leq 5$ where there are flattenings up to $r = \binom{\mathbf{v}+1}{2} - 1$.

Proof. The right hand side is the size of the maximal minors of $\phi_{\delta,d-\delta}$. ☐

Corollary 3.5.1.5 (Clebsch [**91**])**.** *For $3 \leq \mathbf{v} \leq 5$, the generic rank of elements of S^4V is $\binom{\mathbf{v}+1}{2}$.*

In other words, $\hat{\sigma}_{\binom{\mathbf{v}+1}{2}-1,S^4\mathbb{C}^{\mathbf{v}}}$ is not the ambient space when $\mathbf{v} = 3, 4, 5$.

The proof of Corollary 3.5.1.5 will be completed by Exercise 5.3.2.8.

Symmetric flattenings provide enough equations for $\hat{\sigma}_{r,S^d\mathbb{C}^2}$ for all r, d, for $\hat{\sigma}_{2,S^dV}$ for all $\mathbf{v}$, and for $\hat{\sigma}_{r,S^2V}$ for all $r, \mathbf{v}$, as well as $\hat{\sigma}_{3,S^dV}$ for all $\mathbf{v}$ and all $d \geq 4$. However, for $\hat{\sigma}_{3,S^3V}$, new equations are needed.

Exercise 3.5.1.6: Show that $\underline{\mathbf{R}}_S(\det_n) \geq \binom{n}{\lfloor \frac{n}{2} \rfloor}^2$ and $\underline{\mathbf{R}}_S(\mathrm{perm}_n) \geq \binom{n}{\lfloor \frac{n}{2} \rfloor}^2$.

3.5.2. Symmetric ranks and border ranks of elements in $S^d\mathbb{C}^{\mathbf{v}}$.

Theorem 3.5.2.1 (Comas-Seiguer [**95**])**.** *Let $\phi \in S^d\mathbb{C}^2$, so the maximum possible border rank of ϕ is $\lfloor \frac{d+1}{2} \rfloor$. If $\underline{\mathbf{R}}_S(\phi) = r$, then either $\mathbf{R}_S(\phi) = r$ or $\mathbf{R}_S(\phi) = d - r + 2$. In the first case, the expression of ϕ as a sum of $\mathbf{R}_S(\phi)$ d-th powers is unique, unless $r = \frac{d}{2}$, in which case there are finitely many expressions. In the second case, one has parameters worth of choices in the expression.*

A partial generalization of Theorem 3.5.2.1 to higher dimensions is as follows:

Theorem 3.5.2.2 ([**47**])**.** *Let $\phi \in S^dV$. If $\mathbf{R}_S(\phi) \leq \frac{d+1}{2}$, then $\mathbf{R}_S(\phi) = \underline{\mathbf{R}}_S(\phi)$ and the expression of ϕ as a sum of $\mathbf{R}_S(\phi)$ d-th powers is unique.*

3.5.3. Sylvester's algorithm (with a shortcut). By Theorem 3.5.1.3 the equations for $\hat{\sigma}_{r,S^d\mathbb{C}^2}$ are the size $r+1$ minors of any nontrivial flattening. By the Comas-Seguir Theorem 3.5.2.1, any $\phi \in S^d\mathbb{C}^2$ of symmetric border rank r either has symmetric rank r or $d-r+2$. When the symmetric rank is r, the decomposition is unique except for when d is even and $r = \frac{d}{2}$. Here is an algorithm (whose origin dates back to Sylvester) to find the decomposition.

Given $\phi \in S^dW = S^d\mathbb{C}^2$, consider the flattening $\phi_{1,d-1} : W^* \to S^{d-1}W$. If it has a kernel generated by $\ell \in W^*$, then $\phi = x^d$ for some x in the line

$\ell^{\perp} \subset W$. Next, if $\phi_{1,d-1}$ is injective, consider $\phi_{2,d-2} : S^2W^* \to S^{d-2}W$. If it has a kernel, the kernel is one-dimensional, the span of some $Q \in S^2W^*$. If Q has distinct roots, i.e., $Q = \ell_1\ell_2$, then the rank is two, and $\phi = x_1^d + x_2^d$, where $x_j \in \ell_j{}^{\perp}$. Otherwise $R(\phi) = d$, and its decomposition will not be unique. To find a decomposition, consider the linear map $\phi_{d,0} : S^dW^* \to S^0W = \mathbb{C}$. Its kernel will contain (parameters worth of) polynomials $P \in S^dW$ with distinct roots; write $P = \ell_1 \cdots \ell_d$. Then $\phi = x_1^d + \cdots + x_d^d$, where $x_j \in \ell_j{}^{\perp}$ for any such P.

If $\phi_{2,d-2}$ is injective, consider $\phi_{3,d-3}$; if it has a kernel, the kernel will be 1-dimensional and either the kernel $Q = \ell_1\ell_2\ell_3$ with each ℓ_i distinct, in which case the rank is three and $\phi = \sum_{j=1}^3 x_j^d$, where $x_j \in \ell_j{}^{\perp}$, or $R(\phi) = d-1$, in which case one can find a decomposition of ϕ by taking an element of the kernel of $\phi_{d-1,1} : S^{d-1}W^* \to W$ that has distinct roots, say $Q = \ell_1 \cdots \ell_{d-1}$ and $\phi = \sum_{j=1}^{d-1} x_j^d$, where $x_j \in \ell_j{}^{\perp}$. There are parameters worth of choices in this latter case.

One continues similarly: if $\phi_{j,d-j}$ is injective, one considers $\phi_{j+1,d-j-1}$. The algorithm terminates at the latest at $\phi_{\lfloor \frac{d}{2} \rfloor, \lceil \frac{d}{2} \rceil}$. Note that if d is even, the map $\phi_{\frac{d}{2},\frac{d}{2}}$ will not have a kernel for general tensors and one can take any polynomial with distinct roots in the two-dimensional kernel of $\phi_{\frac{d}{2}+1,\frac{d}{2}-1}$, so the decomposition will not be unique.

For example, let $\mathbb{C}^2$ have coordinates x, y and $\mathbb{C}^{2*}$ dual coordinates e, f. If $\ker \phi_{2,d-2}$ is spanned by $(s_1e + t_1f)(s_2e + t_2f)$, then $\phi = \lambda_1(t_1x - s_1y)^d + \lambda_2(t_2x - s_2y)^d$ for some λ_1, λ_2.

For an explicit example, let $\phi = x^3 + xy^2$. Then $\phi_{1,2} = 3x \otimes x^2 + x \otimes y^2 + 2y \otimes xy$, and $\phi_{1,2}$ is injective. Under $\phi_{2,1}$ we have

$$\begin{aligned} e^2 &\mapsto 3x, \\ ef &\mapsto 2y, \\ f^2 &\mapsto x, \end{aligned}$$

so the kernel is generated by $(e^2 - 3f^2) = (e + \sqrt{3}f)(e - \sqrt{3}f)$ and one computes $\phi = \frac{1}{6\sqrt{3}}(\sqrt{3}x + y)^3 + \frac{1}{6\sqrt{3}}(\sqrt{3}x - y)^3$.

Further exact decomposition algorithms are presented in §12.4.

3.6. Partially symmetric tensor rank and border rank

Definition 3.6.0.1. Say $T \in S^{d_1}A_1 \otimes \cdots \otimes S^{d_n}A_n$. Define the *rank of* T *as a partially symmetric tensor* as the smallest r such that there exist $a_j^s \in A_j$, $1 \leq s \leq r$, such that

$$T = (a_1^1)^{d_1} \otimes \cdots \otimes (a_n^1)^{d_n} + \cdots + (a_1^r)^{d_1} \otimes \cdots \otimes (a_n^r)^{d_n}$$

and define the *partially symmetric border rank* similarly. Let

$$\hat{\sigma}_{r,S^{d_1}A_1\otimes\cdots\otimes S^{d_n}A_n}$$

denote the set of elements of $S^{d_1}A_1\otimes\cdots\otimes S^{d_n}A_n$ of partially symmetric border rank at most r.

The expected generic partially symmetric rank of $T\in S^{d_1}A_1\otimes\cdots\otimes S^{d_n}A_n$ is

$$\lceil\frac{\binom{d_1+\mathbf{a}_1-1}{d_1}\cdots\binom{d_n+\mathbf{a}_n-1}{d_n}}{\mathbf{a}_1+\cdots+\mathbf{a}_n-n+1}\rceil. \tag{3.6.1}$$

In [**3, 2**], the authors show that this expected value often occurs. See §5.5.3 for what is known about generic partially symmetric rank.

Considering $S^{d_1}A_1\otimes\cdots\otimes S^{d_n}A_n\subset A_1^{\otimes d_1}\otimes\cdots\otimes A_n^{\otimes d_n}$ one obtains immediately that the partially symmetric (border) rank of a tensor is at least its (border) rank as a tensor. It is an open question as to whether equality holds, see §5.7.2.

Remark 3.6.0.2. Let V be a vector space with a tensor product structure, e.g., $V=S^{d_1}A_1\otimes\cdots\otimes S^{d_n}A_n$. In Chapter 5 we will see uniform definitions for rank, symmetric rank, and partially symmetric rank, as well as border rank, namely the notion of X-rank (and X-border rank) where $\hat{X}\subset V$ is the set of simplest elements in the space V.

3.7. Two useful techniques for determining border rank

In this section I briefly mention two techniques for determining equations of $\hat{\sigma}_r$ that are discussed in detail in Chapter 7. The first, *inheritance*, allows one to reduce the search for equations to certain "atomic" cases; the second, *prolongation*, is a method that in principle finds equations, but can be difficult to implement without the help of other tools such as representation theory.

3.7.1. Inheritance.

Theorem 3.7.1.1. *If* $\dim A_j\geq r$*, then polynomials defining* $\hat{\sigma}_r\subset A_1\otimes\cdots\otimes A_n$ *are given by*

- *The equations for* $\hat{S}ub_r(A_1\otimes\cdots\otimes A_n)$*, i.e.,* $\Lambda^{r+1}A_j^*\otimes\Lambda^{r+1}A_{\hat{j}}^*$*,*
- *and the following equations: fix bases in each* A_j^**, choose all possible subsets of* r *basis vectors in each space and consider the resulting* $\mathbb{C}^{r*}\otimes\cdots\otimes\mathbb{C}^{r*}$*. Take the polynomials defining the corresponding* $\hat{\sigma}_{r,(\mathbb{C}^r)^{\otimes n}}$ *and consider them as equations on* $A_1\otimes\cdots\otimes A_n$*.*

See Proposition 7.4.1.1 for the proof and an invariant (i.e., without choices of bases) statement.

Analogs of the above result hold for symmetric tensor rank as well; see §7.4.2.

3.7.2. Prolongation. Prolongation is a general technique for finding equations of $\hat{\sigma}_r$. It is discussed in §7.5. Here is the basic idea:

Say P is a homogeneous polynomial of degree d on $A_1 \otimes \cdots \otimes A_n$ that vanishes on $\hat{\sigma}_2$, i.e., $P(T_1 + sT_2) = 0$ for all $T_1, T_2 \in \hat{\sigma}_1$ and $s \in \mathbb{C}$. Consider P as a multilinear form:

$$\overline{P}((T_1 + sT_2)^d) = \overline{P}((T_1)^d) + s\overline{P}((T_1)^{d-1}T_2) + \cdots + s^d\overline{P}(T_2^d).$$

Since we are allowed to vary s, each of these terms must vanish separately. This is the essence of the technique.

Consider the special case of $d = 2$. To have a degree two equation for $\hat{\sigma}_2$, one would need a quadratic form $\overline{P}$ such that $\overline{P}(T_1, T_2) = 0$ for all rank one tensors T_1, T_2. But one can take a basis of $A_1 \otimes \cdots \otimes A_n$ consisting of rank one tensors, thus any such $\overline{P}$ must be zero. Thus there are no nonzero degree two homogeneous polynomials vanishing on $\hat{\sigma}_{2,A_1 \otimes \cdots \otimes A_n}$ or $\hat{\sigma}_{2,S^dV}$. More generally (see §7.5):

Proposition 3.7.2.1. *There are no nonzero degree $d \leq r$ homogeneous polynomials vanishing on $\hat{\sigma}_{r,A_1 \otimes \cdots \otimes A_n}$ or $\hat{\sigma}_{r,S^dV}$.*

Now consider the case $d = 3$. Since $\overline{P}(T_1, T_1, T_2) = 0$ for all T_1, T_2, recalling from equation (2.6.6) that $\overline{P}(\cdot, \cdot, T_2)$ is a derivative of P, we conclude

Proposition 3.7.2.2. *A necessary and sufficient condition that a degree three homogeneous polynomial vanishes on $\hat{\sigma}_2$ is that $\frac{\partial P}{\partial x}$ is an equation for $\hat{\sigma}_1$ for all $x \in A_1 \otimes \cdots \otimes A_n$.*

In other words, letting $I_2(\hat{\sigma}_1) \subset S^2(A_1 \otimes \cdots \otimes A_n)^*$ denote the space of quadratic polynomials vanishing on $\hat{\sigma}_1$ (the space spanned by the two by two minors of flattenings), the space of degree three homogenous polynomials vanishing on $\hat{\sigma}_2$ is

$$S^3(A_1 \otimes \cdots \otimes A_n)^* \cap [I_2(\hat{\sigma}_1) \otimes (A_1 \otimes \cdots \otimes A_n)^*].$$

Exercise 3.7.2.3: Verify that the "In other words" assertion is equivalent to the first assertion of Proposition 3.7.2.2.

Exercise 3.7.2.4: State and prove an analogous result for $\hat{\sigma}_{2,S^dV}$.

Using prolongation, one can find degree four equations for $\hat{\sigma}_3 \subset A \otimes B \otimes C$ that are not minors of flattenings, see Proposition 7.5.5.4. In the next section these equations are derived by direct methods.

I record the following equations that were only found by representation theory. They are discussed in §6.8.7:

Proposition 3.7.2.5 ([**205**]). *There are degree six equations that vanish on* $\hat{\sigma}_{4,\mathbb{C}^3\otimes\mathbb{C}^3\otimes\mathbb{C}^4}$.

3.8. Strassen's equations and variants

By Theorem 3.1.1.1, if $T \in A\otimes B\otimes C$, then $\mathbf{R}(T)$ is the smallest number of rank one elements of $B\otimes C$ needed to span a linear space containing $T(A^*) \subset B\otimes C$. V. Strassen [**300**] discovered equations for tensors of bounded border rank beyond $\Lambda^{r+1}A^*\otimes\Lambda^{r+1}(B\otimes C)^*$ and permutations by using $T(A^*)$. I first present a version of (some of) Strassen's equations due to G. Ottaviani, and then Strassen's original formulation.

3.8.1. Ottaviani's derivation of Strassen's equations. To obtain more tests for the border rank of $T \in A\otimes B\otimes C$, i.e., $T : B^* \to A\otimes C$, one can augment it. Consider $T\otimes \mathrm{Id}_A : A\otimes B^* \to A\otimes A\otimes C$. Here $T\otimes \mathrm{Id}_A$ has flattenings, but one now needs to determine if they have any relation to the border rank of T.

Due to the $GL(A) \times GL(B) \times GL(C)$-invariance, one should really consider the two projections of $T\otimes \mathrm{Id}_A$, $T_A^\wedge : A\otimes B^* \to \Lambda^2 A\otimes C$ and $T_A^s : A\otimes B^* \to S^2A\otimes C$.

Assume $\mathbf{a} = 3$. If we choose bases of A, B, C and write $T = a_1\otimes X_1 + a_2\otimes X_2 + a_3\otimes X_3$, where $X_j : B^* \to C$ is a $\mathbf{b} \times \mathbf{c}$ matrix, then the matrix representation of $T_A^\wedge$ is, in terms of block matrices,

$$T_A^\wedge = \begin{pmatrix} 0 & X_3 & -X_2 \\ -X_3 & 0 & X_1 \\ X_2 & -X_1 & 0 \end{pmatrix}. \tag{3.8.1}$$

The odd size minors of this matrix provide a convenient expression for Strassen's equations in coordinates. However, the degree of the first equations will be seven, whereas they have a zero set equivalent to Strassen's degree four equations described below. (The size nine and larger minors have the same degree as Strassen's.)

Theorem 3.8.1.1 ([**256**]). *Let* $T \in A\otimes B\otimes C$. *Assume* $3 \le \mathbf{a} \le \mathbf{b} \le \mathbf{c}$.

(1) *If* $\underline{\mathbf{R}}(T) \le r$, *then* $\mathrm{rank}(T_A^\wedge) \le r(\mathbf{a}-1)$.

(2) *If* $T \in A\otimes B\otimes C$ *is generic and* $\mathbf{a} = \dim A = 3$, $\mathbf{b} = \dim B = \dim C \ge 3$, *then* $\mathrm{rank}(T_A^\wedge) = 3\mathbf{b}$.

Thus for k *even, the* $(k+1)\times(k+1)$ *minors of* $T_A^\wedge$ *furnish equations for* $\hat{\sigma}_{\frac{k}{2}}$. *In particular, one obtains equations for* $\hat{\sigma}_{r,A\otimes B\otimes C}$ *up to* $r = \frac{3\mathbf{b}}{2} - 1$.

Proof. If $T = a\otimes b\otimes c$, then image$(A_T^{\wedge 2}) = a \wedge A\otimes c$ and thus rank$(A_T^{\wedge 2}) = \dim A - 1$ and (1) follows by the remark at the beginning of §3.4 as

$$\mathrm{rank}(A^{\wedge 2}_{T_1+T_2}) \leq \mathrm{rank}(A^{\wedge 2}_{T_1}) + \mathrm{rank}(A^{\wedge 2}_{T_2}).$$

For (2), take, e.g., $T = a_1\otimes \sum_{j=1}^{\mathbf{b}}(b_j\otimes c_j) + a_2\otimes \sum_{j=1}^{\mathbf{b}}(\lambda_j b_j\otimes c_{j-1}) + a_3\otimes \sum_{j=1}^{\mathbf{b}}(\mu_j b_j\otimes c_{j+1})$, where a_s, b_j, c_j are bases of A, B, C respectively with the conventions $c_0 = c_{\mathbf{b}}$, $c_{\mathbf{b}+1} = c_1$ and the μ_j, λ_j are all distinct and not 1, and check that rank$(T_A^{\wedge}) = 3\mathbf{b}$. □

Corollary 3.8.1.2 (Strassen)**.** $\hat{\sigma}_{4,\mathbb{C}^3\otimes\mathbb{C}^3\otimes\mathbb{C}^3} \neq \mathbb{C}^3\otimes\mathbb{C}^3\otimes\mathbb{C}^3$.

Proof. A generic tensor $T \in \mathbb{C}^3\otimes\mathbb{C}^3\otimes\mathbb{C}^3$ will not satisfy $\det(T_A^{\wedge}) = 0$. □

Exercise 3.8.1.3: What equations arise from T_A^s?

Here is a generalization:

Theorem 3.8.1.4. *Let A, B, C be vector spaces with $\mathbf{a} = 2p+1 \leq \mathbf{b} \leq \mathbf{c}$. Consider the map*

$$T_A^{\wedge p} : \Lambda^p A\otimes B^* \to \Lambda^{p+1}A\otimes C.$$

If T has rank one, then rank$(T_A^{\wedge p}) = \binom{2p}{p}$. *If T is generic, then* rank$(T_A^{\wedge p}) = \binom{2p+1}{p}\mathbf{b}$. *Thus the size $(r+1)\binom{2p}{p}$ minors of $T_A^{\wedge p}$ furnish equations for $\hat{\sigma}_r$ up to $r = \frac{2p+1}{p+1}\mathbf{b}$.*

Theorem 3.8.1.4 follows from the results in [**215**].

Other generalizations of these flattened augmentations of T are given in Chapter 7. The polynomials of Proposition 3.8.1.1 are written down explicitly in §3.8.4.

Remark 3.8.1.5. Using inheritance, and exchanging the role of A with B, C, one obtains three different sets of equations. Are they the same equations or different? Using representation theory, we will determine the answer, see §7.6.4.

Aside 3.8.1.6 (A wiring diagram for Ottaviani's derivation of Strassen's equations)**.** Figure 3.8.1 is a wiring diagram that expresses Ottaviani's version of Strassen's equations.

The two strands going into the black skew-symmetrization box, are one from the tensor T, and one from Id_A. The right funnel stands for considering the output of the black box (an element of $\Lambda^2 A$), tensored with the C component of T as an element of a single vector space, and the left funnel, the $A^*\otimes B$ component as a single vector space. One then performs the rank r test.

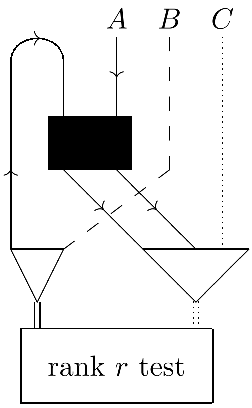


Figure 3.8.1. Ottaviani's Strassen.

3.8.2. Original formulation of Strassen's equations. Before presenting Strassen's original formulation, here is a warm up problem: say we wanted to show directly that the border rank of a generic element $T \in \mathbb{C}^2 \otimes \mathbb{C}^{\mathbf{b}} \otimes \mathbb{C}^{\mathbf{b}}$ was $\mathbf{b}$. We may choose bases such that

$$T = a_1 \otimes \Big(\sum_{i=1}^{\mathbf{b}} b_i \otimes c_i \Big) + a_2 \otimes \Big(\sum_{i=1,j}^{\mathbf{b}} x^{ij} b_i \otimes c_j \Big),$$

where (x^{ij}) is a generic matrix. (I remind the reader that the summation convention is in use.) The change of bases in B and C used to make the coefficient of a_1 the identity still leaves $GL_{\mathbf{b}} \subset GL(B) \times GL(C)$ acting to normalize x by conjugation. But if x is generic, it can be diagonalized by $GL_{\mathbf{b}}$. Say it has eigenvalues $\lambda_1, \dots, \lambda_{\mathbf{b}}$; then

$$T = \sum_{i=1}^{\mathbf{b}} (a_1 + \lambda_i a_2) \otimes (b_i \otimes c_i), \tag{3.8.2}$$

so T has rank at most $\mathbf{b}$.

Now for Strassen's equations: Assume for the moment that $\dim A = 3$ and $\dim B = \dim C \geq 3$, and that T is concise. In particular $\underline{\mathbf{R}}(T) \geq \mathbf{b}$. Under these conditions, there exist bases $(a_s), (b_i), (c_j)$ of A, B, C such that

$$T = a_1 \otimes (b_1 \otimes c_1 + \cdots + b_{\mathbf{b}} \otimes c_{\mathbf{b}}) + a_2 \otimes (x^{ij} b_i \otimes c_j) + a_3 \otimes (y^{ij} b_i \otimes c_j), \tag{3.8.3}$$

where $1 \leq i, j \leq \mathbf{b}$, $x^{ij}, y^{ij} \in \mathbb{C}$. Hence $T(A^*)$, expressed in matrices, is the span of the identity matrix, (x^{ij}) and (y^{ij}). It will be convenient to use the identity matrix to identify $B \simeq C^*$ (so c_j becomes b_j^*), and then one can think of x, y as linear maps $B \to B$. Assume for the moment further that x, y are diagonalizable; if they are moreover simultaneously diagonalizable, then I claim that $\mathbf{R}(T) = \mathbf{b}$. To see this let $(b'_j), (c'_k)$ be bases in which they are diagonalized. Then

$$\begin{aligned} T &= a_1 \otimes (b'_1 \otimes c'_1 + \cdots + b'_{\mathbf{b}} \otimes c'_{\mathbf{b}}) + a_2 \otimes (\lambda_1 b'_1 \otimes c'_1 + \cdots + \lambda_{\mathbf{b}} b'_{\mathbf{b}} \otimes c'_{\mathbf{b}}) \\ &\quad + a_3 \otimes (\mu_1 b'_1 \otimes c'_1 + \cdots + \mu_{\mathbf{b}} b'_{\mathbf{b}} \otimes c'_{\mathbf{b}}) \\ &= (a_1 + \lambda_1 a_2 + \mu_1 a_3) \otimes b'_1 \otimes c'_1 + \cdots + (a_1 + \lambda_{\mathbf{b}} a_2 + \mu_{\mathbf{b}} a_3) \otimes b'_{\mathbf{b}} \otimes c'_{\mathbf{b}}. \end{aligned}$$

I now use this observation to obtain equations, in particular to go from rank to border rank. Recall that a pair of diagonalizable matrices are simultaneously diagonalizable if and only if they commute. Thus one should consider the commutator $[x, y]$. The above discussion implies:

Proposition 3.8.2.1 ([**300**])**.** *Let* $\mathbf{a} = 3$, $\mathbf{b} = \mathbf{c} \geq 3$, *let* $T \in A \otimes B \otimes C$ *be concise and choose bases to express* T *as in* (3.8.3). *Then* $\underline{\mathbf{R}}(T) = \mathbf{b}$ *if and only if* $[x, y] = 0$.

Exercise 3.8.2.2: Prove Proposition 3.8.2.1. ⊚

More generally,

Theorem 3.8.2.3 (Strassen's equations, original formulation, [**300**])**.** *Let* $\mathbf{a} = 3$, $\mathbf{b} = \mathbf{c} \geq 3$, *let* $T \in A \otimes B \otimes C$ *be concise, and choose bases to express* T *as in* (3.8.3). *Then* $\underline{\mathbf{R}}(T) \geq \frac{1}{2} \operatorname{rank}([x, y]) + \mathbf{b}$.

I prove Theorem 3.8.2.3 and show that the two formulations of Strassen's equations are equivalent in §7.6.

It will be useful to rephrase Theorem 3.8.2.3 as follows:

Theorem 3.8.2.4 ([**300**])**.** *Let* $\mathbf{a} = 3$, $\mathbf{b} = \mathbf{c} \geq 3$, *and let* $T \in A \otimes B \otimes C$ *be concise. Let* $\alpha \in A^*$ *be such that* $T_\alpha := T(\alpha) : C^* \to B$ *is invertible. For* $\alpha_j \in A^*$, *write* $T_{\alpha,\alpha_j} := {T_\alpha}^{-1} T_{\alpha_j} : B \to B$. *Then for all* $\alpha_1, \alpha_2 \in A^*$,

$$\operatorname{rank}[T_{\alpha,\alpha_1}, T_{\alpha,\alpha_2}] \leq 2(\underline{\mathbf{R}}(T) - \mathbf{b}).$$

Strassen's test is presented in terms of polynomials in coordinates in §3.8.4. A further discussion and generalizations of Strassen's equations are in §7.6.

Theorem 3.8.2.5 ([**207**])**.** *Let* $\mathbf{b} = 3$ *or* 4. *With the above notations, if there exists* $\alpha \in A^*$ *such that* T_α *is invertible, then Strassen's degree* $\mathbf{b} + 1$ *equations are sufficient to determine if* $T \in \hat{\sigma}_{\mathbf{b}}$.

The proof of Theorem 3.8.2.5 rests on the fact that Strassen's equations are for abelian subalgebras of $\operatorname{End}(C)$, and when $\mathbf{b} = 3, 4$, any abelian subalgebra may be approximated arbitrarily closely by a diagonalizable subalgebra. As we have seen above, if the subalgebra is diagonalizable, the corresponding tensor has rank $\mathbf{b}$.

3.8.3. A lower bound for the border rank of matrix multiplication via Strassen's equations. Recall the notation $M_{m,m,m} : \mathbb{C}^{m^2} \times \mathbb{C}^{m^2} \to \mathbb{C}^{m^2}$ for the matrix multiplication operator.

Corollary 3.8.3.1 (Strassen [**300**]). $\underline{\mathbf{R}}(M_{m,m,m}) \geq \frac{3m^2}{2}$.

Proof. If one writes out $M_{m,m,m}$ explicitly in a good basis and takes a generic $\alpha \in A^* = Mat_{m\times m}$, then the corresponding linear map T_α is a block diagonal matrix with blocks of size m, each block identical and the entries of the block arbitrary. So $\operatorname{rank}([T_{\alpha,\alpha_1}, T_{\alpha,\alpha_2}]) = m^2$, hence $m^2 \leq 2(\underline{\mathbf{R}}(M_{m,m,m}) - m^2)$ and the result follows. ☐

Exercise 3.8.3.2: Note that if $\mathbf{a} = \mathbf{b} = \mathbf{c} = 2p+1$, Theorem 3.8.1.4 provides equations for $\hat{\sigma}_{2\mathbf{a}-2}$. Why do these equations fail to show that the border rank of $M_{3,3,3}$ is at least 18?

Strassen's equations are also the key to the proof of Bläser's $\frac{5}{2}$-Theorem 11.5.0.1.

3.8.4. Strassen's equations in bases. In Theorem 3.8.2.4 one needed to make a choice of α with $T(\alpha)$ invertible, because we wanted to take inverses. For a linear map $f : V \to W$ between vector spaces of the same dimension $\mathbf{v}$, recall from §2.6.9 that $f^{-1} = f^{\wedge \mathbf{v}-1}\otimes(\det f)$, so use $T(\alpha)^{\wedge(\mathbf{b}-1)}$ instead of $T(\alpha)^{-1}$, where $T(\alpha)^{\wedge \mathbf{b}-1} \in \Lambda^{\mathbf{b}-1}B\otimes\Lambda^{\mathbf{b}-1}C$. Write

$$T^{\alpha,\alpha_1} := T(\alpha)^{\wedge \mathbf{b}-1} \circ T(\alpha_1) : \Lambda^{\mathbf{b}}B\otimes C \to \Lambda^{\mathbf{b}}C\otimes C. \tag{3.8.4}$$

Here are polynomials corresponding to Strassen's commutator being of rank at most w: Let $\alpha_1, \alpha_2, \alpha_3$ be a basis of A^*, and $\beta_1, \dots, \beta_{\mathbf{b}}$, $\xi_1, \dots, \xi_{\mathbf{b}}$ bases of B^*, C^*. Consider the element

$$P = \alpha_2 \wedge \alpha_3\otimes(\alpha_1)^{\wedge(\mathbf{b}-1)}\otimes\beta_1 \wedge \cdots \wedge \beta_{\mathbf{b}}\otimes\beta_s\otimes\xi_1 \wedge \cdots \wedge \xi_{\mathbf{b}}\otimes\xi_t.$$

This expands to (ignoring scalars)

$$\begin{aligned} P &= (\alpha_2\otimes\alpha_3 - \alpha_3\otimes\alpha_2)\otimes(\alpha_1)^{\wedge(\mathbf{b}-1)}\otimes\Big(\sum_j (-1)^{j+1}\beta_{\hat{j}}\otimes\beta_j\otimes\beta_s\Big) \\ &\quad \otimes\Big(\sum_k (-1)^{k+1}\xi_{\hat{k}}\otimes\xi_k\otimes\xi_t\Big) \\ &= (-1)^{j+k}[((\alpha_1)^{b-1}\otimes\beta_{\hat{j}}\otimes\xi_{\hat{k}})\otimes(\alpha_2\otimes\beta_j\otimes\xi_t)\otimes(\alpha_3\otimes\beta_s\otimes\xi_k) \\ &\qquad - ((\alpha_1)^{b-1}\otimes\beta_{\hat{j}}\otimes\xi_{\hat{k}})\otimes(\alpha_3\otimes\beta_j\otimes\xi_t)\otimes(\alpha_2\otimes\beta_s\otimes\xi_k)]. \end{aligned}$$

A hat over an index indicates the wedge product of all vectors in that index range except the hatted one. Choose dual bases for A, B, C and write $T = a_1\otimes X + a_2\otimes Y + a_3\otimes Z$, where the a_j are dual to the α_j and X, Y, Z are represented as $\mathbf{b}\times\mathbf{b}$ matrices with respect to the dual bases of B, C. Then $P(T)$ is represented by the matrix

$$P(T)^s_t = \sum_{j,k}(-1)^{j+k}(\det X^{\hat{j}}_{\hat{k}})(Y^j_t Z^s_k - Y^s_k Z^j_t), \tag{3.8.5}$$

where $X^{\hat{j}}_{\hat{k}}$ is X with its j-th row and k-th column removed.

Strassen's commutator has rank at most w if and only if all $(w+1)\times(w+1)$ minors of $P(T)$ are zero.

3.8.5. History of Strassen's equations and variants. The first instance of a Strassen-like equation is probably the *Frahm-Toeplitz invariant* (see [**256**]), which is a degree six equation on $S^2\mathbb{C}^4\otimes\mathbb{C}^3$ for $\hat{\sigma}_{4,S^3\mathbb{C}^4\otimes\mathbb{C}^3}$, i.e., a partially symmetrized version of Strassen's equation for $\hat{\sigma}_{4,\mathbb{C}^3\otimes\mathbb{C}^4\otimes\mathbb{C}^4}$. It is given as the condition that a three-dimensional space of quadratic polynomials can appear as the space of partials of a cubic polynomial in 4 variables $F\in S^3\mathbb{C}^4$, that is, as the image of the polarization $F_{1,2}$. Previous to Strassen, Barth [**16**] rediscovered the Frahm-Toeplitz invariant in the context of his work on vector bundles on $\mathbb{P}^2$. The Aronhold invariant, see §3.10.1, is the fully symmetrized version of Strassen's equation.

3.9. Equations for small secant varieties

3.9.1. Equations for $\hat{\sigma}_2$. When $r=2$, the flattenings of §3.4.1 are enough to cut out $\hat{\sigma}_2$:

Theorem 3.9.1.1 ([**205**]). *$\hat{\sigma}_2\subset A_1\otimes\cdots\otimes A_n$ is the zero set of the 3×3 minors of flattenings, $\Lambda^3A^*_I\otimes\Lambda^3A^*_{I^c}$.*

Proof. I will take an unknown tensor T satisfying the equations and show it is a point of $\hat{\sigma}_2$. Recall from §3.4.1 that the equations of $Sub_{2,\dots,2}$ are $\Lambda^3A^*_j\otimes\Lambda^3A^*_{\hat{j}}$ which are included among the modules obtained by flattenings. Thus there exist $A'_j\subset A_j$, with $\dim A'_j=2$, such that $T\in A'_1\otimes\cdots\otimes A'_n$ and it is sufficient to study the case $\dim A_j=2$ for all j. The three factor case follows from Theorem 3.1.4.3.

What follows is the case $n=4$; the general case is left as an exercise. Write $T\in A\otimes B\otimes C\otimes D$. Using the equations $\Lambda^3(A\otimes B)^*\otimes\Lambda^3(C\otimes D)^*$, one may write $T=M_1\otimes S_1+M_2\otimes S_2$ with $M_j\in A\otimes B$ and $S_j\in C\otimes D$ (or $T=M_1\otimes S_1$, in which case one is easily done). This is because in $\hat{\sigma}_{2,V\otimes W}$ every point is of the form $v_1\otimes w_1+v_2\otimes w_2$ or $v_1\otimes w_1$.

Case 1. M_j both have rank 1 as elements of $A\otimes B$. One is reduced to the three-factor case, as without loss of generality one may write $M_1=a_1\otimes b_1$, $M_2=a_2\otimes b_2$; thus if $T\in\hat{\sigma}_{1,(A\otimes B)\otimes C\otimes D}$, it is in $\hat{\sigma}_{2,A\otimes B\otimes C\otimes D}$. (Were we in the n-factor case, at this point we would be reduced to $n-1$ factors.)

Case 2. M_1 has rank two, and using M_1 to identify $A \simeq B^*$ so it becomes the identity map, assume $M_2 : B \to B$ is diagonalizable. Then write

$$\begin{aligned} T &= (a_1{\otimes}b_1 + a_2{\otimes}b_2){\otimes}S_1 + (\lambda_1 a_1{\otimes}b_1 + \lambda_2 a_2{\otimes}b_2){\otimes}S_2 \\ &= a_1{\otimes}b_1{\otimes}(S_1 + \lambda_1 S_2) + a_2{\otimes}b_2{\otimes}(S_1 + \lambda_2 S_2) =: a_1{\otimes}b_1{\otimes}S + a_2{\otimes}b_2{\otimes}\tilde{S}, \end{aligned}$$

and one is reduced to the previous case.

Case 3. M_1 has rank two, and using M_1 to identify $A \simeq B^*$ so it becomes the identity map, $M_2 : B \to B$ is not diagonalizable. By Jordan canonical form we may write:

$$\begin{aligned} T &= (a_1{\otimes}b_1 + a_2{\otimes}b_2){\otimes}S_1 + (a_1{\otimes}b_2 + \lambda(a_1{\otimes}b_1 + a_2{\otimes}b_2)){\otimes}S_2 \\ &= (a_1{\otimes}b_1 + a_2{\otimes}b_2){\otimes}S + a_1{\otimes}b_2{\otimes}\tilde{S}. \end{aligned}$$

For $\delta \in D^*$, consider $T(\delta) \in A{\otimes}B{\otimes}C$. Since $T \in \hat{\sigma}_{2,(A\otimes B\otimes C)\otimes D}$, $\dim T(D^*)$ is at most two. This implies $S, \tilde{S}$ must both be of rank one. Write $S = c_1{\otimes}d_1$, $\tilde{S} = c_2{\otimes}d_2$.

Finally, since the rank of $T : (A{\otimes}D)^* \to B{\otimes}C$ is at most two, either $c_2 = \lambda c_1$ or $d_2 = \lambda d_1$, so after absorbing the constant λ and possibly relabeling, we have

$$T = a_1{\otimes}b_1{\otimes}c_1{\otimes}d_1 + a_2{\otimes}b_2{\otimes}c_1{\otimes}d_1 + a_1{\otimes}b_2{\otimes}c_1{\otimes}d_2,$$

but this point is tangent to $\hat{\sigma}_1$ at $a_1{\otimes}b_2{\otimes}c_1{\otimes}d_1$, i.e., it is

$$\lim_{t\to 0} \frac{1}{t}[(a_1 + ta_2){\otimes}(b_2 + tb_1){\otimes}(c_1 + tc_2){\otimes}(d_1 + td_2) - a_1{\otimes}b_2{\otimes}c_1{\otimes}d_1]. \quad \square$$

Exercise 3.9.1.2: Write out the proof of the general case. ⊚

3.9.2. Equations for $\hat{\sigma}_{3,A\otimes B\otimes C}$. By Theorem 3.1.4.3, $\hat{\sigma}_{3,\mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^3}$ and $\hat{\sigma}_{3,\mathbb{C}^2\otimes\mathbb{C}^3\otimes\mathbb{C}^3}$, are the ambient spaces so there are no equations.

By Theorem 3.7.1.1, once we have defining equations for $\hat{\sigma}_{3,\mathbb{C}^3\otimes\mathbb{C}^3\otimes\mathbb{C}^3}$, we will have defining equations for all $\hat{\sigma}_{3,A\otimes B\otimes C}$. We have some equations thanks to §3.8.1, and we now determine if they are enough.

Theorem 3.9.2.1 ([207, 130]). *$\hat{\sigma}_{3,\mathbb{C}^{\mathbf{a}}\otimes\mathbb{C}^{\mathbf{b}}\otimes\mathbb{C}^{\mathbf{c}}}$ is the zero set of the size four minors of flattenings and Strassen's degree four equations obtained from 3-dimensional subspaces of $\mathbb{C}^{\mathbf{a}}$.*

The proof is given in Chapter 7.

3.9.3. Equations for $\hat{\sigma}_{4,A\otimes B\otimes C}$. The variety $\hat{\sigma}_{4,\mathbb{C}^3\otimes\mathbb{C}^3\otimes\mathbb{C}^3}$ is defined by the single degree nine equation $\det(T_A^{\wedge}) = 0$. (This will be proved by you in Exercise 7.6.4.5.) Thus the first case to consider is $\sigma_{4,\mathbb{C}^3\otimes\mathbb{C}^3\otimes\mathbb{C}^4}$. Here there are degree 9 equations given by the size three minors of $T_A^{\wedge}$ as in §3.8.1, and a first question is if these equations are enough.

The degree 9 equations are not enough, and this was first realized using representation theory. More precisely, using prolongation, degree six equations were found that were independent of the degree nine equations, see Proposition 3.7.2.5. A geometric model for these degree six equations is not yet known, but they were written down explicitly in coordinates in [**20**], see §6.8.7.

Recently, D. Bates and L. Oeding [**20**] showed, using numerical methods, that the equations of degrees six and nine are enough to define $\sigma_{4,\mathbb{C}^3\otimes\mathbb{C}^3\otimes\mathbb{C}^4}$. They also outline how one could write a computer-free proof, which was accomplished in [**131**]. An exposition of equations of degrees nine and sixteen that are also enough to define $\sigma_{4,\mathbb{C}^3\otimes\mathbb{C}^3\otimes\mathbb{C}^4}$ due to S. Friedland, is given in §7.7.3. While the degree 16 equations are of higher degree, they have the advantage of having a known geometric model. In summary:

Theorem 3.9.3.1 ([**20, 131**]). *The variety $\hat{\sigma}_{4,A\otimes B\otimes C}$ is the zero set of the inherited degree nine and the degree six equations plus the degree five equations of flattenings.*

Theorem 3.9.3.2 ([**130**]). *The variety $\hat{\sigma}_{4,A\otimes B\otimes C}$ is the zero set of the inherited degree nine and the degree sixteen equations plus the degree five equations of flattenings.*

3.9.4. Equations for $\hat{\sigma}_r$, $r > 4$. When $\mathbf{a} \leq \mathbf{b} \leq \mathbf{c}$, one has *some* equations up to $r \leq \frac{3\mathbf{b}}{2} - 1$. These equations are not enough to generate the ideal. In [**207**, Thm. 4.2] (see Chapter 7), additional equations are given that are independent of Strassen's. It is unlikely that these additional equations will be enough to have $\hat{\sigma}_r$ as their common zero locus. The equations from flattenings only generate the ideal in unbalanced cases as described in §3.4.2. There are *no* equations known for $\hat{\sigma}_{6,\mathbb{C}^4\otimes\mathbb{C}^4\otimes\mathbb{C}^4}$ as of this writing.

3.10. Equations for symmetric border rank

3.10.1. The Aronhold invariant. In this subsection I present a first example of equations for $\hat{\sigma}_{r,S^dV}$ beyond symmetric flattenings. These equations are a symmetric version of Strassen's equations. Let $\dim V = 3$.

Map $S^3V \to (V\otimes\Lambda^2V)\otimes(V\otimes V^*)$, by first embedding $S^3V \subset V\otimes V\otimes V$, then tensoring with $\mathrm{Id}_V \in V\otimes V^*$, and then skew-symmetrizing. Thus, when $\mathbf{v} = 3$, $\phi \in S^3V$ gives rise to an element of $\mathbb{C}^9\otimes\mathbb{C}^9$. In bases, if we write

$$\begin{aligned}\phi = \phi_{000}x_0^3 + \phi_{111}x_1^3 + \phi_{222}x_2^3 + 3\phi_{001}x_0^2x_1 + 3\phi_{011}x_0x_1^2 + 3\phi_{002}x_0^2x_2 \\ + 3\phi_{022}x_0x_2^2 + 3\phi_{112}x_1^2x_2 + 3\phi_{122}x_1x_2^2 + 6\phi_{012}x_0x_1x_2,\end{aligned}$$

the corresponding matrix is:

$$\begin{bmatrix} & & & \phi_{002} & \phi_{012} & \phi_{022} & -\phi_{010} & -\phi_{011} & -\phi_{012} \\ & & & \phi_{012} & \phi_{112} & \phi_{122} & -\phi_{011} & -\phi_{111} & -\phi_{112} \\ & & & \phi_{012} & \phi_{112} & \phi_{222} & -\phi_{012} & -\phi_{112} & -\phi_{122} \\ -\phi_{002} & -\phi_{012} & -\phi_{022} & & & & \phi_{000} & \phi_{001} & \phi_{002} \\ -\phi_{012} & -\phi_{112} & -\phi_{122} & & & & \phi_{001} & \phi_{011} & \phi_{012} \\ -\phi_{012} & -\phi_{112} & -\phi_{222} & & & & \phi_{001} & \phi_{011} & \phi_{022} \\ \phi_{010} & \phi_{011} & \phi_{012} & -\phi_{000} & -\phi_{001} & -\phi_{002} & & & \\ \phi_{011} & \phi_{111} & \phi_{112} & -\phi_{001} & -\phi_{011} & -\phi_{012} & & & \\ \phi_{012} & \phi_{112} & \phi_{122} & -\phi_{001} & -\phi_{011} & -\phi_{022} & & & \end{bmatrix}.$$

In other words, letting $H(\psi)$ denote the Hessian of a polynomial ψ, the matrix, in block format, is

$$\begin{pmatrix} 0 & H(\frac{\partial\phi}{\partial x_2}) & -H(\frac{\partial\phi}{\partial x_1}) \\ -H(\frac{\partial\phi}{\partial x_2}) & 0 & H(\frac{\partial\phi}{\partial x_0}) \\ H(\frac{\partial\phi}{\partial x_1}) & -H(\frac{\partial\phi}{\partial x_0}) & 0 \end{pmatrix}.$$

All the principal Pfaffians of size 8 of the this matrix coincide, up to scale, with the classical *Aronhold invariant*. (Redundancy occurs here because one should really work with the submodule $S_{21}V \subset V{\otimes}\Lambda^2 V \simeq V{\otimes}V^*$, where the second identification uses a choice of volume form.)

Proposition 3.10.1.1. *The variety $\hat\sigma_{3,S^3\mathbb{C}^3}$ is the zero set of the Aronhold invariant. More generally, $\hat\sigma_{3,S^3\mathbb{C}^{\mathbf{v}}}$ is the zero set of the equations inherited from the Aronhold invariant and the size four minors of the flattening $\phi_{1,2}$.*

The proposition follows from Theorem 5.4.1.1, which shows that $\hat\sigma_{3,S^3\mathbb{C}^3}$ has codimension one.

The Aronhold invariant is a symmetric version of Strassen's degree nine equation.

Exercise 3.10.1.2: Recover the Aronhold equation from Strassen's equations for $\sigma_3(Seg(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2))$.

3.10.2. A generalization of the Aronhold invariant. Now consider the inclusion $V \subset \Lambda^k V^* {\otimes} \Lambda^{k+1} V$ by, $v \in V$ maps to the map $\omega \mapsto v \wedge \omega$. In bases one obtains a matrix whose entries are basis elements or zero. In the special case where $\mathbf{v} = 2a+1$ is odd and $k = a$, one obtains a square matrix $K_{\mathbf{v}}$, which is skew-symmetric for odd a and symmetric for even a. For example, when $n = 2$, the matrix is

$$K_3 = \begin{pmatrix} 0 & x_2 & -x_1 \\ -x_2 & 0 & x_0 \\ x_1 & -x_0 & 0 \end{pmatrix},$$

and, when $\mathbf{v} = 5$, the matrix is

$$K_5 = \begin{bmatrix} & & & & & & & x_4 & -x_3 & x_2 \\ & & & & & -x_4 & x_3 & & & -x_1 \\ & & & & x_4 & & -x_2 & & x_1 & \\ & & & & -x_3 & x_2 & & -x_1 & & \\ & & x_4 & -x_3 & & & & & & x_0 \\ & -x_4 & & x_2 & & & & & -x_0 & \\ & x_3 & -x_2 & & & & & x_0 & & \\ x_4 & & & -x_1 & & & x_0 & & & \\ -x_3 & & x_1 & & & -x_0 & & & & \\ x_2 & -x_1 & & & x_0 & & & & & \end{bmatrix}.$$

Finally consider the following generalization of both the Aronhold invariant and the $K_{\mathbf{v}}$:

Let $a = \lceil \frac{\mathbf{v}}{2} \rceil$ and let $d = 2\delta+1$. Map $S^dV \to (S^\delta V \otimes \Lambda^a V^*) \otimes (S^\delta V \otimes \Lambda^{a+1} V)$ by first performing the inclusion $S^dV \to S^\delta V \otimes S^\delta V \otimes V$, then using the last factor to obtain the map $\Lambda^a V \to \Lambda^{a+1} V$. Or in the perspective of the Aronhold invariant, tensor with $\mathrm{Id}_{\Lambda^a V}$ to get an element of $(S^\delta V \otimes \Lambda^a V^* \otimes V) \otimes (S^\delta V \otimes \Lambda^a V)$ and then skew-symmetrize to get a map:

$$YF_{d,\mathbf{v}}(\phi) : S^\delta V^* \otimes \Lambda^a V \to S^\delta V \otimes \Lambda^{a+1} V. \tag{3.10.1}$$

If $\mathbf{v}$ is odd, the matrix representing $YF_{d,\mathbf{v}}(\phi)$ is skew-symmetic, so we may take Pfaffians instead of minors. In bases, one obtains a matrix in block form, where the blocks correspond to the entries of $K_{\mathbf{v}}$ and the matrices in the blocks are the $\pm(\frac{\partial \phi}{\partial x_i})_{\delta,\delta}$ in the place of $\pm x_i$. Let

$$\hat{Y}F^r_{d,\mathbf{v}} := \left\{ \phi \in S^dV \mid \mathrm{rank}(YF_{d,\mathbf{v}}(\phi)) \leq \binom{\mathbf{v}-1}{\lceil \frac{\mathbf{v}}{2} \rceil} r \right\}.$$

Exercise 3.10.2.1: Show that $\hat{\sigma}_{r,S^dV} \subset \hat{Y}F^r_{d,\mathbf{v}}$, and thus the minors of $YF_{d,\mathbf{v}}(\phi)$ give equations for $\hat{\sigma}_{r,S^dV}$.

Remark 3.10.2.2. The set of ranks of the maps $YF_{d,\mathbf{v}}(\phi)$ is called the "kappa invariant" in [**122**], where it is used to study the nature of points with symmetric border rank lower than their symmetric rank.

Two interesting cases are $\hat{Y}F^3_{3,3} = \hat{\sigma}_{3,S^3\mathbb{C}^3}$, which defines the quartic Aronhold invariant, and $\hat{Y}F^7_{3,5} = \hat{\sigma}_{7,S^3\mathbb{C}^5}$, which shows

Theorem 3.10.2.3 ([**277, 260**])**.** *The generic rank of an element of $S^3\mathbb{C}^5$ is eight. In other words, $\hat{\sigma}_{7,S^3\mathbb{C}^5}$ is not the ambient space $S^3\mathbb{C}^5$ as predicted by* (3.2.1).

This equation of degree 15 was first written down in 2009 (in [**257**]), but Theorem 3.10.2.3 was originally proved by other methods independently by Richmond [**277**] and Palatini [**260**] in 1902.

Remark 3.10.2.4. Just as with the Aronhold invariant above, there will be redundancies among the minors and Pfaffians of $YF_{d,n}(\phi)$. See §7.8.2 for a description without redundancies.

3.10.3. Summary of what is known. To conform with standard notation in algebraic geometry, in Table 3.10.1 I write $\hat{\sigma}_r(v_d(\mathbb{P}V))$ for $\hat{\sigma}_{r,S^dV}$. More precisely, $\hat{\sigma}_{r,S^dV} \subset S^dV$ is the cone over $\sigma_r(v_d(\mathbb{P}V)) \subset \mathbb{P}(S^dV)$, see §4.2.

See §4.2.2 for an explanation of the words "ideal" and "scheme". The abbreviation "irred. comp." is short for *irreducible component*; see §4.2.3 for its definition. Roughly speaking, it means one has local defining equations, but the zero set of the equations may have other components.

3.11. Tensors in $\mathbb{C}^2\otimes\mathbb{C}^{\mathbf{b}}\otimes\mathbb{C}^{\mathbf{c}}$

Certain spaces of tensors, e.g., $\mathbb{C}^{\mathbf{b}}\otimes\mathbb{C}^{\mathbf{c}}$, have a finite number of orbits under the action of the group of changes of bases. Thus there are finitely many *normal forms* for elements of such spaces. Other spaces, while there are not a finite number of orbits, are such that the orbits admit "nice" parameterizations. A classical example of this situation is $V\otimes V^*$ under the action of $GL(V)$, where one has Jordan canonical form. See Chapter 10 for an extensive discussion. In this section I present one such case, that of $\mathbb{C}^2\otimes\mathbb{C}^{\mathbf{b}}\otimes\mathbb{C}^{\mathbf{c}}$, where a nice parametrization was found by Kronecker, and describe the ranks of each normal form.

3.11.1. Kronecker's normal form. Kronecker determined a normal form for pencils of matrices, i.e., two-dimensional linear subspaces of $B\otimes C$ up to the action of $GL(B) \times GL(C)$. It is convenient to use matrix notation, so choose bases of A, B, C and write the family as $sX + tY$, where $X, Y \in B\otimes C$ and $s, t \in \mathbb{C}$. (Kronecker's classification works over arbitrary fields, as does the Grigoriev/J. Ja'Ja' classification of rank, but I only discuss results over $\mathbb{C}$.) The result is as follows (see, e.g., [**136**, Chap. XII]):

Define the $\epsilon \times (\epsilon + 1)$ matrix

$$L_\epsilon = L_\epsilon(s,t) = \begin{pmatrix} s & t & & \\ & \ddots & \ddots & \\ & & s & t \end{pmatrix}.$$

Table 3.10.1. Known equations for symmetric border rank.

case	**equations**	**cuts out**	**reference**
$\sigma_r(v_2(\mathbb{P}^n))$	size $r+1$ minors	*ideal*	classical
$\sigma_r(v_d(\mathbb{P}^1))$	size $r+1$ minors of any nontrivial $\phi_{s,d-s}$	*ideal*	Gundelfinger [**172**]
$\sigma_2(v_d(\mathbb{P}^n))$	size 3 minors of any $\phi_{1,d-1}$ and $\phi_{2,d-2}$	*ideal*	[**183**]
$\sigma_3(v_3(\mathbb{P}^n))$	Aronhold + size 4 minors of $\phi_{1,2}$	*ideal*	inheritance and Aronhold for $n=2$ [**172**]
$\sigma_3(v_d(\mathbb{P}^n))$, $d\geq 4$	size 4 minors of $\phi_{2,2}$ and $\phi_{1,3}$	*scheme*	[**215**], [**283**] for $n=2, d=4$
$\sigma_4(v_d(\mathbb{P}^2))$	size 5 minors of $\phi_{a,d-a}, a=\lfloor\frac{d}{2}\rfloor$	*scheme*	[**215**] [**283**] for $d=4$
$\sigma_5(v_d(\mathbb{P}^2))$, $d\geq 6$ and $d=4$	size 6 minors of $\phi_{a,d-a}, a=\lfloor\frac{d}{2}\rfloor$	*scheme*	[**215**], Clebsch for $d=4$ [**172**]
$\sigma_r(v_5(\mathbb{P}^2))$, $\rho\leq 5$	size $2r+2$ sub-Pfaff. of $\phi_{31,31}$	*irred. comp.*	[**215**]
$\sigma_6(v_5(\mathbb{P}^2))$	size 14 sub-Pfaff. of $\phi_{31,31}$	*scheme*	[**215**]
$\sigma_6(v_d(\mathbb{P}^2))$, $d\geq 6$	size 7 minors of $\phi_{a,d-a}, a=\lfloor\frac{d}{2}\rfloor$	*scheme*	[**215**]
$\sigma_7(v_6(\mathbb{P}^2))$	symm. flat. + Young flat.	*irred. comp.*	[**215**]
$\sigma_8(v_6(\mathbb{P}^2))$	symm. flat. + Young flat.	*irred. comp.*	[**215**]
$\sigma_9(v_6(\mathbb{P}^2))$	$\det\phi_{3,3}$	*ideal*	classical
$\sigma_j(v_7(\mathbb{P}^2))$, $j\leq 10$	size $2j+2$ sub-Pfaff. of $\phi_{41,41}$	*irred. comp.*	[**215**]
$\sigma_j(v_{2\delta}(\mathbb{P}^2))$, $j\leq\binom{\delta+1}{2}$	rank $\phi_{a,d-a}=\min(j,\binom{a+2}{2})$, $1\leq a\leq\delta$ open and closed conditions	*scheme*	[**172**, Thm. 4.1A]
$\sigma_j(v_{2\delta+1}(\mathbb{P}^2))$, $j\leq\binom{\delta+1}{2}+1$	rank $\phi_{a,d-a}=\min(j,\binom{a+2}{2})$, $1\leq a\leq\delta$ open and closed conditions	*scheme*	[**172**, Thm. 4.5A]
$\sigma_j(v_{2\delta}(\mathbb{P}^n))$, $j\leq\binom{\delta+n-1}{n}$	size $j+1$ minors of $\phi_{\delta,\delta}$	*irred. comp.*	[**172**, Thm. 4.10A]
$\sigma_j(v_{2\delta+1}(\mathbb{P}^n))$, $j\leq\binom{\delta+n}{n}$	size $\binom{n}{a}j+1$ minors of $Y_{d,n}$, $a=\lfloor n/2\rfloor$ if $n=2a, a$ odd, $\binom{n}{a}j+2$ sub-Pfaff. of $Y_{d,n}$	*irred. comp.*	[**215**]

The normal form is

$$sX + tY = \begin{pmatrix} L_{\epsilon_1} & & & & & & \\ & \ddots & & & & & \\ & & L_{\epsilon_q} & & & & \\ & & & L_{\eta_1}^T & & & \\ & & & & \ddots & & \\ & & & & & L_{\eta_p}^T & \\ & & & & & & s\,\mathrm{Id}_f + tF \end{pmatrix}, \tag{3.11.1}$$

where F is an $f \times f$ matrix in Jordan normal form and T denotes the transpose. (One can also use rational canonical form for F.)

For a fixed linear map $F : \mathbb{C}^f \to \mathbb{C}^f$, let $d(\lambda)$ denote the number of Jordan blocks of size at least two associated to the eigenvalue λ, and let $M(F)$ denote the maximum of the $d(\lambda)$.

Theorem 3.11.1.1 (Grigoriev, Ja'Ja', Teichert [**149, 181, 304**])**.** *A pencil of the form* (3.11.1) *has rank* $\sum \epsilon_i + \sum \mu_j + f + q + p + M(F)$.

In particular, the maximum possible rank of a tensor in $\mathbb{C}^2 \otimes \mathbb{C}^{\mathbf{b}} \otimes \mathbb{C}^{\mathbf{b}}$ *is* $\lfloor \frac{3\mathbf{b}}{2} \rfloor$.

Remark 3.11.1.2. In [**181**], Theorem 3.11.1.1 is stated as an inequality (Cor. 2.4.3 and Thm. 3.3), but the results are valid over arbitrary fields (with appropriate hypotheses). In [**149**] the results are stated, but not proved, and the reader is referred to [**148**] for indications towards the proofs. In [**54**] a complete proof is given of an equivalent statement in terms of the elementary divisors of the pair, which is more complicated, and the text states the proof is taken from the unpublished PhD thesis [**304**].

Part 2

Geometry and representation theory

Chapter 4

Algebraic geometry for spaces of tensors

This chapter covers basic definitions from algebraic geometry. Readers familiar with algebraic geometry should skip immediately to Chapter 5 after taking the diagnostic test in §4.1 to make sure there are no gaps to be filled. This chapter is not intended to be a substitute for a proper introduction to the subject, but for the reader willing to accept basic results without proof, enough material is covered to enable a reading of everything up to the advanced topics.

In the introduction, several situations arose where one wanted to test if a tensor or symmetric tensor ϕ could be written as a sum of r rank one tensors, or as a limit of such. The set of tensors in a given space of border rank at most r is an example of an *algebraic variety*, that is a set of vectors in a vector space defined as the zero set of a collection of homogeneous polynomials. Thus, if one has the defining polynomials, one has the desired test by checking if the polynomials vanish on ϕ. The language of algebraic geometry will allow us not only to develop such tests, but to extract qualitative information about varieties of tensors that is useful in applications.

Projective varieties and their ideals are defined in §4.2. §4.3 contains first examples of projective varieties, including the variety of tensors of border rank at most r and several homogeneous varieties. To perform calculations, e.g., of dimensions of varieties, it is often easiest to work infinitesimally, so tangent spaces are defined in §4.6. An aside §4.8 is a series of exercises to understand Jordan normal form for linear maps via algebraic geometry. The chapter concludes in §4.9 with several additional definitions and results that are used in later chapters.

4.1. Diagnostic test for those familiar with algebraic geometry

Those *not* familiar with algebraic geometry should skip this section. The number placed before the problem is where the relevant discussion appears.

§4.3.4 Let $X \subset \mathbb{P}V$ be a variety. Consider $Seg(X \times \mathbb{P}W) \subset Seg(\mathbb{P}V \times \mathbb{P}W) \subset \mathbb{P}(V \otimes W)$. Show that the ideal of $Seg(X \times \mathbb{P}W) \subset \mathbb{P}(V \otimes W)$ in degree d is the span of $I_d(X) \otimes S^d W^*$ and $I_d(Seg(\mathbb{P}V \times \mathbb{P}W))$.

§4.3.7 Show that if $x_1, \dots, x_k \in V$ are such that every subset of $r+1$ of them is a linearly independent set of vectors, then for $k \leq dr+1$, $x_1^d, \dots, x_k^d \in S^d V$ is a linearly independent set of vectors.

§4.4.1 Let $X \subset \mathbb{P}V$ be a variety. Show that $I_d(X) = \langle v_d(X) \rangle^{\perp} \subset S^d V^*$.

§4.4.1 Show that if $I(X)$ is generated in degree two, then $v_2(X)$ can be described as the zero set of the two by two minors of a matrix of linear forms.

§4.6 Show that for $X \subset \mathbb{P}V$, there is a canonical identification $T_x X \simeq \hat{x}^* \otimes (\hat{T}_x X / \hat{x})$, where $T_x X$ is the Zariski tangent space and $\hat{T}_x X \subset V$ is the affine tangent space.

§4.6 If $P \in S^d V^*$, define $dP|_x$ to be the linear form $dP|_x = \overline{P}(x, \dots, x, \cdot)$. Show that $N_x^* X := (N_x X)^* = \hat{x} \otimes \{dP_x \mid P \in I(X)\} \subset T_x^* \mathbb{P}V = \hat{x} \otimes x^{\perp}$, where $N_x X := T_x \mathbb{P}V / T_x X$ is the normal space.

§4.6.2 Given $[p] \in Seg(\mathbb{P}A \times \mathbb{P}B)$, consider $p : A^* \to B$ as a linear map. Show that

$$\hat{N}^*_{[p]} Seg(\mathbb{P}A \times \mathbb{P}B) = \ker(p) \otimes (\mathrm{image}(p))^{\perp}.$$

§4.7 Show that the ideal of a G-variety is a G-module.

§4.8 Let $f : V \to V$ be a linear map. Endow $\mathbb{C}^2$ with a basis e_1, e_2 and consider the map $F : \mathbb{P}V \to \mathbb{P}(\mathbb{C}^2 \otimes V)$ given by $[v] \mapsto v \otimes e_1 + f(v) \otimes e_2$. Show that the eigenvalues of f are distinct if and only if $F(\mathbb{P}V)$ is transverse to the Segre variety.

4.2. First definitions

One of the disorienting things for researchers talking with algebraic geometers, is that the objects of interest occur in a vector space (a space of tensors), but geometers tend to work in projective space. This is because the properties of tensors discussed in this book, such as rank, border rank, multilinear rank, etc., are *invariant under rescaling*, so it is more efficient to quotient by this rescaling.

4.2.1. Varieties and their ideals. Let V be a finite-dimensional complex vector space of dimension $\mathbf{v}$. The *projective space* $\mathbb{P}V = \mathbb{P}^{\mathbf{v}-1}$ associated to V is the set of lines through the origin in V. Its precise definition is as follows:

Definition 4.2.1.1. *Projective space* $\mathbb{P}V$ is the set whose points $[v] \in \mathbb{P}V$ are equivalence classes of nonzero elements $v \in V$, where $[v] = [w]$ if and only if there exists a nonzero $\lambda \in \mathbb{C}$ such that $w = \lambda v$.

Let $\pi : V \backslash 0 \to \mathbb{P}V$ denote the projection map. As a quotient of $V \backslash 0$, projective space inherits aspects of the linear structure on V. When $U \subset V$ is a linear subspace, one also says that $\mathbb{P}U \subset \mathbb{P}V$ is a *linear subspace*. Just as in affine space, given any two distinct points $x, y \in \mathbb{P}V$, there exists a unique *line* $\mathbb{P}^1_{xy}$ containing them. A line in $\mathbb{P}V$ is the image under π of a 2-plane through the origin in V. An essential property of $\mathbb{P}^2$ is that any two distinct lines will intersect in a point.

Remark 4.2.1.2. A striking difference between Italian paintings from medieval times and the Renaissance is the introduction of perspective, which can be traced to the early 1400's (according to Wikipedia, to Brunelleschi around 1425). A key point in providing perspective is to have parallel lines appear to meet at infinity. The projective plane was defined mathematically by Kepler and Desargues in the 1600s.

Definition 4.2.1.3. For a subset $Z \subset \mathbb{P}V$, let $\hat{Z} := \pi^{-1}(Z)$ denote the *(affine) cone* over Z. The image of an affine cone $\mathcal{C}$ in projective space is called its *projectivization*, and I often write $\mathbb{P}\mathcal{C}$ for $\pi(\mathcal{C})$.

Here is a definition of an algebraic variety and its ideal sufficient for many of our purposes:

Definition 4.2.1.4. An *algebraic variety* is the image under

$$\pi : V \backslash 0 \to \mathbb{P}V$$

of the set of common zeros of a collection of homogeneous polynomials on V. The *ideal* $I(X) \subset S^\bullet V^*$ of a variety $X \subset \mathbb{P}V$ is the set of all polynomials vanishing on $\hat{X}$.

Exercise 4.2.1.5: Show that $I(X)$ is indeed an ideal (cf. Definition 2.9.2.5) in the ring $S^\bullet V^*$.

Exercise 4.2.1.6: Show that a polynomial P vanishes on $\hat{X}$ if and only if all its homogeneous components vanish on $\hat{X}$. (In particular, P cannot have a constant term.)

By Exercise 4.2.1.6, there is no loss of generality by restricting our attention to homogeneous polynomials.

Write $I_k(X) = I(X) \cap S^k V^*$, so

$$I_k(X) = \{P \in S^k V^* \mid P(v) = 0 \ \forall v \in \hat{X}\}.$$

Definition 4.2.1.7. For a subset $Z \subset \mathbb{P}V$, define

$$I(Z) := \{P \in S^\bullet V^* \mid P|_{\hat{Z}} \equiv 0\}.$$

Define $\overline{Z}$ to be the set of common zeros of $I(Z)$. $\overline{Z}$ is called the *Zariski closure* of Z.

I often write $P|_Z \equiv 0$ instead of $P|_{\hat{Z}} \equiv 0$ and similarly for $x \in \mathbb{P}V$, that $P(x) = 0$. These expressions are well defined.

Definition 4.2.1.8. Let $X \subset \mathbb{P}V$ and $Y \subset \mathbb{P}W$ be varieties. X and Y are said to be *projectively equivalent* if there exist linear maps $f : V \to W$, $g : W \to V$, such that $f(\hat{X}) = \hat{Y}$ and $g(\hat{Y}) = \hat{X}$, and otherwise they are said to be *projectively inequivalent*.

Remark 4.2.1.9. One can define an intrinsic notion of algebraic varieties and equivalence of algebraic varieties as well, see, e.g., [**161**].

4.2.2. Equations of a variety. There are three possible meanings to the phrase "equations of a variety X":

Definition 4.2.2.1. A collection of homogeneous polynomials $P_1, \dots, P_r \in S^\bullet V^*$ *cuts out X set-theoretically* if the set of common zeros of the polynomials $P_1, \dots, P_r$ is the set of points of X. One says that $P_1, \dots, P_r$ *generate the ideal of X* (or *cut out X ideal-theoretically*) if every $P \in I(X)$ may be written $P = q_1 P_1 + \cdots + q_r P_r$ for some polynomials $q_j \in S^\bullet V^*$. A collection of polynomials $P_1, \dots, P_r$ generates $I(X)$ *scheme-theoretically* if the ideal generated by them, call it J, satisfies $J_d = I_d(X)$ for all sufficiently large $d >> 0$.

A set of generators of the ideal of a variety X is the "best" set of equations possible to define X, with the equations of lowest possible degree. (In particular, generators of the ideal of X cut out X set-theoretically.)

For example the line $(0, y) \in \mathbb{C}^2$ with coordinates x, y is cut out set-theoretically by the polynomial $x^2 = 0$, but the ideal is generated by $x = 0$.

One sometimes says that the zero set of x^2 in $\mathbb{C}^2$ is the line $(0, y)$ counted with *multiplicity* two. More generally, if f is an irreducible polynomial, one sometimes says that $\mathrm{Zeros}(f^d)$ is $\mathrm{Zeros}(f)$ *counted with multiplicity d*.

For a more substantial example, Friedland's equations in §3.9.3 cut out $\hat{\sigma}_{4,\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^4}$ set-theoretically but they cannot generate the ideal because of the degree six equations in the ideal.

For another example of set-theoretic defining equations that do not generate the ideal of a variety, see Exercise 4.5.2(2).

An example of a variety whose ideal is generated in degree 3 by seven polynomials and is cut out scheme-theoretically by four of the seven polynomials is given in Example 4.9.1.8.

The ideal of a variety $X \subset \mathbb{P}V^*$ is generated in degrees at most r if and only if the map

$$I_{d-1}(X) \otimes V \to I_d(X),$$

obtained by restricting the symmetrization map $S^{d-1}V \otimes V \to S^d V$, is surjective for all $d \geq r$.

4.2.3. Reducible and irreducible varieties.

Definition 4.2.3.1. A variety $X \subset \mathbb{P}V$ is said to be *reducible* if there exist varieties $Y, Z \neq X$ such that $X = Y \cup Z$. Equivalently, X is reducible if there exist nontrivial ideals I_Y, I_Z such that $I_X = I_Y \cap I_Z$, and otherwise X is said to be *irreducible*.

For example, the zero set of the equation $xyz = 0$ is reducible, it is the union of the sets $x = 0$, $y = 0$ and $z = 0$, while the zero set of $x^3+y^3+z^3 = 0$ is irreducible. Every homogeneous polynomial in two variables of degree $d > 1$ is reducible. In fact it is a product of degree one polynomials (possibly with multiplicities).

4.2.4. Exercises on ideals of varieties.

(1) Show that if $X \subset Y$, then $I(Y) \subset I(X)$.

(2) Show that $I(Y \cup Z) = I(Y) \cap I(Z)$.

(3) Show that if $X, Y \subset \mathbb{P}V$ are varieties, then $X \cap Y$ is a variety. What is the ideal of $X \cap Y$ in terms of the ideals of X and Y? ⊚

(4) An ideal $J \subset S^\bullet V^*$ is *prime* if $fg \in J$ implies either $f \in J$ or $g \in J$. Show that the ideal of a variety X is prime if and only if X is irreducible.

(5) Prove the following implications:
$P_1, \dots, P_r$ generate $I(X)$
$\Rrightarrow P_1, \dots, P_r$ cut out X scheme-theoretically
$\Rrightarrow P_1, \dots, P_r$ cut out X set-theoretically.

4.3. Examples of algebraic varieties

Most of the algebraic varieties discussed in this book will be natural subvarieties of the projective space of a vector space of tensors, such as the set of matrices of rank at most r. We saw in §2.7.3 that this variety is cut out by the set of $r+1$ by $r+1$ minors. (In §2.7.3 we saw that the $r+1$ by $r+1$ minors cut out the variety set-theoretically; in §6.7.4 we will see that they generate the ideal.)

4.3.1. Hyperplanes and hypersurfaces. Let $\alpha \in V^*$; then its zero set in $\mathbb{P}V$ is $H_\alpha := \mathbb{P}(\alpha^\perp)$, where recall from §2.1 that $\alpha^\perp := \{v \in V \mid \alpha(v) = 0\}$ and that $\alpha^\perp$ is a linear subspace. H_α is called a *hyperplane*. More generally, the zero set of some $P \in S^d V^*$ is called a *hypersurface of degree d*.

4.3.2. Chow's theorem and compact complex manifolds. (For those familiar with manifolds.) Chow's theorem (see e.g., [**247**, Chapter 4]) states that a compact complex submanifold of $\mathbb{P}V$ is also an algebraic variety.

4.3.3. Linear sections. Let $X \subset \mathbb{P}V$ be a variety and let $\mathbb{P}W \subset \mathbb{P}V$ be a linear space. The variety $X \cap \mathbb{P}W \subset \mathbb{P}W$ is called the *linear section of X with $\mathbb{P}W$.*

Exercise 4.3.3.1: Take linear coordinates $x^1, \dots, x^{\mathbf{v}}$ on V such that $W = \{x^{k+1} = \cdots = x^{\mathbf{v}} = 0\}$. Express the equations of $X \cap \mathbb{P}W$ in terms of the equations of X in these coordinates. ⊚

4.3.4. The two-factor Segre varieties: rank one matrices revisited. Let A, B be vector spaces respectively of dimensions $\mathbf{a}, \mathbf{b}$ and let $V = A{\otimes}B$. Recall from §2.7.2 that the set of $\mathbf{a} \times \mathbf{b}$ matrices of rank one is the zero set of the two by two minors, i.e., the $GL(A) \times GL(B)$-module $\Lambda^2 A^* {\otimes} \Lambda^2 B^* \subset S^2(A{\otimes}B)^*$.

Definition 4.3.4.1. Define the *two-factor Segre variety $Seg(\mathbb{P}A \times \mathbb{P}B)$* to be the zero set of the ideal generated by the two by two minors as in §2.7.2.

By Exercises 2.1.2 and 2.1.3, the Segre variety is the projectivization of the set of matrices of rank one, and it may be parametrized by the points of $\mathbb{P}A \times \mathbb{P}B$. This may be seen directly as $([a],[b]) \mapsto [a{\otimes}b]$. Or in bases, writing $a = (a_1, \dots, a_{\mathbf{a}}) \in A$, $b = (b_1, \dots, b_{\mathbf{b}}) \in B$, then the (up to scale) matrix $Seg([a],[b])$ is

$$Seg([a],[b]) = \left[\begin{pmatrix} a_1b_1 & a_1b_2 & \cdots & a_1b_{\mathbf{b}} \\ a_2b_1 & a_2b_2 & \cdots & a_2b_{\mathbf{b}} \\ \vdots & \vdots & \cdots & \vdots \\ a_{\mathbf{a}}b_1 & a_{\mathbf{a}}b_2 & \cdots & a_{\mathbf{a}}b_{\mathbf{b}} \end{pmatrix}\right].$$

4.3.5. The n-factor Segre variety. Let A_j be vector spaces and let $V = A_1{\otimes} \cdots {\otimes}A_n$.

Definition 4.3.5.1. Define the n-factor Segre variety to be the image of the map

$$\begin{aligned} Seg : \mathbb{P}A_1 \times \mathbb{P}A_2 \times \cdots \times \mathbb{P}A_n &\to \mathbb{P}V, \\ ([v_1], \dots, [v_n]) &\mapsto [v_1 \otimes \cdots \otimes v_n]. \end{aligned}$$

This map is called the *Segre embedding* of a product of projective spaces.

It is easy to see that Seg is well defined and a differentiable mapping, so its image is an immersed submanifold, and that the map is an embedding. By Chow's theorem (§4.3.2), $Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n)$ is indeed a variety as the definition asserts.

One way to see directly that $Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n) \subset (A_1 \otimes \cdots \otimes A_n)$ is a variety is to note that it is the set of common zeros of $\Lambda^2 A_j^* \otimes \Lambda^2 A_{\hat{j}}^*$, $1 \leq j \leq n$, where $A_{\hat{j}} = A_1 \otimes \cdots \otimes A_{j-1} \otimes A_{j+1} \otimes \cdots \otimes A_n$. To see this, first note that the zero set of $\Lambda^2 A_1^* \otimes \Lambda^2 A_{\hat{1}}^*$ is $Seg(\mathbb{P}A_1 \times \mathbb{P}(A_2 \otimes \cdots \otimes A_n))$, i.e., tensors of the form $a_1 \otimes T_1$ with $a_1 \in A_1$ and $T_1 \in A_2 \otimes \cdots \otimes A_n$. Then similarly the zero set of $\Lambda^2 A_2^* \otimes \Lambda^2 A_{\hat{2}}^*$ is $Seg(\mathbb{P}A_2 \times \mathbb{P}A_{\hat{2}})$, i.e., tensors of the form $a_2 \otimes T_2$ with $a_2 \in A_2$ and $T_2 \in A_{\hat{2}}$. Thus the zero set of both equations consists of tensors of the form $a_1 \otimes a_2 \otimes U$, with $U \in A_3 \otimes \cdots \otimes A_n$. Continuing, one obtains the result.

$\hat{Seg}(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n)$ is the set of elements of $A_1 \otimes \cdots \otimes A_n$ of rank one, denoted $\hat{\sigma}_1 \subset A_1 \otimes \cdots \otimes A_n$ in Chapter 3.

If A_j has basis $a_j^1, \dots, a_j^{\mathbf{a}_j}$ and we write $v_j = v_{s_j} a_j^{s_j} \in A_j$, $1 \leq s_j \leq \mathbf{a}_j$, then

$$Seg([v_1], \dots, [v_n]) = [v_{s_1} \cdots v_{s_n} a_1^{s_1} \otimes \cdots \otimes a_n^{s_n}],$$

where the summation is over $\mathbf{a}_1 \cdots \mathbf{a}_n$ terms.

4.3.6. The variety σ_r of tensors of border rank at most r.

Let $\sigma_r^0 \subset \mathbb{P}(A_1 \otimes \cdots \otimes A_n)$ denote the (projectivization of the) set of tensors of rank at most r. Define $\sigma_r := \overline{\sigma}_r^0$. In §5.2.1, I show that $\hat{\sigma}_r$ agrees with the previous notation of the set of tensors of *border rank* at most r. So $\underline{\mathbf{R}}(T) = r$ if and only if $[T] \in \sigma_r$ and $[T] \notin \sigma_{r-1}$.

4.3.7. Veronese varieties.

Definition 4.3.7.1. The *quadratic Veronese variety* $v_2(\mathbb{P}V)$ is the image of $\mathbb{P}V$ under the injective mapping

$$\begin{aligned} v_2 : \mathbb{P}V &\to \mathbb{P}S^2V, \\ [v] &\mapsto [v^2] = [vv]. \end{aligned}$$

The variety $v_2(\mathbb{P}V) \subset \mathbb{P}(S^2V)$ is the projectivization of the rank one elements. In bases, these are the rank one symmetric matrices. Explicitly, if we choose bases and $w = (w_1, \dots, w_{\mathbf{v}}) \in V$, then

$$v_2([w]) = \left[\begin{pmatrix} w_1w_1 & w_1w_2 & \cdots & w_1w_{\mathbf{v}} \\ w_2w_1 & w_2w_2 & \cdots & w_2w_{\mathbf{v}} \\ \vdots & \vdots & \vdots & \vdots \\ w_{\mathbf{v}}w_1 & w_{\mathbf{v}}w_2 & \cdots & w_{\mathbf{v}}w_{\mathbf{v}} \end{pmatrix}\right].$$

Definition 4.3.7.2. The *d-th Veronese embedding of* $\mathbb{P}V$, $v_d(\mathbb{P}V) \subset \mathbb{P}S^dV$, is defined by $v_d([v]) = [v^d]$. The variety $v_d(\mathbb{P}^1) \subset \mathbb{P}^d$ is often called the *rational normal curve.*

Intrinsically, all these varieties are $\mathbb{P}V$, but the embeddings are projectively inequivalent. One way to see this is to observe that they all are mapped to projective spaces of different dimensions, but none is contained in a hyperplane.

One can conclude Veronese varieties are indeed varieties by invoking Chow's theorem or by the following exercise:

Exercise 4.3.7.3: Show that $v_d(\mathbb{P}V) = Seg(\mathbb{P}V\times\cdots\times\mathbb{P}V)\cap\mathbb{P}S^dV \subset \mathbb{P}V^{\otimes d}$.

Exercise 4.3.7.4: Give $V = \mathbb{C}^2$ coordinates (x, y). Show that the map $v_d : \mathbb{P}V\to\mathbb{P}S^dV$ may be given in coordinates by $[x, y]\mapsto[x^d, x^{d-1}y, x^{d-2}y^2, \dots, y^d]$. Find degree two polynomials in the ideal of $v_d(\mathbb{P}^2)$. ⊚

Exercise 4.3.7.5: Show that if $P_0, \dots, P_N$ form a basis of S^dV^*, then $v_d(\mathbb{P}V)$ is the image of the map

$$\begin{aligned} v_d : \mathbb{P}V &\to \mathbb{P}(S^dV), \\ [v] &\mapsto [P_0(v), \dots, P_N(v)]. \circledcirc \end{aligned}$$

The following elementary fact about Veronese mappings will be important for our study, as it will imply uniqueness of expressions for symmetric tensors as sums of powers:

Proposition 4.3.7.6. *If $[x_1], \dots, [x_k] \in V$ are distinct, then $x_1^d, \dots, x_k^d \in S^dV$ is a linearly independent set of vectors for all $d \geq k-1$.*

Moreover, if not all the $[x_j]$ are colinear, then $x_1^d, \dots, x_k^d$ are linearly independent for all $d \geq k-2$.

Proof. Without loss of generality, assume that $k = d+1$ and that all points lie on a $\mathbb{P}^1 = \mathbb{P}A \subset \mathbb{P}V$. Say the images of the points were all contained in a hyperplane $H \subset \mathbb{P}S^dA$. Let $h \in S^dA^*$ be an equation for H. Then h is a homogeneous polynomial of degree d in two variables, and thus has at most d distinct roots in $\mathbb{P}A$. But $h(x_j) = \overline{h}(x_j^d) = 0$, $j = 1, \dots, d+1$, a contradiction, as h can only vanish at d distinct points.

Exercise 4.3.7.7: Prove the second assertion of Proposition 4.3.7.6. □

Exercise 4.3.7.8: Show that if $x_1, \dots, x_k \in V$ are such that every subset of $r+1$ of them is a linearly independent set of vectors, then for $k \leq dr+1$, $x_1^d, \dots, x_k^d$ is a linearly independent set of vectors. ⊚

4.4. Defining equations of Veronese re-embeddings

Let $X \subset \mathbb{P}V$ be a variety. Consider $v_d(X) \subset v_d(\mathbb{P}V) \subset \mathbb{P}(S^dV)$, the *d-th Veronese re-embedding of* X. One may study $I_d(X)$ via $v_d(X)$ as I now explain:

4.4.1. Veronese varieties and ideals.

Proposition 4.4.1.1. *Let $X \subset \mathbb{P}V$ be a variety; then $I_d(X) = \langle v_d(X)\rangle^{\perp} \subset S^dV^*$.*

Proof. $P \in S^dV^*$ may be considered as a linear form on S^dV or as a homogeneous polynomial of degree d on $\mathbb{P}V$. Distinguish the first by using a bar. $\overline{P}(v_d(x)) = 0$ if and only if $P(x) = 0$. To finish the proof it remains only to observe that $I_d(X)$ is a linear space so one may take the span of the vectors annihilating $v_d(X)$. □

Remark 4.4.1.2. Note that the case of Proposition 4.4.1.1 where X is a set of $d+1$ points was used in the proof of Proposition 4.3.7.6.

That $I(v_d(\mathbb{P}V))$ is generated in degree two has the following consequence:

Proposition 4.4.1.3 ([**246, 292**])**.** *Let $X \subset \mathbb{P}V$ be a variety, whose ideal is generated in degrees $\leq d$. Then the ideal of $v_d(X) \subset \mathbb{P}S^dV$ is generated in degrees one and two.*

Proof of the set-theoretic assertion. It is enough to observe that $v_d(X) = v_d(\mathbb{P}V) \cap I_d(X)^{\perp}$, that $I(v_d(\mathbb{P}V))$ is generated in degree two, and that the ideal of the linear space $I_d(X)^{\perp}$ is generated in degree one. □

Mumford [**246**] proved the ideal-theoretic statement when X is smooth (and the set-theoretic in general). The ideal-theoretic statement for singular varieties is due to J. Sidman and G. Smith [**292**], where they also proved the ideal-theoretic version of Proposition 4.4.2.1 below.

4.4.2. Determinantal equations. Recall that the set of 2×2 minors of a matrix with variable entries gives defining equations of the Segre variety, and the Veronese variety can be defined as the set of 2×2 minors of a specialization. Say a variety $X \subset \mathbb{P}V$ has *determinantal defining equations* (or more strongly has a *determinantal ideal*) if its equations can be "naturally" realized as a set of minors. The following observation is due to P. Griffiths.

Proposition 4.4.2.1 ([**147**, p. 271])**.** *If $X \subset \mathbb{P}V$ is a variety whose ideal is generated in degree two, then the ideal of $v_2(X)$ is determinantal—in fact, the set of two by two minors of a matrix of linear forms. In particular, if $I(X)$ is generated in degrees at most d, then $v_{2d}(X)$ is the zero set of two by two minors.*

4.5. Grassmannians

4.5.1. The Grassmannian of k-planes.

Definition 4.5.1.1. Let $G(k, V) \subset \mathbb{P}\Lambda^k V$ denote the set of elements that are of the form $[v_1 \wedge \cdots \wedge v_k]$ for some $v_1, \dots, v_k \in V$. $G(k, V)$ is called the *Grassmannian*, and this embedding in projective space, the *Plücker embedding*.

The Grassmannian admits the geometric interpretation as the set of k-dimensional linear subspaces of V. (I often refer to such a subspace as a *k-plane*, although the cumbersome "k-plane through the origin" would be more precise.) To see this, given a k-plane $E \subset V$, let $v_1, \dots, v_k$ be a basis, associate to it the element $[v_1 \wedge \cdots \wedge v_k] \in G(k, V)$, and conclude by Exercise 4.5.2.1 below. Note that $\mathbb{P}V = G(1, V)$ and $\mathbb{P}V^* = G(\dim V - 1, V)$.

I have not yet shown that $G(k, V)$ is a variety. If one is willing to use Chow's theorem, this will follow from Exercise 4.5.2.3, which can be used to obtain charts for the Grassmannian as a complex manifold. Otherwise, equations are provided in §6.10.3. In the exercises below, one at least obtains equations for $G(2, V)$.

Choosing bases, identify $\Lambda^2 V$ with the space of skew-symmetric matrices. Then $G(2, V)$ is the projectivization of the skew-symmetric matrices of rank 2, i.e., of minimal rank. The module $\Lambda^3 V^* \otimes \Lambda^3 V^*$ restricted to $\Lambda^2 V \subset V \otimes V$ thus cuts out $G(2, V)$ set-theoretically by equations of degree three. But in fact $I(G(2, V))$ is generated in degree two, which can partially be seen by Exercise 4.5.2(6) below.

4.5.2. Exercises on Grassmannians.

(1) Prove that $G(k, V)$ indeed admits the geometric interpretation of the set of k-planes in V by showing that if $w_1, \dots, w_k$ is another basis of E, that $[w_1 \wedge \cdots \wedge w_k] = [v_1 \wedge \cdots \wedge v_k]$.

(2) Show that in bases that identify $\Lambda^2 V$ with the space of skew-symmetric $\mathbf{v} \times \mathbf{v}$ matrices, $G(2, V) \subset \mathbb{P}\Lambda^2 V$ admits the interpretation of the projectivization of the set of rank two skew-symmetric matrices. (Recall that a skew-symmetric matrix always has even rank.)

(3) Show that Grassmannians can be locally parametrized as follows: Fix a basis $v_1, \dots, v_{\mathbf{v}}$ of V, and let $E_0 = \langle v_1, \dots, v_k \rangle$. Fix index ranges $1 \le j \le k$, and $k+1 \le s \le \mathbf{v}$. Show that any k-plane which projects isomorphically onto E_0 may be uniquely written as $E(x^s_j) := \langle v_1 + x^s_1 v_s, \dots, v_k + x^s_k v_s \rangle$.

(4) Say $\phi \in \Lambda^2 V$. Then $[\phi] \in G(2, V)$ if and only if $\phi \wedge \phi = 0$.

(5) Write out the resulting quadratic equations from (4) explicitly in coordinates.

(6) Show the equations are the Pfaffians (see §2.7.4) of the 4×4 minors centered about the diagonal.

(7) Show that the cubics obtained by the 3×3 minors are indeed in the ideal generated by the Pfaffians.

4.6. Tangent and cotangent spaces to varieties

4.6.1. Tangent spaces. I will use two related notions of tangent space, both defined over the ground field $\mathbb{C}$. I begin with an auxiliary, more elementary notion of tangent space for subsets of affine space.

Definition 4.6.1.1. Define the *tangent space* to a point x of a subset M of a vector space V, $\hat{T}_x M \subset V$, to be the span of all vectors in V obtained as the derivative $\alpha'(0)$ of a smooth analytic parametrized curve $\alpha : \mathbb{C} \to M$ with $\alpha(0) = x$ considered as a vector in V based at x. Often one translates the vector to the origin, and I will do so here. If $\dim \hat{T}_x M$ is independent of $x \in M$, one says M is a *submanifold* of V.

Exercise 4.6.1.2: If M is a cone through the origin in V, minus the vertex, show that $\hat{T}_x M$ is constant along rays of the cone.

Definition 4.6.1.3. For a variety $X \subset \mathbb{P}V$, and $x \in X$, define the *affine tangent space to X at x*, $\hat{T}_x X := \hat{T}_{\overline{x}} \hat{X}$, where $\overline{x} \in \hat{x}$. By Exercise 4.6.1.2, $\hat{T}_x X$ is well defined.

Definition 4.6.1.4. If $\dim \hat{T}_x X$ is locally constant near x, we say x is a *smooth point* of X. Let X_{smooth} denote the set of smooth points of X. Otherwise, one says that x is a *singular point* of X. Let $X_{sing} = X \backslash X_{smooth}$ denote the singular points of X. If $X_{sing} \neq \emptyset$, one says X is *singular*.

For example, the variety in $\mathbb{P}^2$ given by $\{xyz = 0\}$ is smooth except at the points $x_1 = [1, 0, 0]$, $x_2 = [0, 1, 0]$, and $x_3 = [0, 0, 1]$. At each of these three points the affine tangent space is all of $\mathbb{C}^3$.

Note that X_{smooth} is a complex manifold.

Exercise 4.6.1.5: Show that if $X \subset \mathbb{P}V$ is a variety, then X_{sing} is also a variety. ⊚

Exercise 4.6.1.6: Show that the hypersurface with equation $x_1^d+\cdots+x_{\mathbf{v}}^d = 0$ in $\mathbb{P}V$ is smooth.

Definition 4.6.1.7. Define the *affine conormal space* to X at x, $\hat{N}_x^*X \subset V^*$, by $\hat{N}_x^*X = (\hat{T}_xX)^\perp$.

Definition 4.6.1.8. For an irreducible variety X define the *dimension* of X, $\dim(X)$, to be $\dim(\hat{T}_xX) - 1$ for $x \in X_{smooth}$.

4.6.2. Tangent spaces to Segre varieties. Any curve in $Seg(\mathbb{P}A \times \mathbb{P}B)$ is of the form $[a(t)\otimes b(t)]$ for curves $a(t) \subset A$, $b(t) \subset B$.

Differentiating

$$\left.\frac{d}{dt}\right|_{t=0} a(t)\otimes b(t) = a'(0)\otimes b(0) + a(0)\otimes b'(0)$$

shows that

$$\hat{T}_{[a\otimes b]}Seg(\mathbb{P}A \times \mathbb{P}B) = A\otimes b + a\otimes B,$$

where the sum is not direct. (The intersection is $\langle a\otimes b\rangle$.)

If one prefers to work in bases, choose bases such that

$$a(t) = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + t \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{\mathbf{a}} \end{pmatrix} + O(t^2),$$

$$b(t) = (1, 0, \ldots, 0) + t(y_1, y_2, \ldots, y_{\mathbf{b}}) + O(t^2),$$

so

$$a(0)\otimes b(0) = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 0 & \ddots & 0 \\ \vdots & 0 & \ldots & 0 \end{pmatrix}.$$

then

$$\hat{T}_{[a(0)\otimes b(0)]}Seg(\mathbb{P}A \times \mathbb{P}B) = \begin{pmatrix} x_1 + y_1 & y_2 & \ldots & y_{\mathbf{b}} \\ x_2 & 0 & \ldots & 0 \\ \vdots & 0 & \vdots & 0 \\ x_{\mathbf{a}} & 0 & \ldots & 0 \end{pmatrix}.$$

For future reference, note that for any two points of the two-factor Segre, their affine tangent spaces must intersect in at least a 2-plane. To see this pictorially, consider the following figures, where the first represents the tangent space to $a_1\otimes b_1$, below it is the tangent space to some $a_i\otimes b_j$, and the third shows the two spaces intersect in the span of $a_1\otimes b_j$ and $a_i\otimes b_1$.

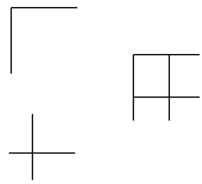


Figure 4.6.1. In a two-factor Segre variety, two tangent spaces must intersect.

Consider the corresponding three-dimensional figure for the 3-factor Segre. One gets that the tangent space to the point $a_1 \otimes b_1 \otimes c_1$ would be represented by three edges of a box coming together in a corner, and the tangent space to another point $a_i \otimes b_j \otimes c_k$ would be represented by three lines intersecting inside the box, and the two tangent spaces would not intersect.

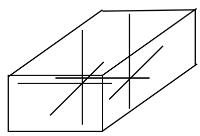


Figure 4.6.2. Two, even three, general tangent spaces of a three-factor Segre variety will not intersect.

4.6.3. Exercises on tangent spaces.

(1) Given $[p] \in Seg(\mathbb{P}A \times \mathbb{P}B)$, consider $p : A^* \to B$ as a linear map. Show that

$$\hat{N}^*_{[p]} Seg(\mathbb{P}A \times \mathbb{P}B) = \ker(p) \otimes (\text{image}(p))^{\perp}. \tag{4.6.1}$$

(2) Similarly, show that

$$\hat{T}_{[p]} Seg(\mathbb{P}A \times \mathbb{P}B) = \{ f : A^* \to B \mid f(\ker(p)) \subseteq \text{image}(p) \}.$$

(3) Considering $\phi \in S^2 V$ as a symmetric linear map $\phi : V^* \to V$, show that

$$\hat{N}^*_{[\phi]} v_2(\mathbb{P}V) = S^2(\ker(\phi)). \tag{4.6.2}$$

(4) Considering $\phi \in \Lambda^2 V$ as a skew-symmetric linear map $\phi : V^* \to V$, show that

$$\hat{N}^*_{[\phi]} G(2, V) = \Lambda^2(\ker(\phi)). \tag{4.6.3}$$

(5) Show that

$$\begin{aligned} \hat{T}_{[v_1 \otimes \cdots \otimes v_n]} & Seg(\mathbb{P}V_1 \times \cdots \times \mathbb{P}V_n) \\ &= V_1 \otimes v_2 \otimes \cdots \otimes v_n + v_1 \otimes V_2 \otimes v_3 \otimes \cdots \otimes v_n + \cdots \\ &\quad + v_1 \otimes \cdots \otimes v_{n-1} \otimes V_n \end{aligned}$$

(the sum is not direct).

(6) Find $\hat{T}_{v_d(x)}v_d(X) \subset S^dV$ in terms of $\hat{T}_xX \subset V$. ⊚

(7) Show that $T_EG(k,V) \simeq E^*{\otimes}V/E$ by differentiating a curve in $G(k,V)$.

4.7. G-varieties and homogeneous varieties

4.7.1. G-varieties.

Definition 4.7.1.1. A variety $X \subset \mathbb{P}V$ is called a *G-variety* if V is a module for the group G and for all $g \in G$ and $x \in X$, $g \cdot x \in X$.

An important property of G-varieties is

The ideals of G-varieties $X \subset \mathbb{P}V$ are G-submodules of $S^\bullet V^*$.

This is because G has an induced action on S^dV^* and $I_d(X)$ is a linear subspace of S^dV^* that is invariant under the action of G, as recall $g \cdot P(x) = P(g^{-1} \cdot x)$.

For an irreducible submodule M, either $M \subset I(X)$ or $M \cap I(X) = 0$. Thus to test if a module M gives equations for X, one only needs to test one polynomial in M.

A special case of G-varieties is the *homogeneous varieties*:

Definition 4.7.1.2. A G-variety X is *homogeneous* if for all $x, y \in X$, there exists $g \in G$ such that $y = g \cdot x$.

Examples of homogeneous varieties include Segre, Veronese, and Grassmann varieties.

Definition 4.7.1.3. A G-variety X is called *quasi-homogeneous* if $X = \overline{G \cdot x}$ for some $x \in X$ such that $G \cdot x$ is a Zariski open subset of X.

For example, the varieties $\sigma_r(Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n))$ are $G = GL(A_1) \times \cdots \times GL(A_n)$ varieties, and $\sigma_2(Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n))$ is quasi-homogeneous with $x = [a_1{\otimes} \cdots {\otimes}\, a_n + a_1'{\otimes} \cdots {\otimes}\, a_n']$, where $a_j, a_j' \in A_j$ are independent vectors.

If $X = \overline{G \cdot x} = \mathbb{P}V$ is quasi-homogeneous, with $G \cdot x \subset X$ a Zariski open subset, the general points of X are the points of $G \cdot x$.

4.7.2. Exercises on quasi-homogeneous varieties.

(1) Show that $\sigma_r(Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n))$ is quasi-homogenous for all $r \leq \min_{j \in 1,\dots,n} \mathbf{a}_j$. ⊚

(2) Show that $\sigma_r(Seg(\mathbb{P}^{r-2} \times \mathbb{P}^{r-2} \times \mathbb{P}^{r-1}))$ is quasi-homogeneous; in

fact, if a_i, b_i, c_s are bases of A, B, C, show that the $GL(A) \times GL(B) \times GL(C)$-orbit of $a_1 \otimes b_1 \otimes c_1 + \cdots + a_{r-1} \otimes b_{r-1} \otimes c_{r-1} + (a_1 + \cdots + a_{r-1}) \otimes (b_1 + \cdots + b_{r-1}) \otimes c_r$ is a Zariski open subset.

4.8. Exercises on Jordan normal form and geometry

These exercises relate the Jordan normal form for a linear map and linear sections of the Segre. While of interest in their own right, they also will serve as a model for algorithms to decompose tensors and symmetric tensors as sums of rank one tensors.

Let $f : V \to V$ be a linear map. Give $\mathbb{C}^2$ basis e_1, e_2 and consider the map

$$\begin{aligned} F : \mathbb{P}V &\to \mathbb{P}(\mathbb{C}^2 \otimes V), \\ [v] &\mapsto [e_1 \otimes v + e_2 \otimes f(v)]. \end{aligned} \tag{4.8.1}$$

(1) Show that $[v]$ is an eigenline for f if and only if $F([v]) \in Seg(\mathbb{P}^1 \times \mathbb{P}V)$. In particular, if the eigenvalues of f are distinct, so f is a "generic" linear map, then there are exactly $\mathbf{v}$ points of intersection with the Segre—this is not an accident; see §4.9.4.

(2) Let $\mathbf{v} = 2$. Show that here $F(\mathbb{P}V)$ either intersects the Segre in two distinct points, or is tangent to the Segre, or is contained in the Segre. Relate this to the possible Jordan normal forms of linear maps $\mathbb{C}^2 \to \mathbb{C}^2$. ⊚

(3) Let $\mathbf{v} = 3$. To each possible Jordan normal form of a map $f : \mathbb{C}^3 \to \mathbb{C}^3$, describe the corresponding intersection of $F(\mathbb{P}V)$ with the Segre.

4.9. Further information regarding algebraic varieties

This section may be skipped until the terms are needed later.

4.9.1. Projections of varieties. Given a vector space V, and a subspace $W \subset V$, one can define the *quotient vector space* V/W whose elements are equivalence classes of elements of V, where $v \equiv u$ if there exists $w \in W$ such that $v = u + w$. One defines the corresponding quotient map $\pi : V \to V/W$, which is a linear map, called the *linear projection*.

Definition 4.9.1.1. Define the *projection* of $X \subset \mathbb{P}V$ such that $X \cap \mathbb{P}W = \emptyset$, onto $\mathbb{P}(V/W)$ by $\mathbb{P}(\pi(\hat{X})) \subset \mathbb{P}(V/W)$, where $\pi : V \to V/W$ is the linear projection.

For such projections, $\dim \pi(X) = \dim X$. In the next paragraph, the notion of projection of a variety is extended to the case where $X \cap \mathbb{P}W \neq \emptyset$.

There is no well-defined map from $\mathbb{P}V$ to $\mathbb{P}(V/W)$. Such a map can only be defined on the open set $\mathbb{P}V\backslash\mathbb{P}W$. Nevertheless, algebraic geometers speak of the *rational map* $\pi_W : \mathbb{P}V \dashrightarrow \mathbb{P}(V/W)$. In general, if X, Y are varieties, and there is a Zariski open subset $U \subset X$ and a regular map $f : U \to Y$, one says there exists a rational map $f : X \dashrightarrow Y$. One calls $\overline{f(U)}$ the *image* of the rational map f. If $X \subset \mathbb{P}V$ is a subvariety and $X \cap \mathbb{P}W \neq \emptyset$, then the *projection* of X onto $\mathbb{P}(V/W)$ is defined to be the image of the rational map from X to $\mathbb{P}(V/W)$, i.e., the Zariski closure of the image of $X\backslash(X \cap \mathbb{P}W)$ under the projection.

A variety is said to be *rational* if it is the image of a rational map of some $\mathbb{P}^n$.

Exercise 4.9.1.2: Show that Segre varieties and Grassmannians are rational.

Exercise 4.9.1.3: Show that the projection of an irreducible variety is irreducible. ⊚

To explain the origin of the terminology "rational map", we need yet another definition:

Definition 4.9.1.4. The *coordinate ring* of $X \subset \mathbb{P}V$ is $\mathbb{C}[X] := S^{\bullet}V^*/I(X)$.

Exercise 4.9.1.5: What is $\mathbb{C}[\mathbb{P}V]$?

Exercise 4.9.1.6: Let $\mathbb{P}W$ be a point. When is $\dim \pi(X) < \dim X$?⊚

In this book, most maps between algebraic varieties arise from maps of the vector spaces of the ambient projective spaces. However, one can define maps more intrinsically as follows: A *regular map* $X \to Y$ is by definition a ring map $\mathbb{C}[Y] \to \mathbb{C}[X]$.

Exercise 4.9.1.7: Show that a ring map $f^* : \mathbb{C}[Y] \to \mathbb{C}[X]$ allows one to define a map $f : X \to Y$.⊚

Similarly, a rational map $X \dashrightarrow Y$ is equivalent to a map from the set of rational functions on Y to the set of rational functions on X. See [**157**, p. 73] for a definition of rational functions and further discussion.

These definitions extend to abstract algebraic varieties, which are defined via their locally defined regular functions.

To compute the equations of $\pi_W(X)$, one generally uses *elimination theory* (see, e.g., [**157**, p. 35] for a brief description or [**141**] for an extended discussion), which is difficult to implement. (Compare with Exercise 4.3.3.1, where the ideal of a linear section is easy to compute.)

For example $Seg(\mathbb{P}V \times \cdots \times \mathbb{P}V) \cap \mathbb{P}(S^dV) = v_d(\mathbb{P}V)$ is the Veronese, which is well understood, but if one projects to the quotient of $V^{\otimes d}$ by the

$GL(V)$-complement to S^dV, which we may identify with S^dV, one obtains the *Chow variety* discussed in §8.6.

Using projections, one obtains an example of a collection of scheme-theoretic defining equations that do not generate the ideal:

Example 4.9.1.8 (The projection of $v_2(\mathbb{P}^2)$)**.** Recall the Veronese surface $v_2(\mathbb{P}^2) \subset \mathbb{P}^5$. Note that $GL_3\mathbb{C}$ has an open orbit in $S^2\mathbb{C}^3$, the orbit of the identity matrix, so the projection of $v_2(\mathbb{P}^2)$ from any point in this orbit will be isomorphic to the projection from any other. The ideal of the projected variety is generated by seven cubic equations. However, four of them are sufficient to cut out the variety scheme-theoretically. See [**255**] for a proof.

4.9.2. Intrinsic tangent and cotangent spaces.

Definition 4.9.2.1. Define the *(Zariski) cotangent space* to be $T^*_xX = \mathfrak{m}_x/\mathfrak{m}_x^2$, where $\mathfrak{m}_x$ is the ideal of functions (e.g., locally defined analytic) $f : U \to \mathbb{C}$ (for some open $U \subset X$ containing x) that are zero at x. Define the *Zariski tangent space* to X at x, T_xX, to be the dual vector space to T^*_xX.

Note that $\mathfrak{m}_x^2 = \mathfrak{m}_x\mathfrak{m}_x$ has the interpretation of the functions vanishing to order two at x.

Remark 4.9.2.2. The Zariski tangent space is the space of (locally defined maps) $f : \mathbb{C} \to X$ with $f(0) = x$, subject to the equivalence relation that two analytic curves $f, g : \mathbb{C} \to X$ are defined to be equivalent if in some choice of local coordinates they agree to first order at x. To verify that T_xX is well defined and a vector space, see, e.g., [**180**, p. 335]. To see that T_xX is the dual space to T^*_xX with this definition, consider the composition of a locally defined function $f : \mathbb{C} \to X$ and a locally defined function $h : X \to \mathbb{C}$ and differentiate at $z = 0$.

Exercise 4.9.2.3: Show that for $X \subset \mathbb{P}V$ and $x \in X_{smooth}$, there is a canonical identification $T_xX \simeq \hat{x}^*{\otimes}(\hat{T}_xX/\hat{x})$. ⊚

Exercise 4.9.2.4: Show that $N^*_xX := (N_xX)^* = \hat{x}{\otimes}\{dP_x | P \in I(X)\} \subset T^*_x\mathbb{P}V = \hat{x}{\otimes}x^{\perp}$.

4.9.3. Hilbert's Nullstellensatz.

Definition 4.9.3.1. An ideal $J \subset S^{\bullet}V^*$ is *radical* if $f^n \in J$ implies $f \in J$.

A basic tool in studying the ideals of varieties is Hilbert's "zero locus theorem":

Theorem 4.9.3.2 (Hilbert's Nullstellensatz in projective space)**.** *Let $J \subset S^{\bullet}V^*$ be an ideal such that $Zeros(J) \neq \emptyset$. Then $I(Zeros(J))$ is a radical ideal.*

There are algorithms for computing the radical of an ideal, see e.g. [**145**]. Unfortunately, the varieties we will deal with typically will have ideals with so many generators, and live in spaces with so many variables, that such algorithms will be ineffective.

4.9.4. Dimension, degree and Hilbert polynomial. A basic property of projective space is that if $X, Y \subset \mathbb{P}V$ are respectively subvarieties of codimension a, b, then $\mathrm{codim}(X \cap Y) \leq a+b$. Equality holds if they intersect transversely, see, e.g., [**247**, Chap. 3].

The codimension of a variety $X \subset \mathbb{P}V$, defined above for irreducible varieties as the codimension of the tangent space at a smooth point, is thus equivalently the largest a such that a general $\mathbb{P}^{a-1} \subset \mathbb{P}V$ will fail to intersect X.

Remark 4.9.4.1. While it is always the case that for nonzero $P \in S^dV^*$, $\mathrm{Zeros}(P) \subset \mathbb{P}V$ has codimension one, codimensions are only subadditive. For example $Seg(\mathbb{P}^1 \times \mathbb{P}^2) \subset \mathbb{P}^5$ is defined by three linearly independent equations, but it has codimension two.

Exercise 4.9.4.2: What is the zero set in $\mathbb{P}^5$ of just two of the 2×2-minors of a 2×3 matrix (x^i_s)?

The *degree* of a variety $X^n \subset \mathbb{P}^{n+a}$ of codimension a is the number of points of intersection of X with a general $\mathbb{P}^a$. A basic fact about degree (see e.g. [**157**, Cor. 18.12]) is that if $X^n \subset \mathbb{P}^{n+a}$ is a variety not contained in a hyperplane, then $\deg(X) \geq a+1$.

Exercise 4.9.4.3: If $X^n \subset \mathbb{P}^{n+1}$ is a hypersurface given by $X = \mathrm{Zeros}(f)$, show that $\deg(X) = \deg(f)$.

Exercise 4.9.4.4: If $\dim L = \mathrm{codim}(X) - 1$, and $X \cap L = \emptyset$, then $\pi_L : X \to \mathbb{P}^n$ defines a finite to one map. Show that the number of points in a general fiber of this map equals the degree of X.

Definition 4.9.4.5. Given a variety $X \subset \mathbb{P}V$, define the *Hilbert function* $Hilb_X(m) := \dim(S^mV^*/I_m(X))$.

If one ignores small values of m, $Hilb_X(m)$ agrees with a polynomial (see, e.g., [**157**, p. 165]) denoted $HilbP_X(m)$ and called the *Hilbert polynomial* of X.

Write $HilbP_X = (d/k!)m^k + O(m^{k-1})$. Then (see, e.g., [**157**, p. 166]), $d = \deg(X)$ and $k = \dim X$.

For example, $\deg(Seg(\mathbb{P}A \times \mathbb{P}B)) = \binom{\mathbf{a}+\mathbf{b}-2}{\mathbf{a}-1}$ and $\deg(v_d(\mathbb{P}^1)) = d$, as you will prove in Exercise 6.10.6.2.

4.9.5. The Zariski topology. In algebraic geometry, one generally uses the *Zariski topology* on $\mathbb{P}V$, where a basis of the closed sets is given by the zero sets of homogenous polynomials. In comparison with the classical manifold topology, the open sets are huge — every nonempty open set is of full measure with respect to any standard measure on $\mathbb{P}V$.

Any open set in the Zariski topology is also open in the classical topology, as the zero set of a polynomial is also closed in the classical topology.

The following consequence of the above remarks will be useful:

Proposition 4.9.5.1. *Any Zariski closed proper subset of $\mathbb{P}V$ has measure zero with respect to any measure on $\mathbb{P}V$ compatible with its linear structure.*

In particular, a tensor not of typical rank has probability zero of being selected at random.

Chapter 5

Secant varieties

The variety of tensors of border rank at most r is a special example of a *secant variety*—the r-th secant variety of the *Segre variety of tensors of border rank one*. Secant varieties have been studied for over a hundred years in algebraic geometry. This chapter covers basic facts about joins and secant varieties.

Secant varieties and joins of arbitrary varieties are introduced in §5.1. The X-rank and X-border rank of a point are introduced in §5.2, as well as several concepts coming from applications. Terracini's lemma, an indispensable tool for computing dimensions of secant varieties, is presented in §5.3. Secant varieties of Veronese and Segre varieties are respectively the varieties of symmetric tensors and of tensors of bounded border rank. Their dimensions are discussed in §5.4 and §5.5 respectively. The Alexander-Hirshowitz theorem determining the dimensions of secant varieties of Veronese varieties, in particular the typical ranks of all spaces of symmetric tensors, is stated in §5.4. (An outline of the proof is given in Chapter 15.) The dimensions of secant varieties of Segre varieties (and in particular, the typical ranks in spaces of tensors) are *not* known. What is known to date is stated in §5.5. Ideas towards the proofs in the Segre cases are given in §5.6. Conjectures of P. Comon and V. Strassen regarding secant varieties of linear sections are discussed in §5.7. The conjectures are respectively of importance in applications to signal processing and computational complexity.

For a variety $X \subset \mathbb{P}V$, $\hat{X} \subset V$ denotes its inverse image under the projection $\pi : V\backslash 0 \to \mathbb{P}V$, the (affine) cone over X in V.

5.1. Joins and secant varieties

5.1.1. Definitions.

Definition 5.1.1.1. The *join* of two varieties $Y, Z \subset \mathbb{P}V$ is

$$J(Y,Z) = \overline{\bigcup_{x\in Y, y\in Z, x\neq y} \mathbb{P}^1_{xy}}.$$

To develop intuition regarding joins, let $C \subset \mathbb{P}V$ be a curve (i.e., a one-dimensional variety) and $q \in \mathbb{P}V$ a point. Let $J(q,C) \subset \mathbb{P}V$ denote the *cone* over C with vertex q, which by definition contains all points on all lines containing q and a point of C. More precisely, $J(q,C)$ denotes the closure of the set of such points. It is only necessary to take the closure when $q \in C$, as in this case one also includes the points on the tangent line to C at q, because the tangent line is the limit of secant lines $\mathbb{P}^1_{q,x_j}$ as $x_j \to q$. Define $J(q,Z)$ similarly for $Z \subset \mathbb{P}V$, a variety of any dimension. Unless Z is a linear space and $q \in Z$, $\dim J(q,Z) = \dim Z + 1$. (Warning: do not confuse a projective variety that is a cone $J(q,X)$ with a cone in affine space.)

One may think of $J(Y,Z)$ as the closure of the union of the cones $\bigcup_{q\in Y} J(q,Z)$ (or as the closure of the the union of the cones over Y with vertices points of Z).

If $Y = Z$, define $\sigma(Y) = \sigma_2(Y) := J(Y,Y)$, the *secant variety* of Y. By the discussion above, $\sigma_2(Y)$ contains all points of all secant and tangent lines to Y. Similarly,

Definition 5.1.1.2. The join of k varieties $X_1, \dots, X_k \subset \mathbb{P}V$ is defined to be the closure of the union of the corresponding $\mathbb{P}^{k-1}$'s, or by induction, $J(Y_1,\dots,Y_k) = J(Y_1, J(Y_2,\dots,Y_k))$. Define *$k$-th secant variety of* Y to be $\sigma_k(Y) = J(Y,\dots,Y)$, the join of k copies of Y.

Example 5.1.1.3. Let $X = Seg(\mathbb{P}V_1 \times \cdots \times \mathbb{P}V_n)$; then $\sigma_r(X) = \sigma_r$ of §4.3.6.

For a proof of the following theorem, see, e.g. [**157**, p. 144].

Theorem 5.1.1.4. *Joins and secant varieties of irreducible varieties are irreducible.*

Corollary 5.1.1.5. *The definitions of σ_r in terms of limits and Zariski closure agree.*

Proof of the corollary. Call the closure of a set by taking limits the "Euclidean closure" and the closure by taking the common zeros of the polynomials vanishing on the set the "Zariski closure". First note that any Zariski closed subset is Euclidean closed. (The zero set of a polynomial is Euclidean closed.) Now if Z is an irreducible variety, and $U \subset Z$ is a Zariski open subset, then $\overline{U} = Z$ in terms of the Zariski closure as well as the Euclidean closure. Now take $Z = \sigma_r(X)$ and U the set elements of X-rank at most r to conclude. $\square$

Exercise 5.1.1.6: Let $p_1, \dots, p_\ell$ be natural numbers with $p_1 + \cdots + p_\ell = k$. Show that

$$\sigma_k(Y) = J(\sigma_{p_1}(Y), \sigma_{p_2}(Y), \dots, \sigma_{p_\ell}(Y)).$$

Proposition 5.1.1.7. *Let $X^n \subset \mathbb{P}V$ be an n-dimensional smooth variety. Let $p \in \sigma(X)$. Then either $p \in X$, or p lies on a tangent line to X, i.e., there exists $x \in X$ such that $p \in \mathbb{P}\hat{T}_x X$, or p lies on a secant line to X, i.e., there exists $x_1, x_2 \in \hat{X}$ and $p = [x_1 + x_2]$.*

Exercise 5.1.1.8: Prove Proposition 5.1.1.7.

While the above proposition is straightforward, a general explicit description for points on $\sigma_r(X)$ is *not known*, even for the case $X = Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)$ (except when $r = 3$, see §10.10). In fact such an understanding would make significant progress on several open questions in computational complexity and statistics.

5.1.2. Expected dimensions of joins and secant varieties. The expected dimension of $J(Y, Z)$ is $\min\{\dim Y + \dim Z + 1, \dim \mathbb{P}V\}$ because a point $x \in J(Y, Z)$ is obtained by picking a point of Y, a point of Z, and a point on the line joining the two points. This expectation fails if and only if every point of $J(Y, Z)$ lies on a one-parameter family of lines intersecting Y and Z, as when this happens, one can vary the points on Y and Z used to form the secant line without varying the point x.

Similarly, the expected dimension of $\sigma_r(Y)$ is $r(\dim Y) + r - 1$, which fails if and only if every point of $\sigma_r(Y)$ lies on a curve of secant $\mathbb{P}^{r-1}$'s to Y.

Definition 5.1.2.1. If $X \subset \mathbb{P}^N$ and $\dim \sigma_r(X) < \min\{rn + r - 1, N\}$, one says $\sigma_r(X)$ is *degenerate*, with *defect* $\delta_r = \delta_r(X) := rn + r - 1 - \dim \sigma_r(X)$. Otherwise one says $\sigma_r(X)$ is *nondegenerate.*

Example 5.1.2.2. Consider $\sigma_2(Seg(\mathbb{P}A \times \mathbb{P}B))$. An open set of this variety may be parametrized as follows: choose bases for A, B and write

$$x_1 = \begin{pmatrix} x_1^1 \\ \vdots \\ x_1^{\mathbf{a}} \end{pmatrix}, \quad x_2 = \begin{pmatrix} x_2^1 \\ \vdots \\ x_2^{\mathbf{a}} \end{pmatrix}.$$

Choose the column vectors x_1, x_2 arbitrarily and then take the matrix

$$p = (x_1, x_2, c_1^3 x_1 + c_2^3 x_2, \dots, c_1^{\mathbf{b}} x_1 + c_2^{\mathbf{b}} x_2)$$

to get a general matrix of rank at most two. Thus $\hat{\sigma}_2(Seg(\mathbb{P}A \times \mathbb{P}B))$ is locally parametrized by $2\mathbf{a} + 2(\mathbf{b} - 2) = 2\mathbf{a} + 2\mathbf{b} - 4$ parameters, i.e.,

$\dim \sigma_2(Seg(\mathbb{P}A \times \mathbb{P}B)) = 2\mathbf{a} + 2(\mathbf{b}-2) = 2\mathbf{a} + 2\mathbf{b} - 5$ compared with the expected $2[(\mathbf{a}-1)+(\mathbf{b}-1)]+1 = 2\mathbf{a}+2\mathbf{b}-3$, so $\delta_2(Seg(\mathbb{P}A \times \mathbb{P}B)) = 2$.

Strassen's equations of §3.8 show:

Theorem 5.1.2.3 (Strassen). *If n is odd, $\sigma_{\frac{3n-1}{2}}(Seg(\mathbb{P}^2 \times \mathbb{P}^{n-1} \times \mathbb{P}^{n-1}))$ is a hypersurface. In particular, it is defective.*

Exercise 5.1.2.4: Compute the dimensions and defects of the variety $\sigma_r(Seg(\mathbb{P}^n \times \mathbb{P}^m))$.

5.1.3. Zak's theorems. (This subsection is not used elsewhere in the book.) Smooth projective varieties $X^n \subset \mathbb{P}^{n+m}$ of small codimension were shown by Barth and Larsen (see, e.g., [**17**]) to behave topologically as if they were *complete intersections*, i.e., the zero set of m homogeneous polynomials. This motivated Hartshorne's famous conjecture on complete intersections [**160**], which says that if $m < \frac{n}{2}$, then X must indeed be a complete intersection. A first approximation to this difficult conjecture was also conjectured by Hartshorne—his conjecture on *linear normality*, which was proved by Zak [**338**]; see [**339**] for an exposition. The linear normality conjecture was equivalent to showing that if $m < \frac{n}{2} + 2$, and X is not contained in a hyperplane, then $\sigma_2(X) = \mathbb{P}^{n+m}$. Zak went on to classify the exceptions in the equality case $m = \frac{n}{2} + 2$. There are exactly four, which Zak called *Severi varieties* (after Severi, who solved the $n = 2$ case [**288**]). Three of the Severi varieties have already been introduced: $v_2(\mathbb{P}^2) \subset \mathbb{P}^5$, $Seg(\mathbb{P}^2 \times \mathbb{P}^2) \subset \mathbb{P}^8$, and $G(2,6) \subset \mathbb{P}^{14}$. The last is the complexified Cayley plane $\mathbb{OP}^2 \subset \mathbb{P}^{26}$. These four varieties admit a uniform interpretation as the rank one elements in a rank three Jordan algebra over a composition algebra; see, e.g., [**202**].

An interesting open question is the *secant defect problem*. For a smooth projective variety $X^n \subset \mathbb{P}V$, not contained in a hyperplane, with $\sigma_2(X) \neq \mathbb{P}V$, let $\delta(X^n) = 2n + 1 - \dim \sigma_2(X)$, the *secant defect of* X. The largest known secant defect is 8, which occurs for the complexified Cayley plane. Problem: Is a larger secant defect than 8 possible? If one does not assume the variety is smooth, the defect is unbounded. (This question was originally posed in [**218**].)

5.2. Geometry of rank and border rank

5.2.1. X-rank.

Definition 5.2.1.1. For a variety $X \subset \mathbb{P}V$ and a point $p \in \mathbb{P}V$, the *X-rank of p*, $\mathbf{R}_X(p)$, is the smallest number r such that p is in the linear span of r points of X. Thus, if $\sigma_{r-1}(X) \neq \mathbb{P}V$, $\sigma_r(X)$ is the Zariski closure of the set of points of X-rank r. The *X-border rank of p*, $\underline{\mathbf{R}}_X(p)$, is the smallest r

such that $p \in \sigma_r(X)$. The *typical X-rank* of $\mathbb{P}V$ is the smallest r such that $\sigma_r(X) = \mathbb{P}V$.

To connect this with our previous notions of tensor rank and symmetric tensor rank, when $V = A_1 \otimes \cdots \otimes A_n$ and $X = Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n)$, $\mathbf{R}_X([p]) = \mathbf{R}(p)$, and when $V = S^d W$ and $X = v_d(\mathbb{P}W)$, $\mathbf{R}_X([p]) = \mathbf{R}_S(p)$. The partially symmetric tensor rank of $\phi \in S^{d_1}A_1 \otimes \cdots \otimes S^{d_n}A_n$ (cf. §3.6) is $R_{Seg(v_{d_1}(\mathbb{P}A_1)\times\cdots\times v_{d_n}(\mathbb{P}A_n))}(\phi)$.

A priori $\mathbf{R}_X(p) \geq \underline{\mathbf{R}}_X(p)$ and, as the following example shows, strict inequality can occur.

Example 5.2.1.2 (Example 1.2.6.1 revisited)**.** By Exercise 5.3.2(3), $\sigma_2(v_3(\mathbb{P}^1)) = \mathbb{P}(S^3\mathbb{C}^2)$ so a general homogeneous cubic polynomial in two variables is the sum of two cubes. We saw in Example 1.2.6.1 that $[x^3 + x^2 y]$ was not. We see now it is on the tangent line to $[x^3]$, which is the limit as $\epsilon \to 0$ of secant lines through the points $[x^3]$ and $[(x+\epsilon y)^3]$.

5.2.2. Examples of expected typical rank.

variety	$v_d(\mathbb{P}^{n-1})$	$Seg(\mathbb{P}^{\mathbf{a}-1}\times\mathbb{P}^{\mathbf{b}-1}\times\mathbb{P}^{\mathbf{c}-1})$	$Seg(\mathbb{P}^{n-1}\times\cdots\times\mathbb{P}^{n-1})$
exp. typ. rk.	$\frac{1}{n}\binom{n+d-1}{d}$	$\frac{\mathbf{abc}}{\mathbf{a}+\mathbf{b}+\mathbf{c}-2}$	$\frac{n^d}{dn-d+1}$

In particular, for $n = 2$, one expects

$$\sigma_r(Seg(\mathbb{P}^1 \times \cdots \times \mathbb{P}^1)) = \mathbb{P}(\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2)$$

when $r \geq \frac{2^d}{d+1}$.

For $\mathbf{a} = \mathbf{b} = \mathbf{c} = 4$, one expects $\sigma_7(Seg(\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3)) = \mathbb{P}(\mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4)$, and this is indeed the case, which explains how Strassen's algorithm for multiplying 2×2 matrices could have been anticipated.

In the case of the Veronese, there are very few cases where the typical rank is larger than expected, all of which are known, see §5.4. In the case of the Segre, most cases are known to be as expected, but the classification is not yet complete. See §5.5 for what is known.

5.2.3. Typical ranks over the reals. When one considers algebraic varieties over the ground field $\mathbb{R}$ instead of $\mathbb{C}$, the *typical X-rank* is defined differently: r is *a typical X-rank* if there is a set of points of nonempty interior in $\mathbb{P}V$ having X-rank r. If one then complexifies, i.e., considers the complex solutions to the same polynomials defining X, and studies their zero set $X^{\mathbb{C}} \subset \mathbb{P}(V \otimes \mathbb{C})$, the smallest typical X-rank will be the typical $X^{\mathbb{C}}$-rank.

For example, when $X = Seg(\mathbb{RP}^1 \times \mathbb{RP}^1 \times \mathbb{RP}^1)$, the typical X-ranks are 2 and 3, which was observed by Kruskal (unpublished); see, e.g., [**101**].

5.2.4. Intermediate notions between rank and border rank. Let $X \subset \mathbb{P}V$ be a variety and let $[p] \in \sigma_r(X)$, so there are curves $[x_1(t)], \dots, [x_r(t)] \subset X$ such that $p \in \lim_{t\to 0}\langle x_1(t), \dots, x_r(t)\rangle$, where the limiting plane exists as a point in the Grassmannian because $G(r, V)$ is compact. Thus there exists an order j such that $p = \frac{1}{t^j}(x_1(t) + \cdots + x_r(t))|_{t=0}$. Assume we have chosen curves $x_k(t)$ to minimize j. Define, for $[y] \in \mathbb{P}V$,

$$R^j_X([y]) := \min_r \left\{ \exists x_1(t), \dots, x_r(t) \subset X \;\middle|\; y = \frac{1}{t^j}(x_1(t) + \cdots + x_r(t))|_{t=0} \right\}.$$

Note that there exists f such that

$$R_X([y]) = R^0_X([y]) \geq R^1_X([y]) \geq \cdots \geq R^f_X([y]) \\ = R^{f+1}_X([y]) = R^\infty_X([y]) = \underline{R}_X([y]).$$

Open Questions: Can f be bounded in terms of invariants of X? Are there explicit bounds when X is a Segre or Veronese variety? See §10.8 for examples when these bounds are known.

Remark 5.2.4.1. The ability to bound the power of t in the discrepancy between rank and border rank is related to the question of whether or not one has the equality of complexity classes $\mathbf{VP} = \overline{\mathbf{VP}}$, see [**50**].

5.3. Terracini's lemma and first consequences

5.3.1. Statement and proof of the lemma. Let $X \subset \mathbb{P}V$ be a variety and $[x_1], \dots, [x_r] \in X$. If $p = [x_1 + \cdots + x_r]$ is a general point, the dimension of $\sigma_r(X)$ is its expected dimension minus the number of ways one can move the $[x_i]$ such that their span still contains p. Terracini's lemma is an infinitesimal version of this remark.

Lemma 5.3.1.1 (Terracini's lemma)**.** *Let $(v, w) \in (\hat{Y} \times \hat{Z})_{general}$ and let $[u] = [v + w] \in J(Y, Z)$; then*

$$\hat{T}_{[u]}J(Y, Z) = \hat{T}_{[v]}Y + \hat{T}_{[w]}Z.$$

Proof. Consider the addition map add : $V \times V \to V$ given by $(v, w) \mapsto v + w$. Then

$$\hat{J}(Y, Z) = \overline{\mathrm{add}(\hat{Y} \times \hat{Z})}.$$

Thus for $(v, w) \in (\hat{Y} \times \hat{Z})_{general}$, $\hat{T}_{[v+w]}J(Y, Z)$ may be obtained by differentiating curves $v(t) \subset \hat{Y}$, $w(t) \subset \hat{Z}$ with $v(0) = v$, $w(0) = w$. The result follows. □

Corollary 5.3.1.2. *If $(x_1, \dots, x_r) \in (X^{\times r})_{general}$, then*

$$\dim \sigma_r(X) = \dim(\hat{T}_{x_1}X + \cdots + \hat{T}_{x_r}X) - 1.$$

Corollary 5.3.1.3. *If $X^n \subset \mathbb{P}^N$ and $\sigma_r(X)$ is of (the expected) dimension $rn+r-1<N$, then a Zariski open subset of points on $\hat{\sigma}_r(X)$ have a finite number of decompositions into a sum of r elements of $\hat{X}$.*

Proof. If there were an infinite number of points, there would have to be a curve's worth (as X is compact algebraic), in which case the secant variety would have to be degenerate. □

Proposition 5.3.1.4. $\dim \sigma_r(Seg(\mathbb{P}A\times\mathbb{P}B)) = \min\{\mathbf{ab}-1, r(\mathbf{a}+\mathbf{b}-r)-1\}$, *so* $\delta_r(Seg(\mathbb{P}A\times\mathbb{P}B)) = 2r^2-2r-2$.

Proof. Let $[p]\in\sigma_r(Seg(\mathbb{P}A\times\mathbb{P}B))$ be a general point and write $p = a_1\otimes b_1+\cdots+a_r\otimes b_r$. We may assume that all the a_i are linearly independent as well as all the b_i (otherwise one would have $[p]\in\sigma_{r-1}(Seg(\mathbb{P}A\times\mathbb{P}B))$). Terracini's lemma gives

$$\begin{aligned}\hat{T}_p\sigma_r &= A\otimes\langle b_1,\dots,b_r\rangle + \langle a_1,\dots,a_r\rangle\otimes B\\ &\simeq \langle a_1,\dots,a_r\rangle\otimes\langle b_1,\dots,b_r\rangle + (A/\langle a_1,\dots,a_r\rangle)\otimes\langle b_1,\dots,b_r\rangle\\ &\quad + \langle a_1,\dots,a_r\rangle\otimes(B/\langle b_1,\dots,b_r\rangle),\end{aligned}$$

where the quotient spaces in the last term are well defined modulo the first term, and $\langle x_j\rangle$ denotes the span of the vectors x_j. These give disjoint spaces of dimensions $r^2, r(\mathbf{a}-r), r(\mathbf{b}-r)$, and one concludes. □

Note that in the proof above we were lucky that there was a normal form for a point of $\sigma_r(Seg(\mathbb{P}A\times\mathbb{P}B))$. A difficulty in applying Terracini's lemma already to 3-factor Segre varieties is that there is no easy way to write down general points when $r>\mathbf{a},\mathbf{b},\mathbf{c}$.

Example 5.3.1.5. Let $X = Seg(\mathbb{P}A\times\mathbb{P}B\times\mathbb{P}C)$. Write $p = a_1\otimes b_1\otimes c_1 + a_2\otimes b_2\otimes c_2$ for a general point of $\hat{\sigma}_2(X)$, where a_1, a_2, etc. are linearly independent. Then

$$\begin{aligned}\hat{T}_{[p]}\sigma_2(X) = \hat{T}_{[a_1\otimes b_1\otimes c_1]}X + \hat{T}_{[a_1\otimes b_1\otimes c_1]}X\\ = a_1\otimes b_1\otimes C + a_1\otimes B\otimes c_1 + A\otimes b_1\otimes c_1\\ + a_2\otimes b_2\otimes C + a_2\otimes B\otimes c_2 + A\otimes b_2\otimes c_2,\end{aligned}$$

and it is easy to see that these affine tangent spaces intersect only at the origin. (See Figure 4.6.2.) Thus $\dim\sigma_2(Seg(\mathbb{P}A\times\mathbb{P}B\times\mathbb{P}C)) = 2\dim Seg(\mathbb{P}A\times\mathbb{P}B\times\mathbb{P}C)+1$, i.e., $\delta_2(Seg(\mathbb{P}A\times\mathbb{P}B\times\mathbb{P}C)) = 0$.

Proposition 5.3.1.6. *Let $r\le\min\{\dim V_i\}$ and assume $m>2$. Then $\sigma_r(Seg(\mathbb{P}V_1\times\cdots\times\mathbb{P}V_m))$ is of the expected dimension $r(\dim V_1+\cdots+\dim V_m-m)+(r-1)$.*

5.3.2. Exercises on dimensions of secant varieties.

(1) Prove Proposition 5.3.1.6.

(2) Show that $\sigma_3(Seg(\mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^2)) = \mathbb{P}(\mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3)$.

(3) If $C \subset \mathbb{P}^N$ is an irreducible curve not contained in a hyperplane, show that $\dim \sigma_r(C) = \min\{2r-1, N\}$. ⊚

(4) Show that if $X^n \subset \mathbb{P}^N$ is a variety and $\sigma_r(X)$ is nondegenerate and not equal to the entire ambient space, then $\sigma_k(X)$ is nondegenerate for all $k < r$. Show that if $\sigma_r(X)$ equals the ambient space and $rn + r - 1 = N$, the same conclusion holds.

(5) Calculate $\dim \sigma_r(G(2, V))$.

(6) Show that the maximum dimension of a linear subspace of the $\mathbf{a} \times \mathbf{b}$ matrices of rank bounded below by r is $(\mathbf{a}-r)(\mathbf{b}-r)$. Show that for symmetric matrices of size $\mathbf{a}$ it is $\binom{\mathbf{a}-r+1}{2}$ and for skew-symmetric matrices (with r even) it is $\binom{\mathbf{a}-r}{2}$. ⊚

(7) Show that $\sigma_2(Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n))$ is always of the expected dimension when $n > 2$.

(8) Show that $\sigma_{\binom{n+2}{2}-1}(v_4(\mathbb{P}^n))$ is of codimension at most one for $2 \leq n \leq 4$ to complete the proof of Corollary 3.5.1.5.

5.3.3. Conormal spaces. The standard method to prove that a module of equations locally defines $\sigma_r(v_d(\mathbb{P}^n))$ is by showing that the conormal space at a smooth point of the zero set of the module equals the conormal space to $\sigma_r(v_d(\mathbb{P}^n))$ at that point.

Let $Rank^r_{(a,d-a)}(S^dV) \subset \mathbb{P}(S^dV)$ denote the zero set of the size $(r+1)$ minors of the flattenings $S^aV^* \to S^{d-a}V$. One has

Proposition 5.3.3.1. *Let $[\phi] \in Rank^r_{(a,d-a)}(S^dV)$ be a sufficiently general point; then*

$$\hat{N}^*_{[\phi]} Rank^r_{(a,d-a)}(S^dV) = \ker(\phi_{a,d-a}) \circ \operatorname{image}(\phi_{a,d-a})^{\perp} \subset S^dV^*.$$

Exercise 5.3.3.2: Verify Proposition 5.3.3.1.

Proposition 5.3.3.3 (Lasker, 1900)**.** *Let $[y^d] \in v_d(\mathbb{P}V)$; then*

$$\begin{aligned}\hat{N}^*_{[y^d]} v_d(\mathbb{P}V) &= \{P \in S^dV^* \mid P(y) = 0,\ dP_y = 0\}\\ &= S^{d-2}V^* \circ S^2 y^{\perp}\\ &= \{P \in S^dV^* \mid [y] \in (\mathrm{Zeros}(P))_{sing}\}.\end{aligned}$$

Exercise 5.3.3.4: Verify Proposition 5.3.3.3.

Applying Terracini's lemma yields the following.

Proposition 5.3.3.5. *Let* $[\phi] = [y_1^d + \cdots + y_r^d] \in \sigma_r(v_d(\mathbb{P}V))$. *Then*

$$\begin{aligned}\hat{N}^*_{[\phi]}\sigma_r(v_d(\mathbb{P}V)) &\subseteq (S^{d-2}V^* \circ S^2 {y_1}^{\perp}) \cap \cdots \cap (S^{d-2}V^* \circ S^2 {y_r}^{\perp})\\ &= \{P \in S^d V^* \mid \mathrm{Zeros}(P) \text{ is singular at } [y_1], \dots, [y_r]\},\end{aligned}$$

and equality holds if ϕ *is sufficiently general.*

5.4. The polynomial Waring problem

5.4.1. Dimensions of secant varieties of Veronese varieties. Waring asked if there exists a function $s(k)$ such that every natural number n is expressible as at most s k-th powers of integers. Hilbert answered Waring's question affirmatively, and this function has essentially been determined; see [**321**] for a survey of results and related open questions.

The *polynomial Waring problem* is as follows:

What is the smallest $r_0 = r_0(d, n)$ *such that a general homogeneous polynomial* $P(x^1, \dots, x^n)$ *of degree* d *in* n *variables is expressible as the sum of* r_0 d*-th powers of linear forms? (I.e.,* $P = l_1^d + \cdots + l_{r_0}^d$ *with* $l_j \in V^*$*.)*

The image of the Veronese map $v_d : \mathbb{P}V^* \to \mathbb{P}(S^d V^*)$ is the set of (projectivized) d-th powers of linear forms, or equivalently the hypersurfaces of degree d whose zero sets are hyperplanes counted with multiplicity d. Similarly $\sigma_p(v_d(\mathbb{P}V))$ is the Zariski closure of the set of homogeneous polynomials that are expressible as the sum of p d-th powers of linear forms. So the polynomial Waring problem may be restated as follows:

Let $X = v_d(\mathbb{P}V^*)$*. What is the typical* X*-rank of an element of* $\mathbb{P}(S^d V^*)$*, that is, what is the smallest* $r_0 = r_0(d, \mathbf{v})$ *such that* $\sigma_{r_0}(v_d(\mathbb{P}V^*)) = \mathbb{P}S^d V^*$*?*

Recall from §5.1.2 that the expected r_0 is $\lceil \frac{\binom{\mathbf{v}+d-1}{d}}{\mathbf{v}} \rceil$. We have seen the following exceptions:

- $\sigma_7(v_3(\mathbb{P}^4))$, which is a hypersurface, Theorem 3.10.2.3.
- $\sigma_{\binom{n+2}{2}-1}(v_4(\mathbb{P}^n))$, $n = 2, 3, 4$, which is a hypersurface, Corollary 3.5.1.5.
- $\sigma_r(v_2(\mathbb{P}^n))$, $2 \le r \le n$, where $\dim \sigma_r(v_2(\mathbb{P}^n)) = \binom{r+1}{2} + r(n+1-r) - 1$, Exercise 5.1.2.4.

The polynomial Waring problem was solved by Alexander and Hirschowitz [**7**] in 1995: all $\sigma_r(v_d(\mathbb{P}^n))$ are of the expected dimension $\min\{r(n+1)-1, \binom{n+d}{d} - 1\}$ except the examples above.

Theorem 5.4.1.1 ([**7**]). *A general homogeneous polynomial of degree d in m variables is expressible as the sum of*

$$r_0(d,m) = \lceil \frac{\binom{m+d-1}{d}}{m} \rceil$$

d-th powers with the exception of the cases $r_0(3,5) = 8$, $r_0(4,3) = 6$, $r_0(4,4) = 10$, $r_0(4,5) = 15$, and $r_0(2,m) = m$.

The Alexander-Hirschowitz theorem had been conjectured by Palatini in 1901, which, as G. Ottaviani remarks, is impressive, considering how many exceptions occur in the beginning. The case of $m = 3$ was proved by Terracini in 1916.

5.4.2. Rephrasing of Alexander-Hirschowitz theorem in terms of prescribing hypersurface singularities. Theorem 5.4.1.1 is presented in [**7**] in the following form:

Theorem 5.4.2.1 (AH theorem, alternative formulation). *Let $x_1, \dots, x_k \in V = \mathbb{C}^{n+1}$ be a general collection of points. Let $I_d \subset S^d V^*$ consist of all polynomials P such that $P(x_i) = 0$ and $(dP)_{x_i} = 0$, $1 \leq i \leq k$, i.e., such that $[x_j] \in Zeros(P)_{sing}$. Then*

$$\operatorname{codim}(I_d) = \min\left\{(n+1)k, \binom{n+d}{n}\right\} \tag{5.4.1}$$

except in the following cases:

- *$d = 2,\ 2 \leq k \leq n$;*
- *$n = 2,3,4,\ d = 4,\ k = n(n+1)/3$;*
- *$n = 4,\ d = 3,\ k = 7$.*

If (5.4.1) holds, one says "the points $[x_i]$ impose independent conditions on hypersurfaces of degree d."

To relate this formulation to the previous ones, consider the points $v_d([x_j])$. I_d is just the annihilator of the points $(x_j)^d, (x_j)^{d-1}y$, $1 \leq j \leq k$, with $y \in V$ arbitrary. Since

$$\hat{N}^*_{v_d(x_j)} v_d(X) = [\hat{T}_{v_d(x_j)} v_d(X)]^\perp = \langle (x_j)^{d-1} y \mid y \in V \rangle^\perp,$$

by Terracini's lemma I_d is the affine conormal space to $\sigma_k(v_d(\mathbb{P}V))$ at the point $[z] = [x_1 + \cdots + x_k]$.

Generalizations of the polynomial Waring problem and their uses are discussed in [**89**].

An outline of the proof of the Alexander-Hirschowitz theorem is given in Chapter 15.

Exercise 5.4.2.2: This exercise outlines the classical proof that $\sigma_7(v_3(\mathbb{P}^4))$ is defective. Fix seven general points $p_1, \dots, p_7 \in \mathbb{P}^4$ and let $I_3 \subset S^3\mathbb{C}^5$ denote their ideal in degree three. Show that one expects $I_3 = 0$. But then show that any such seven points must lie on a rational normal curve $v_4(\mathbb{P}^1) \subset \mathbb{P}^4$. Consider the equation of $\sigma_2(v_4(\mathbb{P}^1))$ in $S^3\mathbb{C}^5$; it is a cubic hypersurface singular at $p_1, \dots, p_7$, but in fact it is singular on the entire rational normal curve. Conclude.

5.5. Dimensions of secant varieties of Segre varieties

5.5.1. Triple Segre products. The expected dimension of σ_r = the variety $\sigma_r(Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C))$ is $r(\mathbf{a}-1+\mathbf{b}-1+\mathbf{c}-1)+r-1 = r(\mathbf{a}+\mathbf{b}+\mathbf{c}-2)-1$. The dimension of the ambient space is $\mathbf{abc}-1$, so one expects σ_r to fill $\mathbb{P}(A \otimes B \otimes C)$ as soon as $r(\mathbf{a}+\mathbf{b}+\mathbf{c}-2)-1 \geq \mathbf{abc}-1$, i.e.,

$$r \geq \frac{\mathbf{abc}}{\mathbf{a}+\mathbf{b}+\mathbf{c}-2}. \tag{5.5.1}$$

Theorem 5.5.1.1. *Facts about dimensions of secant varieties of triple Segre products:*

(1) ([**300**]) $\sigma_4(Seg(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2))$ *is a hypersurface and thus defective with defect two (see* §7.6*).*

(2) ([**220**]) *For all* $n \neq 3$, $\sigma_r(Seg(\mathbb{P}^{n-1} \times \mathbb{P}^{n-1} \times \mathbb{P}^{n-1}))$ *is of the expected dimension* $\min\{r(3n-2)-1, n^3-1\}$.

(3) $\sigma_{\mathbf{b}}(Seg(\mathbb{P}^1 \times \mathbb{P}^{\mathbf{b}-1} \times \mathbb{P}^{\mathbf{b}-1})) = \mathbb{P}(\mathbb{C}^2 \otimes \mathbb{C}^{\mathbf{b}} \otimes \mathbb{C}^{\mathbf{b}})$ *as expected.*

(4) ([**81**]) *Assume that* $\mathbf{a} \leq \mathbf{b} \leq \mathbf{c}$. *If* $\mathbf{c} > r > \mathbf{ab}-\mathbf{a}-\mathbf{b}-2$, *then* $\sigma_r(Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C))$ *is defective with defect* $r^2 - r(\mathbf{ab}-\mathbf{a}-\mathbf{b}+2)$. *If* $r \leq \mathbf{ab}-\mathbf{a}-\mathbf{b}-2$, *then* $\sigma_r(Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C))$ *is of the expected dimension* $r(\mathbf{a}+\mathbf{b}+\mathbf{c}-2)$. *(See* §7.3.1 *for a proof.)*

(5) ([**220**]) $\sigma_r(Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C))$ *is nondegenerate for all* r *whenever* $\mathbf{a} \leq \mathbf{b} \leq \mathbf{c}$, $\mathbf{b}, \mathbf{c}$ *are even and* $\mathbf{abc}/(\mathbf{a}+\mathbf{b}+\mathbf{c}-2)$ *is an integer.*

Other cases have been determined to be nondegenerate by Lickteig and Strassen (see [**220**]), but a complete list is still not known.

In [**4**] they ask if the only defective $\sigma_r(Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C))$ are either unbalanced (i.e., $\mathbf{a}, \mathbf{b} << \mathbf{c}$) or $Seg(\mathbb{P}^2 \times \mathbb{P}^n \times \mathbb{P}^n)$ with n even, or $Seg(\mathbb{P}^2 \times \mathbb{P}^3 \times \mathbb{P}^3)$. They then go on to ask if the only defective secant varieties of any Segre product are either as in Theorem 5.5.2.1 below or $Seg(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^n \times \mathbb{P}^n)$.

Exercise 5.5.1.2: Show that a general point in $\mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4$ admits a six-parameter family of expressions as a sum of 7 rank one elements. (Note that this is less than the nine-parameter family for matrix multiplication $M_{2,2,2}$.)

Remark 5.5.1.3 (Typical rank and matrix multiplication)**.** In the case $\mathbf{a} = \mathbf{b} = \mathbf{c}$, formula (5.5.1) specializes to $r \geq \mathbf{a}^3/(3\mathbf{a} - 2)$, which for large $\mathbf{a}$ is roughly $\mathbf{a}^2/3$. Taking $\mathbf{a} = n^2$, it is roughly $n^4/3$, illustrating that matrix multiplication is far from being a generic bilinear map, as even the standard algorithm gives $\underline{\mathbf{R}}(M_{n,n,n}) \leq n^3$. (The actual typical X-rank cannot be smaller than the expected typical X-rank.) However, for $n = 2$ one obtains $r \geq 64/10$ and thus $r = 7$ is expected to, and by Theorem 5.5.1.1.2 does, fill, so $M_{2,2,2}$ is generic in this sense.

5.5.2. More general Segre products.

Theorem 5.5.2.1 ([**4**])**.** *Consider $\sigma_r(Seg(\mathbb{P}^{a_1-1} \times \cdots \times \mathbb{P}^{\mathbf{a}_n-1}))$. The following are the only defective cases for $r \leq 6$:*

r	n	$(\mathbf{a}_1, \ldots, \mathbf{a}_n)$	
2	2	$(\mathbf{a}_1, \mathbf{a}_2)$,	$\mathbf{a}_j > 2$
3	3	$(2, 2, \mathbf{a}_3)$,	$\mathbf{a}_3 \geq 4$
3	4	$(2, 2, 2, 2)$	
4	3	$(2, 3, \mathbf{a}_3)$,	$\mathbf{a}_3 \geq 5$
4	3	$(3, 3, 3)$	
5	3	$(2, 3, \mathbf{a}_3)$,	$\mathbf{a}_3 \geq 6$
5	3	$(2, 4, \mathbf{a}_3)$,	$\mathbf{a}_3 \geq 6$
5	4	$(2, 2, 3, 3)$	
6	3	$(2, 4, \mathbf{a}_3)$,	$\mathbf{a}_3 \geq 7$
6	3	$(2, 5, \mathbf{a}_3)$,	$\mathbf{a}_3 \geq 7$
6	3	$(3, 3, \mathbf{a}_3)$,	$\mathbf{a}_3 \geq 7$
6	4	$(2, 2, 2, \mathbf{a}_4)$,	$\mathbf{a}_4 \geq 7$

Theorem 5.5.2.2 ([**81**, Thm 2.4.2])**.** *Consider $Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n)$, where $\dim A_s = \mathbf{a}_s$, $1 \leq s \leq n$. Assume that $\mathbf{a}_n \geq \prod_{i=1}^{n-1} \mathbf{a}_i - \sum_{i=1}^{n-1} \mathbf{a}_i - n + 1$.*

(1) *If $r \leq \prod_{i=1}^{n-1} \mathbf{a}_i - \sum_{i=1}^{n-1} \mathbf{a}_i - n + 1$, then $\sigma_r(Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n))$ has the expected dimension $r(\mathbf{a}_1 + \cdots + \mathbf{a}_n - n + 1) - 1$.*

(2) *If $\mathbf{a}_n > r \geq \prod_{i=1}^{n-1} \mathbf{a}_i - \sum_{i=1}^{n-1} \mathbf{a}_i - n + 1$, then $\sigma_r(Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n))$ has defect $\delta_r = r^2 - r(\prod \mathbf{a}_j - \sum \mathbf{a}_j + n - 1)$.*

(3) *If $r \geq \min\{\mathbf{a}_n, \mathbf{a}_1, \ldots, \mathbf{a}_{n-1}\}$, then $\sigma_r(Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n)) = \mathbb{P}(A_1 \otimes \cdots \otimes A_n)$.*

For a proof and more detailed statement, see §7.3.1.

Theorem 5.5.2.3 ([**248**])**.** *The secant varieties of the Segre product of k copies of $\mathbb{P}^1$, $\sigma_r(Seg(\mathbb{P}^1 \times \cdots \times \mathbb{P}^1))$, have the expected dimension except when $k = 2, 4$.*

In the past few years there have been several papers on the dimensions of secant varieties of Segre varieties, e.g., [**77, 78, 75, 74, 4, 248**]. These papers use methods similar to those of Strassen and Lickteig, but the language is more geometric (fat points, degeneration arguments). Some explanation of the relation between the algebreo-geometric and tensor language is given in [**4**].

5.5.3. Dimensions of secant varieties of Segre-Veronese varieties. In [**3**] the dimensions of secant varieties of $Seg(\mathbb{P}A \times v_2(\mathbb{P}B))$ are classified in many cases. In particular, there is a complete classification when $\mathbf{b} = 2$ or 3. Such varieties are never defective when $\mathbf{b} = 2$, which dates back at least to [**70**], and are only defective when $\mathbf{b} = 3$ if $(\mathbf{a}, r) = (2k+1, 3k+2)$ with $k \geq 1$. The defective examples had been already observed in [**70, 256**]. Also see [**80**].

Theorem 5.5.3.1 ([**3**]). (1) *$\sigma_r(Seg(\mathbb{P}^1 \times v_2(\mathbb{P}^n)))$ is never defective.*

(2) *$\sigma_r(Seg(\mathbb{P}^2 \times v_2(\mathbb{P}^n)))$ has the expected dimension unless $n = 2k+1$ and $r = \lceil \frac{3\binom{2k+2}{2}}{2k+4} \rceil$.*

(3) *$\sigma_r(Seg(\mathbb{P}^3 \times v_2(\mathbb{P}^n)))$ has the expected dimension if $r \leq \lfloor \frac{4\binom{n+2}{2}}{n+4} \rfloor$ (other cases are not known).*

(4) *$\sigma_r(Seg(\mathbb{P}^m \times v_2(\mathbb{P}^n)))$ has the expected dimension if $r \leq \lfloor \frac{(m+1)\binom{n+2}{2}}{m+n+1} \rfloor$ (other cases are not known).*

(5) *([**1**]) $\sigma_r(Seg(\mathbb{P}^{n+1} \times v_2(\mathbb{P}^n)))$ and $\sigma_r(Seg(\mathbb{P}^n \times v_2(\mathbb{P}^n)))$ always have the expected dimension except for $\sigma_6(Seg(\mathbb{P}^4 \times v_2(\mathbb{P}^3)))$.*

Here is a table of two-factor cases (courtesy of [**3**]). Dimensions (n, m) and degrees (p, q) mean $S^p\mathbb{C}^n \otimes S^q\mathbb{C}^m$.

dimensions	degrees	defective for $r =$	References
$(3, 2k+3)$	$(1, 2)$	$3k+2$	[**256**]
$(5, 4)$	$(1, 2)$	6	[**70**]
$(2, 3)$	$(1, 3)$	5	[**115**], [**70**]
$(2, n+1)$	$(2, 2)$	$n+2 \leq s \leq 2n+1$	[**77**], [**81**], [**68**]
$(3, 3)$	$(2, 2)$	7, 8	[**77**], [**81**]
$(3, n+1)$	$(2, 2)$	$\lfloor \frac{3n^2+9n+5}{n+3} \rfloor \leq s \leq 3n+2$	[**77**], [**30**]
$(4, 4)$	$(2, 2)$	14, 15	[**77**], [**81**]
$(4, 5)$	$(2, 2)$	19	[**30**]
$(n+1, 1)$	$(2, 2k)$	$kn+k+1 \leq s \leq kn+k+n$	[**5**]

In Chapter 7 we will need the following case so I isolate it.

Proposition 5.5.3.2. $\sigma_4(Seg(v_2(\mathbb{P}^2)\times\mathbb{P}^3)) = \mathbb{P}(S^2\mathbb{C}^3\otimes\mathbb{C}^4)$.

5.5.4. Dimensions of secant varieties of Grassmannians. Several examples of secant defective Grassmannians were known "classically". For $G(k,V)$, assume $2k \le \mathbf{v}$ to avoid redundancies and $k > 2$ to avoid the known case of $G(2,V)$.

Proposition 5.5.4.1. *The varieties* $\sigma_3(G(3,7))$, $\sigma_3(G(4,8))$, $\sigma_4(G(4,8))$, $\sigma_4(G(3,9))$ *are defective. They are respectively of dimensions* $33, 49, 63, 73$.

In [**21**] the authors make explicit a "folklore" conjecture that these are the only examples, based on computer calculations which verify the conjecture up to $\mathbf{v} = 15$, and they prove that there are no other defective $\sigma_r(G(k,V))$ when r is the smallest number such that $rk > \mathbf{v}$. Also see [**79**].

5.6. Ideas of proofs of dimensions of secant varieties for triple Segre products

This section is included here for completeness. It is technical and best skipped on a first reading.

By Exercise 5.3.2(4), to show that all the secant varieties for given $\mathbf{a}_1, \dots, \mathbf{a}_n$ are of the expected dimensions, it would be sufficient to first determine if the typical rank is the expected one. If the right hand side of (5.5.1) is an integer, then all the smaller secant varieties must be of the expected dimension. Otherwise one must also check that the next to last secant variety is also of the expected dimension.

While the above method is good for showing the expected dimensions hold, it is not clear how to show that secant varieties are degenerate. One method is to find explicit nonzero polynomials vanishing on a secant variety that is supposed to fill, which is how Strassen showed that the variety $\sigma_4(Seg(\mathbb{P}^2\times\mathbb{P}^2\times\mathbb{P}^2))$ is degenerate.

The proof of Theorem 5.5.1.1(2) goes by showing that the last secant variety that does not fill always has the expected dimension. The cases where $n^3/(3n-2)$ is an integer are the easiest. For the next cases, one examines nearby cases which have the property that the right hand side of (5.5.1) is an integer, and then devotes a considerable amount of work to prove the general result.

The arguments that follow are elementary, but somewhat involved. I begin with an easier case to illustrate:

Proof of Theorem 5.5.1.1(3). Write $\mathbb{C}^2\otimes\mathbb{C}^{\mathbf{b}}\otimes\mathbb{C}^{\mathbf{b}} = A\otimes B\otimes C$. It is sufficient to prove the case $r = \mathbf{b}$. We pick a general element $x \in \hat{\sigma}_{\mathbf{b}}$ and show that $\hat{T}_{[x]}\sigma_r = A\otimes B\otimes C$. Write $x = a_1\otimes b_1\otimes c_1 + \cdots + a_{\mathbf{b}}\otimes b_{\mathbf{b}}\otimes c_{\mathbf{b}}$, where we assume that any two of the a_j's span A and the b_j's and c_j's are respectively bases of B, C. Let a, a' be a basis of A. It is sufficient to show that all monomials $a\otimes b_i\otimes c_j, a'\otimes b_i\otimes c_j$ lie in $\hat{T}_{[x]}\sigma_{\mathbf{b}}$. If $i = j$, we just differentiate the i-th term in the A-factor. Otherwise, differentiating the i-th term in the C-factor, we may obtain $a_i\otimes b_i\otimes c_j$, and differentiating the j-th term in the B-th factor, we may obtain $a_j\otimes b_i\otimes c_j$. But both a, a' are linear combinations of a_i, a_j. □

I remind the reader that $J(X, \sigma_r(X)) = \sigma_{r+1}(X)$ and similar facts from §5.1 that are used below.

If $Y_1, \dots, Y_p \subset X$ are subvarieties, then of course $\hat{T}_{y_1}Y_1 + \cdots + \hat{T}_{y_p}Y_p \subseteq \hat{T}_{[y_1+\cdots+y_p]}\sigma_p(X)$. So if one can show that the first space is the ambient space, then one has shown that $\sigma_p(X)$ is the ambient space. Lickteig and Strassen show that just taking at most three of the Y_i to be the Segre itself and taking the other Y_i to be linear spaces in it is sufficient for many cases.

Lemma 5.6.0.2 (Lickteig [**220**]). *Adopt the notation* $\mathbb{P}A_i = \mathbb{P}(A\otimes b_i\otimes c_i) \subset Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)$, $\mathbb{P}B_j = \mathbb{P}(a_j\otimes B\otimes c'_j) \subset Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)$.

(1) *One may choose points* $a_1, \dots, a_s \in A$, $b_1, \dots, b_q \in B$, $c_1, \dots, c_q$, $c'_1, \dots, c'_s \in C$, *such that*

$$\hat{J}(\mathbb{P}A_1, \dots, \mathbb{P}A_q, \mathbb{P}B_1, \dots, \mathbb{P}B_s) = A\otimes B\otimes C$$

when $q = \mathbf{b}l_1$, $s = \mathbf{a}l_2$ *and* $\mathbf{c} = l_1 + l_2$ *and when* $\mathbf{a} = \mathbf{b} = 2$, $q + s = 2\mathbf{c}$, $s, q \geq 2$.

(2) *One may choose points* $a_1, \dots, a_s \in A$, $b_1, \dots, b_q \in B$, $c_1, \dots, c_q$, $c'_1, \dots, c'_s \in C$, *such that*

$$\hat{J}(\sigma_2(Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)), \mathbb{P}A_1, \dots, \mathbb{P}A_q, \mathbb{P}B_1, \dots, \mathbb{P}B_s) = A\otimes B\otimes C$$

when $q + s + 2 = \mathbf{c}$ *and* $\mathbf{a} = \mathbf{b} = 2$.

(3) *One may choose points* $a_1, \dots, a_s \in A$, $b_1, \dots, b_q \in B$, $c_1, \dots, c_q$, $c'_1, \dots, c'_s \in C$, *such that*

$$\hat{J}(\sigma_3(Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)), \mathbb{P}A_1, \dots, \mathbb{P}A_q, \mathbb{P}B_1, \dots, \mathbb{P}B_s) = A\otimes B\otimes C$$

when $q = s = \mathbf{c} - 2 \geq 2$ *and* $\mathbf{a} = \mathbf{b} = 3$.

Proof. To prove the first case in (1), write $C = C_1 \oplus C_2$ with $\dim C_1 = l_1$, $\dim C_2 = l_2$. Then choose the vectors above such that $(b_i\otimes c_i)$ form a basis of $B\otimes C_1$ and $(a_j\otimes c'_j)$ form a basis of $A\otimes C_1$. But $A\otimes B\otimes C = A\otimes B\otimes C_1 \oplus A\otimes B\otimes C_2$.

To prove the second case in (1), first note that by the first part one may assume q is odd. Write $C = C_1 \oplus C_2$ with $\dim C_1 = 3$. Take $c_1, c_2, c_3, c'_1 c'_2, c'_3 \in C_1$ with the rest of elements of C taken from C_2. Split the sum into a component in $A\otimes B\otimes C_1$ and another in $A\otimes B\otimes C_2$. The second component is taken care of by the first part of the lemma. It remains to verify that the 12-dimensional vector space $A\otimes B\otimes C_1$ is spanned by the six 2-dimensional vector spaces $A\otimes b_1\otimes c_1, A\otimes b_2\otimes c_2, A\otimes b_3\otimes c_3, B\otimes a_1\otimes c_1', B\otimes a_2\otimes c_2', B\otimes a_3\otimes c_3'$.

To prove (2), write $C = C_1 \oplus C_2 \oplus C_3$ respectively of dimensions $2, q, s$ and respectively let $c_1, c_2, c_1', \dots, c_q'$, and $c_1'', \dots, c_s''$ be bases. Then $A\otimes B\otimes C_1$ is filled because $\sigma_2(Seg(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)) = \mathbb{P}^7$ and the other two factors are filled thanks to the first part of (1) by having A play the role of C with $l_1 = l_2 = 1$.

To prove (3), write $C = C_1 + C_2$ with $\dim C_1 = 2$, $\dim C_2 = q$, and take $c_1, \dots, c_q, c_1', \dots, c_s' \in C_2$. Then $A\otimes B\otimes C_1$ is filled because $\hat{T}_p\sigma_3(Seg(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^1))$ fills the ambient space. To fill $A\otimes B\otimes C_2$, which is of dimension $9q$, it is sufficient to show that the spaces $a_1\otimes b_1\otimes C_2, a_2\otimes b_2\otimes C_2, a_3\otimes b_3\otimes C_2$, $A\otimes b_1\otimes c_1, \dots, A\otimes b_q\otimes c_q, a_1\otimes B\otimes c_1', \dots, a_s\otimes B\otimes c_s'$ are such that no one of them has a nonzero intersection with the span of the others, as the sum of their dimensions is also $9q$. This verification is left to the reader. $\square$

Exercise 5.6.0.3: Finish the proof.

Here is a special case that illustrates the general method:

Theorem 5.6.0.4 ([**220**]). *Let $a \le b = 2\lambda \le c = 2\mu$; then $\sigma_{2\lambda\mu}(Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)) = \mathbb{P}(A\otimes B\otimes C)$.*

Proof. Write $B = \bigoplus_{i=1}^{\lambda} B_i$, $C = \bigoplus_{j=1}^{\mu} C_j$ with $\dim B_i = \dim C_j = 2$. For each fixed i, j consider elements $a_1^{ij}, a_2^{ij} \in A$, $b_1^{ij}, b_2^{ij} \in B_i$, $c_1^{ij}, c_2^{ij} \in C_j$. I show that $\hat{T}_{[\sum_{i,j} a_1^{ij}b_1^{ij}c_1^{ij} + a_2^{ij}b_2^{ij}c_2^{ij}]}\sigma_{2\lambda\mu}(Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)) = A\otimes B\otimes C$. To see that $A\otimes B_i\otimes C_j$ is filled for each fixed i, j, use

$$\hat{T}_{a_1^{ij}b_1^{ij}c_1^{ij} + a_2^{ij}b_2^{ij}c_2^{ij}}\sigma_2 + \sum_{k\neq i} a_1^{kj}B_ic_1^{kj} + a_2^{kj}B_ic_2^{kj} \subset A\otimes B_i\otimes C_j$$

and

$$\sum_{l\neq j} a_1^{il}b_1^{il}C_j + a_2^{il}b_2^{il}C_j.$$

There are $\lambda - 1 + \mu - 1 \ge \mu - 2$ terms in the second two sums so Lemma 5.6.0.2(2) applies.

Exercise 5.6.0.5: Finish the proof. $\square$

Similar methods are used to prove the other cases.

5.7. BRPP and conjectures of Strassen and Comon

The purpose of this section is to give a common geometric formulation to conjectures from complexity theory and signal processing, following [**47**]. Here are the conjectures.

5.7.1. Strassen's conjecture. Recall the central problem of determining the rank and border rank of the matrix multiplication operator $M_{p,q,r}: Mat_{p\times q}\times Mat_{q\times r}\to Mat_{p\times r}$. V. Strassen asked if there exists an algorithm that simultaneously computes two different matrix multiplications that costs less than the sum of the best algorithms for the individual matrix multiplications. If not, one says that *additivity* holds for matrix multiplication. Similarly, define additivity for arbitrary bilinear maps.

Conjecture 5.7.1.1 (Strassen [**302**]). *Additivity holds for bilinear maps. That is, given $T'\in A'\otimes B'\otimes C'$ and $T''\in A''\otimes B''\otimes C''$, then letting $A=A'\oplus A''$, etc., we have*

$$\begin{aligned}&\mathbf{R}_{Seg(\mathbb{P}A\times\mathbb{P}B\times\mathbb{P}C)}(T'+T'')\\&\quad=\mathbf{R}_{Seg(\mathbb{P}A'\times\mathbb{P}B'\times\mathbb{P}C')}(T')+\mathbf{R}_{Seg(\mathbb{P}A''\times\mathbb{P}B''\times\mathbb{P}C'')}(T'').\end{aligned}$$

5.7.2. Comon's conjecture. In signal processing one is interested in expressing a given tensor as a sum of a minimal number of decomposable tensors. Often the tensors that arise have symmetry or at least partial symmetry. Much more is known about symmetric tensors than tensors, so it would be convenient to be able to reduce questions about tensors to questions about symmetric tensors. In particular, if one is handed a symmetric tensor that has rank r as a symmetric tensor, can it have lower rank as a tensor?

Conjecture 5.7.2.1 (P. Comon [**102**, §4.1]). *The tensor rank of a symmetric tensor equals its symmetric tensor rank.*

5.7.3. Uniform formulation of conjectures. Consider $\mathbf{R}_X$ and $\underline{\mathbf{R}}_X$ as functions $\langle X\rangle\to\mathbb{N}$, and if $L\subset\langle X\rangle$, then $\mathbf{R}_X|_L$ and $\underline{\mathbf{R}}_X|_L$ denote the restricted functions.

Definition 5.7.3.1. Let $X\subset\mathbb{P}V$ be a variety and $L\subset\mathbb{P}V$ a linear subspace. Let $Y:=(X\cap L)_{\mathrm{red}}$, which denotes $X\cap L$ with its reduced structure; namely, even if $X\cap L$ has components with multiplicities, we count each component with multiplicity one.

- (X,L) is a *rank preserving pair* or RPP for short, if $\langle Y\rangle=L$ and $R_X|_L=R_Y$ as functions.
- (X,L) is a *border rank preserving pair* or BRPP for short, if $\langle Y\rangle=L$ and $\underline{\mathbf{R}}_X|_L=\underline{\mathbf{R}}_Y$ as functions, i.e., $\sigma_r(X)\cap L=\sigma_r(Y)$ for all r.
- Similarly (X,L) is RPP$_r$ (respectively, BRPP$_r$) if $\mathbf{R}_X(p)=\mathbf{R}_{X\cap L}(p)$ for all $p\in L$ with $\mathbf{R}_X(p)\le r$ (respectively, $\sigma_s(X)\cap L=\sigma_s(Y)$ for all $s\le r$).

Note that one always has $\mathbf{R}_X(p)\le\mathbf{R}_{X\cap L}(p)$ and $\underline{\mathbf{R}}_X(p)\le\underline{\mathbf{R}}_{X\cap L}(p)$.

Remark 5.7.3.2. Work on a conjecture of Eisenbud, Koh, and Stillmann led to the following result.

Theorem 5.7.3.3 ([**47**])**.** *For all smooth subvarieties $X \subset \mathbb{P}V$ and all $r \in \mathbb{N}$, there exists an integer d_0 such that for all $d \geq d_0$, the pair $(v_d(\mathbb{P}V), \langle v_d(X)\rangle)$ is BRPP$_r$.*

Conjecture 5.7.3.4 (Rephrasing and extending Conjecture 5.7.1.1 to multilinear maps)**.** *Let A_j be vector spaces. Write $A_j = A_j' \oplus A_j''$ and let $L = \mathbb{P}((A_1' \otimes \cdots \otimes A_k') \oplus (A_1'' \otimes \cdots \otimes A_k''))$. Then*

$$(X, L) = (Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_k), \mathbb{P}((A_1' \otimes \cdots \otimes A_k') \oplus (A_1'' \otimes \cdots \otimes A_k'')))$$

is RPP.

Conjecture 5.7.3.5 (Rephrasing Conjecture 5.7.2.1)**.** *Let $\dim A_j = \mathbf{a}$ for each j and identify each A_j with a vector space A. Consider $L = \mathbb{P}(S^k A) \subset A_1 \otimes \cdots \otimes A_k$. Then*

$$(X, L) = (Seg(\mathbb{P}A \times \cdots \times \mathbb{P}A), \mathbb{P}S^k A)$$

is RPP.

Border rank versions. It is natural to ask questions for border rank corresponding to the conjectures of Comon and Strassen. For Strassen's conjecture, this has already been answered negatively:

Theorem 5.7.3.6 (Schönhage [**282**])**.** *BRPP fails for*

$$(X, L) = (Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C), (A' \otimes B' \otimes C') \oplus (A'' \otimes B'' \otimes C''))$$

starting with the case $\mathbf{a} \geq 5 = 2+3$, $\mathbf{b} \geq 6 = 3+3$, $\mathbf{c} \geq 7 = 6+1$ and σ_7, where the splittings into sums give the dimensions of the subspaces.

I discuss Shönhage's theorem in §11.2.

Proposition 5.7.3.7 ([**47**])**.** *Strassen's conjecture and its border rank version hold for $Seg(\mathbb{P}^1 \times \mathbb{P}B \times \mathbb{P}C)$.*

Proof. In this case $X \cap L = \mathbb{P}^0 \times \mathbb{P}B' \times \mathbb{P}C' \sqcup \mathbb{P}^0 \times \mathbb{P}B'' \times \mathbb{P}C''$. So any element in the span of $X \cap L$ is of the form:

$$p := e_1 \otimes (f_1 \otimes g_1 + \cdots + f_k \otimes g_k) + e_2 \otimes (f_{k+1} \otimes g_{k+1} + \cdots + f_{k+l} \otimes g_{k+l}).$$

We can assume that the f_i's are linearly independent and the g_i's as well so that $\underline{\mathbf{R}}_{X \cap L}(p) = \mathbf{R}_{X \cap L}(p) = k + l$. After projection $\mathbb{P}^1 \to \mathbb{P}^0$ which maps both e_1 and e_2 to a single generator of $\mathbb{C}^1$, this element therefore becomes clearly of rank $k + l$. Hence both rank and border rank of p are at least $k + l$. $\square$

Example 5.7.3.8 (Cases where the BRPP version of Comon's conjecture holds [**47**])**.** If $\sigma_r(v_d(\mathbb{P}^n))$ is defined by flattenings, or more generally by equations inherited from the tensor product space, such as the Aronhold invariant (which is a symmetrized version of Strassen's equations), then BRPP$_r$ will hold. However, defining equations for $\sigma_r(v_d(\mathbb{P}^n))$ are only known in very few cases. In the known cases, including $\sigma_r(v_d(\mathbb{P}^1))$ for all r, d, the equations are indeed inherited.

Regarding the rank version, it holds trivially for general points (as the BRPP version holds) and for points in $\sigma_2(v_d(\mathbb{P}^n))$, as a point not of honest rank two is of the form $x^{d-1}y$, which gives rise to $x\otimes\cdots\otimes x\otimes y + x\otimes\cdots\otimes x\otimes y\otimes x + \cdots + y\otimes x\otimes\cdots\otimes x$. By examining the flattenings of this point and using induction one concludes.

If one would like to look for counterexamples, it might be useful to look for linear spaces M such that $M \cap Seg(\mathbb{P}^n \times \cdots \times \mathbb{P}^n)$ contains more than $\dim M + 1$ points, but $L \cap M \cap Seg(\mathbb{P}^n \times \cdots \times \mathbb{P}^n)$ contains the expected number of points, as these give rise to counterexamples to the BRPP version of Strassen's conjecture.

Chapter 6

Exploiting symmetry: Representation theory for spaces of tensors

When computing with polynomials on a vector space V^*, one often splits the computation according to degree. This may be thought of as breaking up the space of polynomials $S^\bullet V$ into isotypic (in fact irreducible) submodules for the group $\mathbb{C}^*$, where $\lambda \in \mathbb{C}^*$ acts by scalar multiplication on V. This decomposition is also the irreducible decomposition of $S^\bullet V$ as a $GL(V)$-module. When a smaller group acts, e.g., when $V = A_1 \otimes \cdots \otimes A_n$ and the group $G = GL(A_1) \times \cdots \times GL(A_n)$ acts, one expects to be able to decompose the space of polynomials even further. The purpose of this chapter is to explain how to obtain and exploit that splitting.

An open problem to keep in mind while reading the chapter is to determine equations for the set of tensors of border rank at most r, $\sigma_r = \sigma_r(Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n)) \subset \mathbb{P}(A_1 \otimes \cdots \otimes A_n)$. This variety is invariant under the group $G = GL(A_1) \times \cdots \times GL(A_n)$, so its ideal is a G-submodule of $S^\bullet(A_1 \otimes \cdots \otimes A_n)^*$. The representation theory of this chapter will be used to find equations for σ_r in Chapter 7. Some modules of equations will arise as elementary consequences of the decompositions discussed here, and others will arise from the construction of G-module mappings.

The topics covered are as follows: Schur's lemma, an easy, but fundamental result is presented in §6.1. As was suggested in Chapter 2, the decomposition of $V^{\otimes d}$ into irreducible $GL(V)$-modules may be understood via representations of $\mathfrak{S}_d$, the group of permutations on d elements. Basic representation theory for finite groups in general, and the group of permutations in particular, are discussed respectively in §6.2 and §6.3. The study of groups acting on spaces of tensors begins in §6.4 with the decomposition of the tensor powers of a vector space. How to decompose the space of homogeneous polynomials on spaces of tensors (in principle) is explained in §6.5. To accomplish this decomposition, one needs to compute with the characters of $\mathfrak{S}_n$, which are briefly discussed in §6.6. Some useful decomposition formulas are presented in §6.7, including the Littlewood-Richardson rule. One benefit of representation theory is that, in a given module, the theory singles out a set of preferred vectors in a module, which are the highest weight vectors (under some choice of basis). Highest weight vectors are discussed in §6.8. Homogeneous varieties are introduced in §6.9. These include Segre, Veronese, flag, and Grassmann varieties. The equations of homogeneous varieties are discussed in §6.10, with special attention paid to the ideals of Segre and Veronese varieties. For later use, §6.11 contains a brief discussion of symmetric functions.

6.1. Schur's lemma

6.1.1. Definitions and statement.

Recall the definitions from §2.2.

Definition 6.1.1.1. If W_1 and W_2 are G-modules, i.e., if $\rho_j : G \to GL(W_j)$ are linear representations, a *G-module homomorphism*, or *G-module map*, is a linear map $f : W_1 \to W_2$ such that $f(\rho_1(g) \cdot v) = \rho_2(g) \cdot f(v)$ for all $v \in W_1$ and $g \in G$.

One says W_1 and W_2 are *isomorphic G-modules* if there exists a G-module homomorphism $W_1 \to W_2$ that is a linear isomorphism. A module W is *trivial* if for all g, $\rho(g) = \mathrm{Id}_W$.

For a group G and G-modules V and W, let $\mathrm{Hom}_G(V,W) \subset V^* \otimes W$ denote the vector space of G-module homomorphisms $V \to W$.

Lemma 6.1.1.2 (Schur's lemma)**.** *Let G be a group, let V and W be irreducible G-modules, and let $f : V \to W$ be a G-module homomorphism. Then either $f = 0$ or f is an isomorphism. If further $V = W$, then $f = \lambda\,\mathrm{Id}$ for some constant λ.*

6.1.2. Exercises.

(1) Show that the image and kernel of a G-module homomorphism are G-modules.

(2) Prove Schur's lemma. ⊚

(3) Show that if V and W are irreducible G-modules, then $f : V \to W$ is a G-module isomorphism if and only if $\langle f \rangle \subset V^* \otimes W$ is a

trivial G-submodule of $V^*\otimes W$. More generally, if V and W are G-modules, show that $f \in V^*\otimes W$ is an invariant tensor under the induced action G if and only if the corresponding map $f : V \to W$ is a G-module homomorphism.

(4) Let $\phi : V \times W \to \mathbb{C}$ be a G-invariant bilinear pairing. That is, $\phi(g \cdot v, g \cdot w) = \phi(v, w)$ for all $v \in V$, $w \in W$, and $g \in G$. Show that if V and W are irreducible, then either ϕ is zero or $W \simeq V^*$ as a G-module.

(5) Show that if V is an irreducible G-module, then the isotypic component of the trivial representation in $V\otimes V^*$ is one-dimensional.

(6) If V is an irreducible G-module and W is any G-module, show that the multiplicity of V in W (cf. Definition 2.8.2.6) is $\dim \mathrm{Hom}_G(V, W)$.

(7) Let $\rho : G \to GL(V)$ and $\psi : G \to GL(W)$ be irreducible representations of a finite group G, and let $h : V \to W$ be any linear map. Form a new linear map $h^0 : V \to W$ by

$$h^0 = \sum_{g\in G} \psi(g^{-1}) \circ h \circ \rho(g).$$

Show that if V is not isomorphic to W, then $h^0 = 0$.

(8) Let G, V and W be as in (7). Fix bases (v_s) and (w_j) of V and W and their dual bases. By making judicious choices of h, show that if V and W are not isomorphic, then $\sum_{g\in G}(\psi(g^{-1}))^i_j \rho(g)^s_t = 0$ for all i, j, s, t. In particular, conclude the basis-free assertion $\sum_{g\in G} \mathrm{tr}(\psi(g^{-1})) \mathrm{tr}(\rho(g)) = 0$. Similarly, note that if $V = W$, then $\frac{1}{|G|} \sum_{g\in G} \mathrm{tr}(\psi(g^{-1})) \mathrm{tr}(\rho(g)) = \dim V$. ⊚

6.2. Finite groups

Let G be a finite group with r elements $g_1, \dots, g_r$. (G is called a *finite group of order r*.) Let $\mathbb{C}[G] := \mathbb{C}^r$ with basis indexed $e_{g_1}, \dots, e_{g_r}$. Give $\mathbb{C}[G]$ the structure of an algebra by defining $e_{g_i} e_{g_j} = e_{g_i g_j}$ and extending linearly. $\mathbb{C}[G]$ is called the *group algebra* of G.

Define a representation of G on $\mathbb{C}[G]$ by $\rho(g)e_{g_i} = e_{gg_i}$ and extending the action linearly. A representation $\rho : G \to GL(V)$ corresponds to an algebra homomorphism $\mathbb{C}[G] \to \mathrm{End}(V)$, so it is equivalent to say that V is a G-module or that V is a left $\mathbb{C}[G]$-module.

I will use the following standard facts from the representation theory of finite groups (see, e.g., [**135, 287**] for proofs):

- The set of isomorphism classes of irreducible G-modules is in one to one correspondence with the set of conjugacy classes of G. (The *conjugacy class* of $g \in G$ is the set of $h \in G$ such that there exists $k \in G$ with $h = kgk^{-1}$.)

- Each irreducible G-module is isomorphic to a submodule of $\mathbb{C}[G]$. (In fact, the multiplicity of an irreducible G-module in $\mathbb{C}[G]$ equals the number of elements in its conjugacy class.)

Thus one may obtain explicit realizations of each irreducible G-module by finding G-equivariant projection operators acting on $\mathbb{C}[G]$. If the conjugacy class has more than one element, there will be different choices of projection operators.

For example, the conjugacy classes of $\mathfrak{S}_3$ are $[\mathrm{Id}], [(12)]$, and $[(123)]$. The projection operators associated to each conjugacy class in $\mathfrak{S}_3$ are right multiplication on $\mathbb{C}[\mathfrak{S}_3]$ by the elements

$$\rho_{\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}} := e_{\mathrm{Id}} + e_{(12)} + e_{(13)} + e_{(23)} + e_{(123)} + e_{(132)}, \tag{6.2.1}$$

$$\rho_{\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}} := e_{\mathrm{Id}} - e_{(12)} - e_{(13)} - e_{(23)} + e_{(123)} + e_{(132)}, \tag{6.2.2}$$

$$\rho_{\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}} := e_{\mathrm{Id}} + e_{(12)} - e_{(13)} + e_{(132)}. \tag{6.2.3}$$

6.2.1. Exercises.

(1) Show that $v\rho_{\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}} = \rho_{\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}}$ for all $v \in \mathbb{C}[\mathfrak{S}_3]$. In particular, the image of the projection $\mathbb{C}[\mathfrak{S}_3] \to \mathbb{C}[\mathfrak{S}_3]$ given by $v \mapsto v\rho_{\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}}$ is the trivial representation.

(2) Show that $e_\sigma\rho_{\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}} = \mathrm{sgn}(\sigma)\rho_{\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}}$ for each $e_\sigma \in \mathbb{C}[\mathfrak{S}_3]$. In particular, the image of the projection $\mathbb{C}[\mathfrak{S}_3] \to \mathbb{C}[\mathfrak{S}_3]$ given by $v \mapsto v\rho_{\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}}$ is the sign representation.

(3) Show that the map $\mathbb{C}[\mathfrak{S}_3] \to \mathbb{C}[\mathfrak{S}_3]$ given by $v \mapsto v\rho_{\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}}$ is a projection with two-dimensional image. This image corresponds to the irreducible representation that is the complement to the trivial representation in the three-dimensional permutation representation.

(4) Write out the conjugacy classes of $\mathfrak{S}_4$ and determine the number of elements in each class.

6.3. Representations of the permutation group $\mathfrak{S}_d$

The group $\mathfrak{S}_d$ of permutations on d elements is the key to decomposing $V^{\otimes d}$ as a $GL(V)$-module.

A *partition* $\pi = (p_1, \ldots, p_r)$ of d is a set of integers $p_1 \geq p_2 \geq \cdots \geq p_r$, $p_i \in \mathbb{Z}_+$, such that $p_1 + \cdots + p_r = d$.

The main fact about $\mathfrak{S}_d$ that we will need is:

- The conjugacy class of a permutation is determined by its decomposition into a product of disjoint cycles. The conjugacy classes (and therefore irreducible $\mathfrak{S}_d$-modules) are in 1-1 correspondence with the set of partitions of d. To a partition $\pi = (p_1, \ldots, p_r)$ one associates the conjugacy class of an element with disjoint cycles of lengths $p_1, \ldots, p_r$.

For example, the module associated to $(1, \ldots, 1)$ is the sign representation. The module associated to (d) is the trivial representation, and the module associated to $(d-1, 1)$ is the standard representation on $\mathbb{C}^{d-1}$, the complement of the trivial representation in the permutation representation on $\mathbb{C}^d$.

6.3.1. Notation. Given a partition $\pi = (p_1, \ldots, p_r)$ of d, write $[\pi]$ for the associated irreducible $\mathfrak{S}_d$-module. (This module is constructed explicitly in Definition 6.3.4.2 below.) Write $|\pi| = d$, and $\ell(\pi) = r$. Form a *Young diagram* associated to π which is defined to be a collection of boxes with p_j boxes in the j-th row, as in Figure 6.3.1.

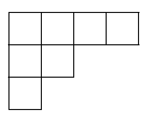


Figure 6.3.1. Young diagram for $\pi = (4, 2, 1)$.

6.3.2. $\mathfrak{S}_d$ acts on $V^{\otimes d}$. Recall from Chapter 2 that $\mathfrak{S}_d$ (and therefore $\mathbb{C}[\mathfrak{S}_d]$) acts on $V^{\otimes d}$. We will construct projection operators that will simultaneously enable us to decompose $V^{\otimes d}$ as a $GL(V)$-module and obtain realizations of the irreducible $\mathfrak{S}_d$-modules. These projection operators are elements of $\mathbb{C}[\mathfrak{S}_d]$. We have already seen the decomposition of $V^{\otimes 2}$ and $V^{\otimes 3}$ via these projection operators in Chapter 2.

The projection maps from §2.8 given by the elements: $\rho_{\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}}$, $\rho_{\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}}$, $\rho_{\begin{array}{|c|c}\hline 1&\multicolumn{1}{c|}{2}\\\hline 3 \\\cline{1-1}\end{array}}$, and $\rho_{\begin{array}{|c|c}\hline 3&\multicolumn{1}{c|}{2}\\\hline 1 \\\cline{1-1}\end{array}}$ enabled us to decompose $V^{\otimes 3}$ as a $GL(V)$-module. They were built by composing symmetrizations and skew-symmetrizations corresponding to different partitions of $\{1, 2, 3\}$. Our goal is to obtain a similar decomposition of $V^{\otimes d}$ for all d. To do this we will need to (i) show that such a decomposition is possible via symmetrizations and skew-symmetrizations, and (ii) determine the relevant projection maps, called *Young symmetrizers*, obtained from symmetrizations and skew-symmetrizations.

6.3.3. Young symmetrizers. Let $\pi = (p_1, \ldots, p_k)$ be a partition of d. Label The boxes of π with elements of $[d] = \{1, \ldots, d\}$. For example

$$\begin{array}{|c|c|c|c|}\hline 1 & 3 & 4 & 2 \\ \hline 6 & 5 & \multicolumn{2}{c}{} \\ \cline{1-2} 7 & \multicolumn{3}{c}{} \\ \cline{1-1}\end{array} \tag{6.3.1}$$

Definition 6.3.3.1. A labelled Young diagram $T_\pi = (t^i_j)$, with $t^i_j \in \{1, \ldots, |\pi|\}$ is called a *Young tableau.* If each element of $\{1, \ldots, |\pi|\}$ appears exactly once, call it a *Young tableau without repetitions.*

Let

$$T_\pi = \begin{array}{llll} \boxed{t^1_1} & \cdots\cdots\cdots & \cdots\cdots & \boxed{t^1_{p_1}} \\ \boxed{t^2_1} & \cdots\cdots\cdots & \boxed{t^2_{p_2}} & \\ & \vdots & & \\ \boxed{t^k_1} & \cdots \boxed{t^k_{p_k}} & & \end{array}$$

be a Young tableau without repetitions for a partition $\pi = (p_1, \ldots, p_k)$. Introduce the notation $\mathfrak{S}(t^i)$ to indicate the group of permutations on the indices appearing in row t^i of T_π, and use $\mathfrak{S}(t_j)$ for the group of permutations on the indices appearing in column t_j of T_π. For example, in (6.3.1), $\mathfrak{S}(t^1) \sim \mathfrak{S}_4$ is the group of permutations of $\{1, 3, 4, 2\}$ and $\mathfrak{S}(t_1) \sim \mathfrak{S}_3$ is the group of permutations of $\{1, 6, 7\}$.

For each row and column, define elements of $\mathbb{C}[\mathfrak{S}_d]$,

$$\rho_{t^i} = \sum_{g \in \mathfrak{S}(t^i)} e_g$$

and

$$\rho_{t_j} = \sum_{g \in \mathfrak{S}(t_j)} \operatorname{sgn}(g) e_g.$$

Definition 6.3.3.2. For a Young tableau T_π without repetitions, define the *Young symmetrizer* of T_π to be

$$\rho_{T_\pi} = \rho_{t^1} \cdots \rho_{t^k} \rho_{t_1} \cdots \rho_{t_{p_1}} \in \mathbb{C}[\mathfrak{S}_d].$$

For example,

$$\rho_{\begin{array}{|c|c|c|}\hline 1&2&5\\ \hline 3&4&\multicolumn{1}{c}{}\\ \cline{1-2}\end{array}} = \rho_{\begin{array}{|c|c|c|}\hline 1&2&5\\ \hline\end{array}} \rho_{\begin{array}{|c|c|}\hline 3&4\\ \hline\end{array}} \rho_{\begin{array}{|c|}\hline 1\\ \hline 3\\ \hline\end{array}} \rho_{\begin{array}{|c|}\hline 2\\ \hline 4\\ \hline\end{array}}.$$

Since the Young symmetrizer ρ_{T_π} is an element of the group algebra $\mathbb{C}[\mathfrak{S}_d]$, it will act on *any* $\mathfrak{S}_d$-module. See Definition 6.4.2.1 for an important example.

Proposition 6.3.3.3. *Right multiplication by ρ_{T_π} as a map $\mathbb{C}[\mathfrak{S}_d] \to \mathbb{C}[\mathfrak{S}_d]$ is a projection operator, i.e., $\mathbb{C}[\mathfrak{S}_d]\rho_{T_\pi} = \mathbb{C}[\mathfrak{S}_d]\rho_{T_\pi}\rho_{T_\pi}$. In particular, $\mathbb{C}[\mathfrak{S}_d]\rho_{T_\pi}$ is a left ideal.*

For a proof, see, e.g., [**135**, §4.2].

Exercise 6.3.3.4: Verify Proposition 6.3.3.3 for $\rho_{\begin{smallmatrix}1&2\\3&4\end{smallmatrix}}$ and $\rho_{\begin{smallmatrix}1&2&3\\4\end{smallmatrix}}$.

6.3.4. Action of ρ_{T_π} on $\mathbb{C}[\mathfrak{S}_d]$.

Proposition 6.3.4.1. *Given Young tableaux without repetitions T_π and $\tilde{T}_\pi$, the modules $\mathbb{C}[\mathfrak{S}_d]\rho_{T_\pi}$ and $\mathbb{C}[\mathfrak{S}_d]\rho_{\tilde{T}_\pi}$ are isomorphic.*

Proof. Given T_π and $\tilde{T}_\pi$ one obtains an explicit isomorphism of the modules $\mathbb{C}[\mathfrak{S}_d]\rho_{T_\pi}$ and $\mathbb{C}[\mathfrak{S}_d]\rho_{\tilde{T}_\pi}$ by multiplying $\mathbb{C}[\mathfrak{S}_d]\rho_{T_\pi}$ by the permutation of $\{1, \ldots, d\}$ taking the tableau T_π to the tableau $\tilde{T}_\pi$. □

Definition 6.3.4.2. The $\mathfrak{S}_d$-module $[\pi]$ is defined to be the representation corresponding to (any of the) $\mathbb{C}[\mathfrak{S}_d]\rho_{T_\pi}$.

Remark 6.3.4.3. I emphasize that there is nothing canonical about the type of projection operator one uses (e.g., skew-symmetrizing, then symmetrizing, or symmetrizing, then skew-symmetrizing), and the specific projection operators within a type (our ρ_{T_π}'s). There is a vector space's worth of copies of the module $[\pi]$ in $\mathbb{C}[\mathfrak{S}_d]$ corresponding to its isotypic component and each ρ_{T_π} projects to exactly one of them.

6.3.5. The hook length formula. A Young tableau T_π is *standard* if the boxes of the diagram are filled with elements $1, \ldots, |\pi|$ such that all rows and columns are strictly increasing. The dimension of $[\pi]$ is given by the *hook-length formula* (6.3.2) and is equal to the number of standard Young tableaux T_π of shape π. The recipe is as follows: given a Young diagram, and a square x in the diagram, we define $h(x)$, the *hook length* of x, to be the number of boxes to the right of x in the same row, plus the number of boxes below x in the same column, plus one. Then

$$\dim[\pi] = \frac{d!}{\prod_{x\in\pi} h(x)}. \tag{6.3.2}$$

See any of, e.g., [**135, 268, 225**] for a proof.

For example, $\dim[(2,1)] = 2$ because there are two standard Young tableaux, associated to $(2,1)$: $\begin{smallmatrix}1&3\\2\end{smallmatrix}$ and $\begin{smallmatrix}1&2\\3\end{smallmatrix}$.

Exercise 6.3.5.1: Compute the dimensions of the irreducible $\mathfrak{S}_4$-modules.

6.4. Decomposing $V^{\otimes d}$ as a $GL(V)$-module with the aid of $\mathfrak{S}_d$

6.4.1. $\Lambda^d V$ revisited. Recall from §2.2.1 that

$$\Lambda^d V = \{X \in V^{\otimes d} \mid X(\alpha_{\sigma(1)}, \dots, \alpha_{\sigma(d)}) \\ = \operatorname{sgn}(\sigma) X(\alpha_1, \dots, \alpha_d)\ \forall \alpha_j \in V^*,\ \forall \sigma \in \mathfrak{S}_d\}.$$

Keeping in mind that $V^{\otimes d}$ is a $\mathfrak{S}_d$-module, we may interpret this equality as saying that $\Lambda^d V$ is the isotypic component of the sign representation $[(1, \dots, 1)]$ in $V^{\otimes d}$.

Similarly, $S^d V$ may be interpreted as the isotypic component of the trivial representation $[(d)]$ of $\mathfrak{S}_d$ in $V^{\otimes d}$.

Let $v_1 \otimes \cdots \otimes v_d \in V^{\otimes d}$ and consider the action of $\rho_{\substack{\boxed{1}\\ \boxed{2}\\ \vdots\\ \boxed{d}}}$ on $v_1 \otimes \cdots \otimes v_d$:

$$\rho_{\substack{\boxed{1}\\ \boxed{2}\\ \vdots\\ \boxed{d}}}(v_1 \otimes \cdots \otimes v_d) = \sum_{\sigma \in \mathfrak{S}_d} \operatorname{sgn}(\sigma) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(d)} \\ = (d!) v_1 \wedge \cdots \wedge v_d.$$

Thus $\Lambda^d V$ is the image of the map $\rho_{\substack{\boxed{1}\\ \boxed{2}\\ \vdots\\ \boxed{d}}} : V^{\otimes d} \to V^{\otimes d}$. Similarly $S^d V$ is the image of $V^{\otimes d}$ under $\rho_{\boxed{1}\boxed{2}\boxed{\cdots}\boxed{d}}$.

In Exercise 2.8.1(3) the operator $\rho_{\substack{\boxed{1}\boxed{2}\\ \boxed{3}\phantom{\boxed{2}}}}$ and others associated to the partition $\pi = (2, 1)$, along with $\rho_{\substack{\boxed{1}\\ \boxed{2}\\ \boxed{3}}}$ and $\rho_{\boxed{1}\boxed{2}\boxed{3}}$, were used to decompose $V^{\otimes 3}$ into irreducible $GL(V)$-modules. I emphasize the word "a", because the decomposition is not unique.

6.4.2. The modules $S_\pi V$. We now obtain a complete isotypic decomposition of $V^{\otimes d}$ into irreducible $GL(V)$-modules via $\mathfrak{S}_d$-invariant projection operators ρ_{T_π}.

Definition 6.4.2.1. Fix a Young tableau T_π with $|\pi| = d$ and let

$$S_{T_\pi} V := \rho_{T_\pi}(V^{\otimes d}).$$

The vector space $S_{T_\pi} V$ is a $GL(V)$-submodule because the actions of $\mathfrak{S}_d$ and $GL(V)$ commute. In more detail, say $X \in S_{T_\pi} V$, $g \in GL(V)$.

To see that $g \cdot X \in S_{T_\pi}V$, let $v_1, \dots, v_{\mathbf{v}}$ be a basis of V and write $X = x^{i_i \dots i_d} v_{i_1} \otimes \cdots \otimes v_{i_d}$, where the $x^{i_i \dots i_d}$ satisfy the commutation and skew-commutation identities to ensure membership in $S_{T_\pi}V$. Write $w_i = g \cdot v_i$, so $g \cdot X = x^{i_i \dots i_d} w_{i_1} \otimes \cdots \otimes w_{i_d}$, which satisfies the same commutation and skew-commutation identities as X; thus $g \cdot X \in S_{T_\pi}V$.

Proposition 6.4.2.2. *Given Young tableaux without repetitions T_π and $\tilde{T}_\pi$, the $GL(V)$-modules $\rho_{T_\pi}(V^{\otimes d})$ and $\rho_{\tilde{T}_\pi}(V^{\otimes d})$ are isomorphic.*

The proof is similar to that of Proposition 6.3.4.1.

Definition 6.4.2.3. Let $S_\pi V$ denote the $GL(V)$-module that is any of the of $S_{T_\pi}V$.

Proposition 6.4.2.4. *The $GL(V)$-module $S_\pi V$ is irreducible.*

Proposition 6.4.2.4 is a consequence of the Schur duality theorem below. Proofs can be found in, e.g., [**135, 268**].

The $S_{T_\pi}V$'s are only certain special realizations of $S_\pi V$. They can be used to furnish a basis of the isotypic component. Thus a general incidence of the module can be achieved as the image of a linear combination of ρ_{T_π}'s. I will use the notation $S_{\overline{\pi}}V$ for a specific realization of $S_\pi V$ in $V^{\otimes|\pi|}$.

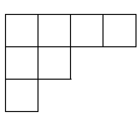
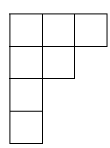
For a box x occurring in a Young tableau, define the *content* of x, denoted $c(x)$, to be 0 if x is on the diagonal, s if x is s steps above the diagonal, and $-s$ if x is s steps below the diagonal. Recall the hook length $h(x)$ defined in §6.3.5.

Proposition 6.4.2.5.

$$\dim S_\pi \mathbb{C}^n = \prod_{x \in \pi} \frac{n + c(x)}{h(x)}. \tag{6.4.1}$$

See, e.g., [**229**] or [**135**, p 78] for a proof of Proposition 6.4.2.5.

Define the *conjugate partition* $\pi' = (q_1, \dots, q_l)$ of π to be the partition whose Young diagram is the reflection of the Young diagram for π about the diagonal. For example, the conjugate partition of $[4, 2, 1]$ is $[3, 2, 1, 1]$:

 has conjugate partition  .

6.4.3. Exercises.

(1) Show that allowing ρ_{T_π} to act on the right, i.e., performing symmetrization first, then skew-symmetrization on $V^{\otimes d}$, also gives a projection from $V^{\otimes d}$ with image isomorphic to $S_\pi V$.

(2) Write $\pi' = (q_1, \dots, q_t)$ and take ρ_{T_π} acting on the right (i.e., symmetrizing first). Show that

$$(e_1 \wedge \cdots \wedge e_{q_1}) \otimes (e_1 \wedge \cdots \wedge e_{q_2}) \otimes \cdots \otimes (e_1 \wedge \cdots \wedge e_{q_t}) \tag{6.4.2}$$

occurs in the corresponding copy of $S_\pi V$.

(3) Verify that the dimensions for $S_{(d)}\mathbb{C}^n$, $S_{(1^d)}\mathbb{C}^n$ and $S_{(2,1)}\mathbb{C}^n$ obtained via (6.4.1) agree with our previous calculations in Chapter 2.

(4) Assume $\dim V$ is large. Order the irreducible submodules appearing in $V^{\otimes 4}$ by their dimensions. Now order the isotypic components by their dimensions.

(5) Give a formula for $\dim S_{a,b}(\mathbb{C}^3)$. ⊚

(6) Give a formula for $S_{2,2}V$.

(7) Let $a_{T_\pi} = \prod_{i=1}^{k} \rho_{t^i}$ and $b_{T_\pi} = \prod_{j=1}^{p_1} \rho_{t_j}$ for any tableau T_π without repetitions associated to $\pi = (p_1, \dots, p_k)$. Show that

$$a_{T_\pi}(V^{\otimes d}) \cong S^{p_1}V \otimes S^{p_2}V \otimes \cdots \otimes S^{p_k}V \subset V^{\otimes d}$$

and

$$b_{T_\pi}(V^{\otimes d}) \cong \Lambda^{q_1}V \otimes \Lambda^{q_2}V \otimes \cdots \otimes \Lambda^{q_l}V \subset V^{\otimes d},$$

where $(q_1, \dots, q_l) = \pi'$ is the conjugate partition to π.

(8) Show that, as an $SL(V)$-module, $S_{p_1,\dots,p_{\mathbf{v}}}(V)$ coincides with $S_{p_1 - p_{\mathbf{v}},\dots,p_{\mathbf{v}-1}-p_{\mathbf{v}},0}(V)$. ⊚

(9) The $GL(V)$-module $S_\pi V$ remains irreducible as an $SL(V)$-module. Let π^* denote the partition whose Young diagram is obtained by placing the Young diagram of π in a box with $\dim V$ rows and p_1 columns, and taking the 180°-rotated complement of π in the box. Show that as an $SL(V)$-module, $(S_\pi V)^*$ is $S_{\pi^*}V$.

6.4.4. Wiring diagrams for the action of ρ_{T_π} on $V^{\otimes d}$. For those who have read §2.11, here are wiring diagrams to describe Young symmetrizers acting on $V^{\otimes d}$.

Since the row operators ρ_{t^i} commute with each other, as do the column operators ρ_{t_j}, one may first do all the skew-symmetrizing simultaneously, and then do all the symmetrizing simultaneously. Thus to each Young tableau T_π, one associates a wiring diagram that encodes ρ_{T_π}.

6.4.5. Schur-Weyl duality. We have seen that the images of the projection operators ρ_{T_π} applied to $V^{\otimes d}$ are submodules. It remains to see (i) that the images $S_{T_\pi}V$ are irreducible, and (ii) that there are enough projection operators to obtain a complete decomposition of $V^{\otimes d}$ into irreducible submodules. The following theorem is the key to establishing these facts.

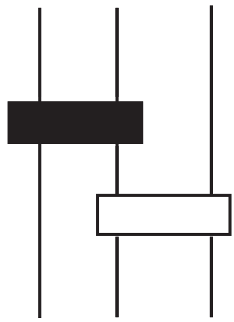


Figure 6.4.1. The wiring diagram corresponding to the Young symmetrizer $\rho_{\begin{array}{|c|c|}\hline 2&3\\\hline 1\\\cline{1-1}\end{array}} = \rho_{\begin{array}{|c|c|}\hline 2&3\\\hline\end{array}}\rho_{\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}}$.

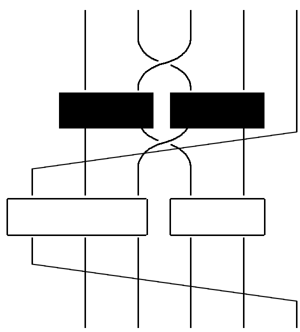


Figure 6.4.2. A wiring diagram for the map corresponding to the Young tableau $\begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4\\\cline{1-2}\end{array}$.

Theorem 6.4.5.1 (Double-commutant theorem)**.** *Let $\rho : GL(V) \to GL(V^{\otimes d})$, and $\psi : \mathfrak{S}_d \to GL(V^{\otimes d})$ denote the natural representations. Then*

$$\psi(\mathfrak{S}_d) = \{g \in GL(V^{\otimes d}) \mid g\rho(A) = \rho(A)g,\ \forall A \in GL(V)\},$$

and

$$\rho(GL(V)) = \{g \in GL(V^{\otimes d}) \mid g\psi(\sigma) = \psi(\sigma)g,\ \forall \sigma \in \mathfrak{S}_d\}.$$

For a proof and extensive discussion of the double-commutant theorem, see [**143**], and [**268**, p. 158]. More generally, if a semisimple algebra $\mathcal{A}$ acts on a vector space U, let $\mathcal{B} = Comm(\mathcal{A})$ denote its *commutant*, i.e., $Comm(\mathcal{A}) := \{b \in \mathrm{End}(U) \mid ba = ab\ \forall a \in \mathcal{A}\}$. Then $\mathcal{A} = Comm(\mathcal{B})$ and the isotypic components of $\mathcal{A}$ and $\mathcal{B}$ in U coincide.

Theorem 6.4.5.2 (Schur duality)**.** *The irreducible decomposition of $V^{\otimes d}$ as an $\mathfrak{S}_d \times GL(V)$-module is*

$$V^{\otimes d} \cong \bigoplus_{|\pi|=d} [\pi] \otimes S_\pi V. \tag{6.4.3}$$

In particular, it is multiplicity-free. Thus the isotypic decomposition of $V^{\otimes d}$ as a $GL(V)$-module is

$$V^{\otimes d} \cong \bigoplus_{|\pi|=d} (S_\pi V)^{\oplus \dim[\pi]}. \tag{6.4.4}$$

For example, the decomposition $V^{\otimes 3} = S^3V \oplus (S_{(21)}V)^{\oplus 2} \oplus \Lambda^3 V$ corresponds to the partitions $(3), (2,1), (1,1,1)$. (Recall that the $\mathfrak{S}_3$-modules $[(3)], [(2,1)]$, and $[(1,1,1)]$ are respectively of dimensions $1, 2, 1$.)

Exercise 6.4.5.3: What is the multiplicity of $[\pi]$ in $V^{\otimes d}$ as an $\mathfrak{S}_d$-module? ⊚

Remark 6.4.5.4. The double commutant theorem allows for a more invariant definition of $S_\pi V$, namely, if $|\pi| = d$,

$$S_\pi V := \mathrm{Hom}_{\mathfrak{S}_d}([\pi], V^{\otimes d}).$$

Here the action of $GL(V)$ on $S_\pi V$ is induced from the action on $V^{\otimes d}$.

Example 6.4.5.5 (For those familiar with Riemannian geometry)**.** Let M be a semi-Riemannian manifold and let $V = T_xM$. Consider the Riemann curvature tensor as an element of $V^{\otimes 4}$. The symmetries $R_{ijkl} = R_{klij} = -R_{jikl}$ of the Riemann curvature tensor imply that it takes values in $S^2(\Lambda^2 V)$. The *Bianchi identity* $R_{ijkl} + R_{iklj} + R_{iljk} = 0$ implies further that it takes values in the irreducible $GL(V)$-module $S_{22}V \subset V^{\otimes 4}$. It is remarkable that it takes values in an irreducible $GL(V)$-module. The semi-Riemannian metric allows the orthogonal group (with the appropriate signature) to act, and as an $O(V)$-module $S_{22}V$ decomposes further into three irreducible submodules, the trivial representation (home of scalar curvature), the traceless quadratic forms (home of the traceless Ricci curvature Ric^0) and the rest, which is totally trace-free, is the home of the Weyl curvature. The famous *Einstein's equations* are $Ric^0 = 0$.

6.4.6. The role of $\dim V$ in the decomposition of $V^{\otimes d}$. Notice that the dimension of V has played a minimal role: it only becomes relevant if one skew-symmetrizes over a set of indices of cardinality greater than the dimension of V.

The modules occurring in the $GL(V)$-isotypic decomposition of $V^{\otimes d}$ and their multiplicities are independent of the dimension of V, as long as $\dim V$ is sufficiently large.

While the dimensions of the modules $S_\pi V$ do depend on $\dim V$*, they are given by closed formulas in the dimension of* V*.*

This yields a very significant decrease in complexity, in that one can deal with very large modules in terms of relatively low-dimensional prototypes. (See for example the inheritance theorems in §3.7.1 and §7.4.)

Remark 6.4.6.1. This observation has given rise to "categorical" generalizations of GL_n to "GL_t", with $t \in \mathbb{Q}$. See [**111**].

6.5. Decomposing $S^d(A_1 \otimes \cdots \otimes A_n)$ as a $G = GL(A_1) \times \cdots \times GL(A_n)$-module

6.5.1. Schur-Weyl duality and the trivial $\mathfrak{S}_d$-action. Let $V = A_1 \otimes \cdots \otimes A_n$. In order to study $G = GL(A_1) \times \cdots \times GL(A_n)$-varieties in $\mathbb{P}V^*$, it will be necessary to decompose $S^d V$ as a G-module. We already know how to decompose $V^{\otimes d}$ as a G-module, namely

$$\begin{aligned}(A_1 \otimes \cdots \otimes A_n)^{\otimes d} &= A_1^{\otimes d} \otimes \cdots \otimes A_n^{\otimes d} \\ &= \Big(\bigoplus_{|\pi_1|=d} [\pi_1] \otimes S_{\pi_1} A_1\Big) \otimes \cdots \otimes \Big(\bigoplus_{|\pi_n|=d} [\pi_n] \otimes S_{\pi_n} A_n\Big) \\ &= \bigoplus_{|\pi_j|=d} ([\pi_1] \otimes \cdots \otimes [\pi_n]) \otimes (S_{\pi_1} A_1 \otimes \cdots \otimes S_{\pi_n} A_n).\end{aligned}$$

Here G acts trivially on $([\pi_1] \otimes \cdots \otimes [\pi_n])$. Now $S^d V \subset V^{\otimes d}$ is the set of elements invariant under the action of $\mathfrak{S}_d$. The group $\mathfrak{S}_d$ only acts on the $[\pi_j]$; it leaves the $S_{\pi_j} A_j$'s invariant.

Definition 6.5.1.1. Let V be a module for the group G. Write V^G for the isotypic component of the trivial representation in V, which is called the *space of G-invariants* in V.

Proposition 6.5.1.2 ([**205**])**.** *As a $G = GL(A_1) \times \cdots \times GL(A_n)$-module,*

$$S^d(A_1 \otimes \cdots \otimes A_n) = \bigoplus_{|\pi_1| = \cdots = |\pi_n| = d} (S_{\pi_1} A_1 \otimes \cdots \otimes S_{\pi_n} A_n)^{\oplus \dim([\pi_1] \otimes \cdots \otimes [\pi_n])^{\mathfrak{S}_d}},$$

where $([\pi_1] \otimes \cdots \otimes [\pi_n])^{\mathfrak{S}_d}$ denotes the space of $\mathfrak{S}_d$-invariants.

To obtain the explicit decomposition, one would need a way to calculate $\dim([\pi_1] \otimes \cdots \otimes [\pi_n])^{\mathfrak{S}_d}$. In the next two subsections I discuss the cases $n = 2$ and $n = 3$.

6.5.2. The two-factor case. Consider $([\pi_1] \otimes [\pi_2])^{\mathfrak{S}_d} = \mathrm{Hom}_{\mathfrak{S}_d}([\pi_1]^*, [\pi_2])$. The $[\pi_j]$ are self-dual $\mathfrak{S}_d$-modules, see Exercise 6.6.1.4. Thus $([\pi_1] \otimes [\pi_2])^{\mathfrak{S}_d} = \mathrm{Hom}_{\mathfrak{S}_d}([\pi_1], [\pi_2])$. The modules $[\pi_j]$ are irreducible, so Schur's Lemma

6.1.1.2 implies $\dim(\mathrm{Hom}_{\mathfrak{S}_d}([\pi_1],[\pi_2])) = \delta_{\pi_1,\pi_2}$. We deduce the $GL(A_1) \times GL(A_2)$-decomposition:

$$S^d(A_1 \otimes A_2) = \bigoplus_{|\pi|=d} S_\pi A_1 \otimes S_\pi A_2. \tag{6.5.1}$$

Note that this decomposition is multiplicity-free. Note also that, although we know the isomorphism class of the modules in the decomposition on the right hand side, we are not given a recipe for how to obtain the inclusion of each of these modules in $S^d(A_1 \otimes A_2)$.

Exercise 6.5.2.1: Show that $S_{\young(12,3)}A \otimes S_{\young(12,3)}B \not\subset S^3(A \otimes B)$, but that it does have a nonzero projection to $S^3(A \otimes B)$. Thus Young symmetrizers do not provide the most natural basis elements when studying tensor products of vector spaces.

6.5.3. Remark on writing down the modules $S_{\pi_1}A_1 \otimes \cdots \otimes S_{\pi_n}A_n \subset S^d(A_1 \otimes \cdots \otimes A_n)$ in practice. I do not know of any general method to write down the isotypic component of a module $S_{\pi_1}A_1 \otimes \cdots \otimes S_{\pi_n}A_n$ in $S^d(A_1 \otimes \cdots \otimes A_n)$. In practice I take projection operators

$$A_1^{\otimes d} \otimes \cdots \otimes A_n^{\otimes d} \to S_{\pi_1}A_1 \otimes \cdots \otimes S_{\pi_n}A_n$$

and then project the result to $S^d(A_1 \otimes \cdots \otimes A_n)$. When doing this, one needs to check that one does not get zero, and that different projection operators, when symmetrized, give independent copies of the module.

6.5.4. The three factor case: Kronecker coefficients. Triple tensor products play a central role in applications. Define the *Kronecker coefficients* $k_{\pi\mu\nu}$ by

$$S^d(A \otimes B \otimes C) = \bigoplus_{\{\pi_1,\pi_2,\pi_3 \colon |\pi_j|=d\}} (S_{\pi_1}A \otimes S_{\pi_2}B \otimes S_{\pi_3}C)^{\oplus k_{\pi_1\pi_2\pi_3}}.$$

By Proposition 6.5.1.2,

$$k_{\pi_1\pi_2\pi_3} = \dim([\pi_1] \otimes [\pi_2] \otimes [\pi_3])^{\mathfrak{S}_d}.$$

Using the same perspective as in the two-factor case, a nontrivial element $f \in ([\pi_1] \otimes [\pi_2] \otimes [\pi_3])^{\mathfrak{S}_d}$ may be viewed as an $\mathfrak{S}_d$-module homomorphism $[\pi_1] \to [\pi_2] \otimes [\pi_3]$.

Exercise 6.5.4.1: Show that the $GL(E) \times GL(F)$-decomposition of $S_\pi(E \otimes F)$ is:

$$S_\pi(E \otimes F) = \bigoplus_{\mu,\nu} (S_\mu E \otimes S_\nu F)^{\oplus k_{\pi\mu\nu}}. \tag{6.5.2}$$

Kronecker coefficients can be computed using *characters* as described in Section 6.6. For more information on Kronecker coefficients see [**225**, I.7].

Exercise 6.5.4.2: Show that the multiplicity of $[\pi_1]$ in $[\pi_2]\otimes[\pi_3]$ is the same as the multiplicity of $[\pi_2]$ in $[\pi_1]\otimes[\pi_3]$.

6.6. Characters

6.6.1. Characters for finite groups. Let G be a group and $\rho : G \to GL(V)$ a representation of G. Define $\chi_V : G \to \mathbb{C}$ by $\chi_V(g) = \mathrm{tr}(\rho(g))$, the *character* associated to V. Note that $\dim V = \chi_V(\rho(\mathrm{Id}))$.

Exercise 6.6.1.1: Show that χ_V is a class function. That is, χ_V takes the same value on all elements of G in the same conjugacy class.

Exercise 6.6.1.2: Let V and W be representations of a finite group G with characters χ_V and χ_W respectively. Show that

(i) $\chi_{V\oplus W} = \chi_V + \chi_W$,

(ii) $\chi_{V\otimes W} = \chi_V\chi_W$.

By (i), to determine the character of any module, it is sufficient to know the characters of irreducible G-modules. Here is a character table for $\mathfrak{S}_3$:

(6.6.1)

class	[Id]	[(12)]	[(123)]
#	1	3	2
$\chi_{(3)}$	1	1	1
$\chi_{(21)}$	2	0	-1
$\chi_{(111)}$	1	-1	1

Notational warning. The reader should not confuse the irreducible $\mathfrak{S}_d$-module $[\pi]$ associated to a partition, with the cycle type of a permutation, which is denoted (π) in the above chart.

For finite groups G, there is a natural G-invariant Hermitian inner product on the space of characters,

$$\langle \chi_V, \chi_W \rangle = \frac{1}{|G|}\sum_{g\in G} \chi_V(g)\bar{\chi}_W(g).$$

Since we will be concerned exclusively with the permutation group $\mathfrak{S}_d$ where the characters are all real-valued, the reader just interested in $\mathfrak{S}_d$ can ignore the word "Hermitian" and the complex conjugation in the formula.

Exercise 6.6.1.3: Using Exercise 6.1.2(8), show that

$$\langle \chi_V, \chi_W \rangle = \dim(\mathrm{Hom}_G(V, W)).$$

In particular, $\langle \chi_{triv}, \chi_V \rangle = \dim V^G$ (cf. Definition 6.5.1.1).

Exercise 6.6.1.4: Show that $\mathfrak{S}_d$-modules are indeed self-dual.

6.6.2. Characters and the decomposition of $S^d(A_1 \otimes \cdots \otimes A_n)$. For V and W irreducible, Schur's lemma, via Exercise 6.6.1.3, implies that $\langle \chi_V, \chi_W \rangle = 0$ unless V and W are isomorphic. Hence characters of irreducible representations are orthogonal. For W not necessarily irreducible, $\langle \chi_V, \chi_W \rangle$ counts the multiplicity of V in W, or equivalently the dimension of the space of trivial representations in the tensor product $V^* \otimes W$. Applying this discussion to the group $\mathfrak{S}_d$ (for which $\chi_V = \bar{\chi}_V$ and all modules are self-dual) yields the following result.

Proposition 6.6.2.1. *For $\mathfrak{S}_d$-modules $[\pi_1], \dots, [\pi_n]$, we have*

$$\begin{aligned}\dim([\pi_1] \otimes \cdots \otimes [\pi_n])^{\mathfrak{S}_d} &= \langle \chi_{\pi_1} \cdots \chi_{\pi_n}, \chi_{(d)} \rangle \\ &= \frac{1}{d!} \sum_{\sigma \in \mathfrak{S}_d} \chi_{\pi_1}(\sigma) \cdots \chi_{\pi_n}(\sigma).\end{aligned}$$

Remark 6.6.2.2. If one is programming the inner products for characters of $\mathfrak{S}_d$, it might be better to take representatives of each conjugacy class to reduce the number of terms in the summation, since characters are class functions.

Exercise 6.6.2.3: Decompose $S^3(A \otimes B \otimes C)$ as a $GL(A) \times GL(B) \times GL(C)$-module using the character table (6.6.1).

Exercise 6.6.2.4: Write out the character table for $\mathfrak{S}_4$. ⊚

The character table of $\mathfrak{S}_d$ can be obtained, e.g., from the *Frobenius formula* (see, e.g., [**135**, p. 49]).

Proposition 6.6.2.5 ([**205**]). *The decomposition of $S^3(A_1 \otimes \cdots \otimes A_k)$ into irreducible $GL(A_1) \times \cdots \times GL(A_k)$-modules is:*

$$\begin{aligned}&S^3(A_1 \otimes \cdots \otimes A_k) \\ &\quad = \bigoplus_{\substack{I+J+L=\{1,\dots,k\},\\ j=|J|>1}} \frac{2^{j-1} - (-1)^{j-1}}{3} S_3 A_I \otimes S_{21} A_J \otimes S_{111} A_L \\ &\qquad \oplus \bigoplus_{\substack{I+L=\{1,\dots,k\},\\ |L|\, even}} S_3 A_I \otimes S_{111} A_L,\end{aligned}$$

where I, J, L are subsets of $[k]$, and $S_\pi A_I := \bigotimes_{i \in I} S_\pi A_i$.

In particular, $S^3(A \otimes B \otimes C)$ decomposes as

$$S_3 S_3 S_3 \oplus S_3 S_{21} S_{21} \oplus S_3 S_{111} S_{111} \oplus S_{21} S_{21} S_{21} \oplus S_{21} S_{21} S_{111}$$

and thus is multiplicity-free. Here $S_\lambda S_\mu S_\nu$ is to be read as $S_\lambda A \otimes S_\mu B \otimes S_\nu C$ plus exchanges of the roles of A, B, C that give rise to distinct modules.

Remark 6.6.2.6. The coefficient $\frac{2^{j-1}-(-1)^{j-1}}{3}$ is the j-th *Jacobsthal number*. These were originally defined recurrently by $J_0 = 1, J_1 = 1$ and $J_n = J_{n-1} + 2J_{n-2}$.

Proof. Calculate

$$\langle \chi_{21}^j, \chi_3 \rangle = \langle \chi_{21}^j, \chi_{111} \rangle = \frac{1}{6}(2^j + 2(-1)^j) = \frac{1}{3}(2^{j-1} - (-1)^{j-1}).$$

Exercise 6.6.2.7: Finish the proof. □

Proposition 6.6.2.8. *The decomposition of $S^4(A{\otimes}B{\otimes}C)$ into irreducible $GL(A) \times GL(B) \times GL(C)$-modules is:*

(6.6.2)
$$\begin{aligned} S^4(A{\otimes}B{\otimes}C) = {} & S_4S_4S_4 \oplus S_4S_{31}S_{31} \oplus S_4S_{22}S_{22} \oplus S_4S_{211}S_{211} \\ & \oplus S_4S_{1111}S_{1111} \oplus S_{31}S_{31}S_{31} \oplus S_{31}S_{31}S_{22} \oplus S_{31}S_{31}S_{211} \\ & \oplus S_{31}S_{22}S_{211} \oplus S_{31}S_{211}S_{211} \oplus S_{31}S_{211}S_{1111} \oplus S_{22}S_{22}S_{22} \\ & \oplus S_{22}S_{22}S_{1111} \oplus S_{22}S_{211}S_{211} \oplus S_{211}S_{211}S_{211}. \end{aligned}$$

Here $S_\lambda S_\mu S_\nu$ is to be read as $S_\lambda A{\otimes}S_\mu B{\otimes}S_\nu C$ plus exchanges of the roles of A, B, C that give rise to distinct modules. In particular, $S^4(A{\otimes}B{\otimes}C)$ is multiplicity-free.

Exercise 6.6.2.9: Prove Proposition 6.6.2.8.

Remark 6.6.2.10. In $S^5(A{\otimes}B{\otimes}C)$ all submodules occurring have multiplicity one except for $S_{311}S_{311}S_{221}$, which has multiplicity two. For higher degrees there is a rapid growth in multiplicities.

6.7. The Littlewood-Richardson rule

6.7.1. Littlewood-Richardson coefficients. Often one would like to decompose $S_\pi V{\otimes}S_\mu V$ under the action of $GL(V)$. The Littlewood-Richardson coefficients $c^\nu_{\pi\mu}$ are defined to be the multiplicity of $S_\nu V$ in $S_\pi V{\otimes}S_\mu V$. That is,

$$S_\pi V{\otimes}S_\mu V = \sum_\nu (S_\nu V)^{\oplus c^\nu_{\pi\mu}}$$

for $c^\nu_{\pi\mu} \in \mathbb{Z}_{\geq 0}$, by definition symmetric in its lower arguments. If $S_\pi V \subset V^{\otimes|\pi|}$ and $S_\mu V \subset V^{\otimes|\mu|}$, then any $S_\nu V$ appearing in $S_\pi V{\otimes}S_\mu V$ must necessarily satisfy $|\nu| = |\pi| + |\mu|$ as one would have to have $S_\nu V \subset V^{\otimes|\pi|}{\otimes}V^{\otimes|\mu|} = V^{\otimes(|\pi|+|\mu|)}$.

Note that

$$\begin{aligned} S_\pi V{\otimes}S_\mu V &= \mathrm{Hom}_{\mathfrak{S}_{|\pi|}}([\pi], V^{|\pi|}){\otimes}\,\mathrm{Hom}_{\mathfrak{S}_{|\mu|}}([\mu], V^{|\mu|}) \\ &= \mathrm{Hom}_{\mathfrak{S}_{|\pi|}\times\mathfrak{S}_{|\mu|}}([\pi]{\otimes}[\mu], V^{\otimes(|\pi|+|\mu|)}). \end{aligned}$$

So

$$c^{\nu}_{\pi\mu} = \dim(\mathrm{Hom}_{\mathfrak{S}_{|\pi|}\times\mathfrak{S}_{|\mu|}}([\nu],[\pi]\otimes[\mu])),$$

where we have chosen an inclusion $\mathfrak{S}_{|\pi|} \times \mathfrak{S}_{|\mu|} \subset \mathfrak{S}_{|\pi|+|\mu|}$.

The coefficients $c^{\nu}_{\pi\mu}$ may be computed as follows: Fix projections $V^{\otimes|\pi|} \to S_\pi V$, $V^{\otimes|\mu|} \to S_\mu V$, to obtain a projection $\psi : V^{\otimes(|\pi|+|\mu|)} \to S_\pi V \otimes S_\mu V \subset V^{\otimes(|\pi|+|\mu|)}$. Fix a projection $T_\nu : V^{\otimes(|\pi|+|\mu|)} \to S_\nu V$. Then $c^{\nu}_{\pi\mu}$ is the number of distinct permutations $\sigma \in \mathfrak{S}_{|\nu|}$ such that the $T_\nu \circ \sigma \circ \psi$ give rise to distinct realizations of $S_\nu V$.

Exercise 6.7.1.1: Show that $S_\pi(A \oplus B) = \sum_{\mu,\nu}(S_\mu A \otimes S_\nu B)^{\oplus c^{\nu}_{\pi\mu}}$.

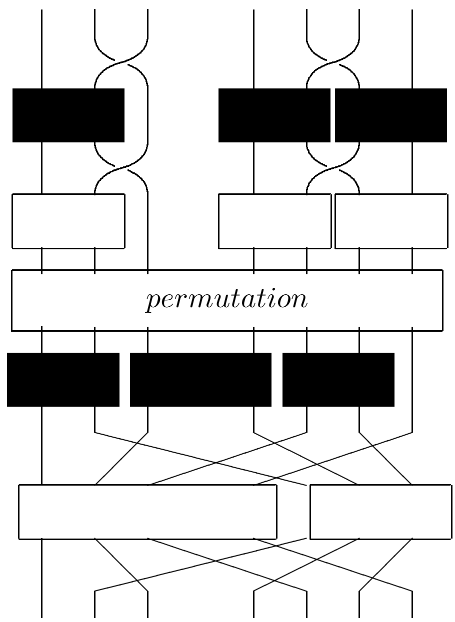


Figure 6.7.1. Towards the Littlewood-Richardson rule. Wiring diagrams for the projection maps $\rho_{\begin{smallmatrix}1&2\\3&\end{smallmatrix}}$, $\rho_{\begin{smallmatrix}1&2\\3&4\end{smallmatrix}}$, followed by a permutation and $\rho_{\begin{smallmatrix}1&3&5&7\\2&4&6&\end{smallmatrix}}$.

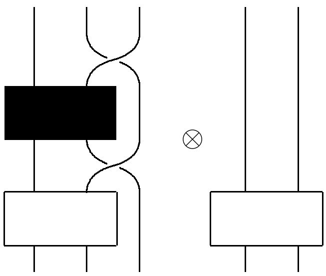
6.7.2. The Pieri formula. Consider the special case $c^{\nu}_{\lambda,(d)}$ where λ and ν are arbitrary partitions. From the perspective of wiring diagrams, since all the strands from the second set are symmetrized, one cannot feed any two of them in to a skew-symmetrizer without getting zero. As long as one avoids such a skew-symmetrization, the resulting projection will be nonzero. This nearly proves:

Theorem 6.7.2.1 (The Pieri formula).

$$c^{\nu}_{\lambda,(d)} = \begin{cases} 1 & \text{if } \nu \text{ is obtained from } \lambda \text{ by adding } d \text{ boxes to} \\ & \text{the rows of } \lambda \text{ with no two in the same column;} \\ 0 & \text{otherwise.} \end{cases}$$

Above I identified partitions with their associated Young diagrams.

It remains to show that different allowable permutations of strands in the wiring diagram give rise to the same module. But this is also clear as every strand coming from the second factor is symmetrized.



$$\cong \quad S_{41} \oplus S_{32} \oplus S_{311} \oplus S_{221}$$

Figure 6.7.2. The Pieri rule for $S_{21}V \otimes S^2 V$.

Introduce the notation $\pi \mapsto \pi'$ to mean that $\pi_1 \geq \pi'_1 \geq \pi_2 \geq \pi'_2 \geq \cdots \geq 0$.

The following proposition is a consequence of the Pieri formula.

Proposition 6.7.2.2. *For* $\dim A' = 1$, *one has the* $GL(A) \times GL(A')$*-module decomposition*

$$S_\pi(A \oplus A') = \bigoplus_{\pi \mapsto \pi'} S_{\pi'} A \otimes S^{|\pi|-|\pi'|} A'.$$

6.7.3. Pieri formula exercises.

(1) Show that $S_{21}V \otimes S_2 V = S_{41}V \oplus S_{32}V \oplus S_{311}V \oplus S_{221}V$.

(2) Show that
$$S_a V \otimes S_b V = \sum_{\substack{0 \leq t \leq s \\ s+t=a+b}} S_{s,t} V.$$

(3) Show that
$$\bigoplus_{\substack{|\mu|=d \\ \ell(\mu) \geq k}} S_\mu V \subseteq \Lambda^k V \otimes \Big(\bigoplus_{|\pi|=d-k} S_\pi V \Big).$$

(4) Write a Pieri-type recipe for decomposing $S_\pi V \otimes S_{1^k} V$.

(5) Write down a rule to determine if $S_d V \subset S_\pi V \otimes S_\mu V$ as $\mathfrak{sl}(V)$-modules. ⊚

(6) Prove Proposition 6.7.2.2.

6.7.4. Application: the ideal of $\sigma_r(Seg(\mathbb{P}A^* \times \mathbb{P}B^*))$. Recall from §2.7.3 that $\Lambda^{r+1}A\otimes\Lambda^{r+1}B \subset I_{r+1}(\sigma_r(Seg(\mathbb{P}A^* \times \mathbb{P}B^*)))$.

Exercises.

(1) Show that $\bigoplus_{\{\pi:\,|\pi|=d,\ \ell(\pi)>r\}} S_\pi A\otimes S_\pi B \subseteq I_d(\sigma_r(Seg(\mathbb{P}A^* \times \mathbb{P}B^*)))$.

(2) Show that $\bigoplus_{\{\pi:\,|\pi|=d,\ \ell(\pi)>r\}} S_\pi A\otimes S_\pi B = I_d(\sigma_r(Seg(\mathbb{P}A^* \times \mathbb{P}B^*)))$.

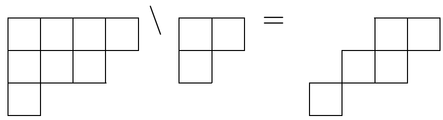
(3) Using Exercise 6.7.3(3), show that $\Lambda^{r+1}A\otimes\Lambda^{r+1}B$ generates the ideal of $\sigma_r(Seg(\mathbb{P}A^* \times \mathbb{P}B^*))$.

In summary, the following theorem holds.

Theorem 6.7.4.1. *The ideal of $\sigma_r(Seg(\mathbb{P}A^* \times \mathbb{P}B^*))$ is generated in degree $r+1$ by the module $\Lambda^{r+1}A\otimes\Lambda^{r+1}B \subset S^{r+1}(A\otimes B)$.*

6.7.5. Littlewood-Richardson: the general case. For the general case, it is clear that each row in μ will be subject to constraints that when adding boxes to the diagram of λ, one cannot add two boxes from the same row of μ to the same column. However, there is a second issue that will arise here: some of the strands coming from μ will be anti-symmetric, and one needs an accounting method that avoids sending two of them through a symmetrizing box.

Introduce the notion of a *skew Young diagram* $\nu\backslash\lambda$ by subtracting the diagram of λ from that of ν.



Introduce the notion of a *skew Young tableau* by numbering the boxes of $\nu\backslash\lambda$. Define a skew tableau $\nu\backslash\lambda$ to be *Yamonuchi* if, when one orders the boxes of $\nu\backslash\lambda$ by $x_1, \dots, x_f$, where the ordering is top to bottom and *right* to left, for all $x_1, \dots, x_s$, and all i, that i occurs at least as many times as $i+1$ in $x_1, \dots, x_s$.

Here is the recipe for the Littlewood-Richardson rule: begin with the diagram for λ. Then add μ_1 boxes to λ as in Pieri, and label each box with a 1. Then add μ_2 boxes to the new diagram as in Pieri, labeling each new box with a 2. Continue in this fashion. Now consider the resulting skew tableau (obtained by deleting the unlabeled boxes from λ). I claim that if the skew tableau is Yamonuchi, then the concatenated diagram appears in $S_\nu V\otimes S_\lambda V$. That is, Yamonuchi-ness ensures that nothing that has been skew-symmetrized is symmetrized and nothing that has been symmetrized is skew-symmetrized.

It is also true that the distinct Yamonuchi fillings of $\nu\backslash\lambda$ give rise to inequivalent submodules. That is, the following theorem holds.

Theorem 6.7.5.1 (The Littlewood-Richardson rule). *The number of Yamonuchi fillings of the skew tableau $\nu\backslash\lambda$ equals $c^{\nu}_{\lambda\mu}$.*

For a proof of the Littlewood-Richardson rule see, e.g., [**229, 133**].

Remark 6.7.5.2. From this perspective the symmetry in λ, μ in $c^{\nu}_{\lambda\mu}$ is far from clear.

Exercise 6.7.5.3: Compute $S_{31}V{\otimes}S_{22}V$.

6.7.6. A few handy decomposition formulas. Most of these can be found in [**225**]:

$$S^d(V \oplus W) = \bigoplus_{a+b=d} S^aV{\otimes}S^bW, \tag{6.7.1}$$

$$\Lambda^d(V \oplus W) = \bigoplus_{a+b=d} \Lambda^aV{\otimes}\Lambda^bW, \tag{6.7.2}$$

$$S^d(V{\otimes}W) = \bigoplus_{|\pi|=d} S_\pi V{\otimes}S_\pi W, \tag{6.7.3}$$

$$\Lambda^d(V{\otimes}W) = \bigoplus_{|\pi|=d} S_\pi V{\otimes}S_{\pi'} W, \tag{6.7.4}$$

$$S^2(S^dV) = \bigoplus_{j\colon \text{ even}} S_{2d-j,j}V, \tag{6.7.5}$$

$$\Lambda^2(S^dV) = \bigoplus_{j\colon \text{ odd}} S_{2d-j,j}V, \tag{6.7.6}$$

$$S^d(S^2V) = \bigoplus_{\alpha_1+\cdots+\alpha_n=d} S_{2\alpha_1,\dots,2\alpha_n}V, \tag{6.7.7}$$

$$S^2(\Lambda^dV) = S_{2^d}V \oplus S_{2^{d-2},1^4}V \oplus S_{2^{d-4},1^8}V \oplus \cdots, \tag{6.7.8}$$

$$\Lambda^aV{\otimes}\Lambda^bV = \bigoplus_{2t+s=a+b,\ t\le\min(a,b)} S_{2^t,1^s}V, \tag{6.7.9}$$

$$\Lambda^2(\Lambda^kV) = \bigoplus_{j\colon \textit{even}} S_{2^j1^{2(k-j)}}V, \tag{6.7.10}$$

$$S^d(\Lambda^2V) = \bigoplus_{|\lambda|=d} S_{\lambda_1\lambda_1\lambda_2\lambda_2\dots\lambda_d\lambda_d}V. \tag{6.7.11}$$

Here $S_{p^a,q^b}V = S_{p,\dots,p,q,\dots,q}V$.

Many of these formulas have pictorial interpretations. For example, $\Lambda^2(\Lambda^kV)$ is obtained by taking a Young diagram with k boxes and its conjugate diagram and gluing them together symmetrically along the diagonal. For example, $\Lambda^3(\Lambda^2W) = S_{3111}W \oplus S_{222}W$.

Exercise 6.7.6.1: Show that

$$S_{21}(V\otimes W) = (S_3V\otimes S_{21}W) \oplus (S_{21}V\otimes S_3W) \oplus (S_{21}V\otimes S_{21}W) \\ \oplus (S_{21}V\otimes S_{111}W) \oplus (S_{111}V\otimes S_{21}W).$$

6.8. Weights and weight spaces: a generalization of eigenvalues and eigenspaces

Recall from §4.7 that the ideals of G-varieties $X \subset \mathbb{P}V^*$ are G-submodules of $S^\bullet V$, and to test if a given irreducible module $M \subset S^dV$ is in $I_d(X)$, it is sufficient to test on a single vector.

The purpose of this section is to describe a method, given an irreducible G-module M, for finding a "simplest" vector in M. The key to this will be to find generalizations of the eigenvalues and eigenspaces of a linear map $f : V \to V$, which will respectively be called *weights* and *weight spaces.*

This section will be used extensively to study the ideals of G-varieties.

6.8.1. Weights.

A diagonalizable linear map $f : W \to W$ is characterized by its eigenvalues (which are numbers) and eigenspaces (which are linear subspaces of W). Given a space $\mathfrak{t} = \langle f_1, \dots, f_r\rangle$ of linear maps $f_j : W \to W$, usually one cannot make sense of eigenvalues and eigenspaces for $\mathfrak{t}$. However, if all the f_j commute, they will have simultaneous eigenspaces, and there is still a notion of an "eigenvalue" associated to an eigenspace. This "eigenvalue" will no longer be a number, but a linear function $\lambda : \mathfrak{t} \to \mathbb{C}$, where $\lambda(f)$ is the eigenvalue of f on the eigenspace associated to λ.

Now we are concerned with $GL(W)$, whose elements do not commute, so at first this idea seems useless. However, instead of giving up, one can try to salvage as much as possible by *choosing* a largest possible subgroup $T \subset GL(W)$ where this idea does work. A maximal abelian diagonalizable subgroup $T \subset GL(W)$ is called a *maximal torus* of $GL(W)$. After having made this choice, for any representation $\rho : GL(W) \to GL(V)$, V can be decomposed into eigenspaces for T. These eigenspaces will be called *weight spaces* and the corresponding eigenvalues for these eigenspaces $\lambda : T \to \mathbb{C}^*$ are called the *weights* (or *T-weights*) of V. *Lie's theorem* (see e.g., [**135**, p. 126] or [**268**, p. 96]) guarantees that the elements of $\rho(T)$ are simultaneously diagonalizable.

Choose a basis $e_1, \dots, e_{\mathbf{w}}$ of W and parametrize T by

$$T = \left\{ \begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_{\mathbf{w}} \end{pmatrix} \,\middle|\, x_j \in \mathbb{C}^* \right\}.$$

Then, e_1 is a weight vector with weight x_1, $e_1^{\otimes i_1}\otimes\cdots\otimes e_{\mathbf{w}}^{\otimes i_{\mathbf{w}}}$ is a weight vector with weight $x_1^{i_1}\cdots x_{\mathbf{w}}^{i_{\mathbf{w}}}$, and $e_1{\otimes}e_2+e_2{\otimes}e_1$ is a weight vector with weight x_1x_2. On the other hand, $e_1{\otimes}e_2+e_1{\otimes}e_3$ is not a weight vector.

Having chosen a basis of W, it makes sense to talk about the group $B\subset GL(W)$ of upper-triangular matrices. Such a group is called a *Borel* subgroup.

The subgroup B may be defined geometrically as the group preserving the *complete flag*

$$\langle e_1\rangle\subset\langle e_1,e_2\rangle\subset\cdots\subset\langle e_1,\dots,e_{\mathbf{w}-1}\rangle\subset W.$$

In general, a *flag* in a vector space V is a sequence of subspaces $0\subset V_1\subset V_2\subset\cdots\subset V$. It is called a *complete flag* if $\dim V_j=j$ and the sequence has length $\mathbf{v}$.

Note that in W, the only line preserved by B is $\langle e_1\rangle$. In $W{\otimes}W$ there are two lines preserved by B, $\langle e_1{\otimes}e_1\rangle$ and $\langle e_1{\otimes}e_2-e_2{\otimes}e_1\rangle$. It is not a coincidence that these respectively lie in S^2W and Λ^2W.

Facts.

(1) If V is an irreducible $GL(W)$-module, then there exists a unique line $\ell\subset V$ with the property $B\cdot\ell=\ell$. This line is called a *highest weight line*. See, e.g., [**268**, p. 192].

(2) An irreducible module V is uniquely determined up to isomorphism by the weight λ of its highest weight line.

(3) Analogous statements hold for any reductive group G, in particular for $GL(A_1)\times\cdots\times GL(A_n)$.

A highest weight vector of S^dW is $(e_1)^{\otimes d}$, as if $b\in B$, $b\cdot(e_1)^d=(b\cdot e_1)^d=(b_1^1)^d(e_1)^d$. A highest weight vector of Λ^dW is $e_1\wedge e_2\wedge\cdots\wedge e_d=\frac{1}{d!}\sum_{\sigma\in\mathfrak{S}_d}\mathrm{sgn}(\sigma)e_{\sigma(1)}\otimes\cdots\otimes e_{\sigma(d)}$ as

$$\begin{aligned}b\cdot(e_1\wedge e_2\wedge\cdots\wedge e_d)&=(b_1^1e_1)\wedge(b_2^1e_1+b_2^2e_2)\wedge\cdots\\&\qquad\wedge(b_d^1e_1+b_d^2e_2+\cdots+b_d^de_d)\\&=(b_1^1b_2^2\cdots b_d^d)e_1\wedge e_2\wedge\cdots\wedge e_d.\end{aligned}$$

Note in particular that a highest weight vector of an irreducible submodule of $W^{\otimes d}$ does not correspond to a rank one tensor in general.

A highest weight vector for $S_{\begin{smallmatrix}1&2\\3&\end{smallmatrix}}W$ is

$$2e_1{\otimes}e_1{\otimes}e_2-e_2{\otimes}e_1{\otimes}e_1-e_1{\otimes}e_2{\otimes}e_1. \tag{6.8.1}$$

Exercise 6.8.1.1: Verify that (6.8.1) is indeed a highest weight vector for $S_{\begin{smallmatrix}1&2\\3&\end{smallmatrix}}W$.

Exercise 6.8.1.2: Show that the only two highest weight lines in $W{\otimes}W$ (with respect to our choice of basis) are $\langle e_1{\otimes}e_1\rangle$ and $\langle e_1{\otimes}e_2 - e_2{\otimes}e_1\rangle$.

Exercise 6.8.1.3: Show that the highest weight of $S^p(\Lambda^k W)$ is equal to $(x_1 \wedge \cdots \wedge x_p)^p$.

6.8.2. Lie algebras and highest weight vectors. For many calculations in geometry, one works infinitesimally. The infinitesimal analog of a Lie group is a *Lie algebra*. To compute highest weight vectors in practice, it is often easier to work with Lie algebras instead of Lie groups, because addition is easier than multiplication.

Associated to any Lie group G is a *Lie algebra* $\mathfrak{g}$, which is a vector space that may be identified with $T_{\mathrm{Id}}G$. By definition, a Lie algebra is a vector space $\mathfrak{g}$ equipped with a bilinear map $[\,,]: \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ that is skew-symmetric and satisfies the *Jacobi identity*, which is equivalent to requiring that the map $\mathfrak{g}\mapsto End(\mathfrak{g})$ given by $X\mapsto [X,\cdot]$ is a derivation. The *Lie algebra* associated to a Lie group G is the set of left-invariant vector fields on G. Such vector fields are determined by their vector at a single point, e.g., at $T_{\mathrm{Id}}G$. For basic information on Lie algebras associated to a Lie group, see any of [**295, 180, 268**].

When $G = GL(V)$, then $\mathfrak{g} = \mathfrak{gl}(V) := V^*{\otimes}V$ and the Lie bracket is simply the commutator of matrices: $[X,Y] = XY - YX$. The Lie algebra of a product of Lie groups is the sum of their Lie algebras.

If $G \subseteq GL(V)$, so that G acts on $V^{\otimes d}$, there is an induced action of $\mathfrak{g}\subseteq\mathfrak{gl}(V)$ given by, for $X\in\mathfrak{g}$,

$$\begin{aligned}X.(v_1{\otimes}v_2{\otimes}\cdots\otimes v_d)\\ &= (X.v_1){\otimes}v_2{\otimes}\cdots\otimes v_d + v_1{\otimes}(X.v_2){\otimes}\cdots\otimes v_d+\cdots\\ &\quad + v_1{\otimes}v_2{\otimes}\cdots\otimes v_{d-1}{\otimes}(X.v_d).\end{aligned}$$

To see why this is a natural induced action, consider a curve $g(t)\subset G$ with $g(0)=\mathrm{Id}$ and $X = g'(0)$ and take

$$\frac{d}{dt}\bigg|_{t=0} g\cdot(v_1{\otimes}\cdots\otimes v_d) = \frac{d}{dt}\bigg|_{t=0}(g\cdot v_1){\otimes}\cdots\otimes(g\cdot v_d).$$

One concludes by applying the Leibniz rule.

Write $\mathfrak{t}$ for the Lie algebra of T, so

$$\mathfrak{t} = \left\{\begin{pmatrix} z_1 & & \\ & \ddots & \\ & & z_{\mathbf{w}}\end{pmatrix} \;\middle|\; z_j\in\mathbb{C}\right\}.$$

By definition, the $\mathfrak{t}$-weights are the logs of the T weights, i.e., if a vector v has T-weight $x_1^{i_1}\cdots x_{\mathbf{w}}^{i_{\mathbf{w}}}$, it will have $\mathfrak{t}$-weight $i_1z_1+\cdots+i_{\mathbf{w}}z_{\mathbf{w}}$. It will often be convenient to work with the $\mathfrak{t}$-weights because it is often easier to work with Lie algebras than Lie groups. (The slight notational discrepancy in the begining of this section is because the discussion began with $\mathfrak{t}$-weights, and then switched to T-weights.)

For $\mathfrak{sl}(W)$, one can use the same torus as for $\mathfrak{gl}(W)$, only one imposes the condition $z_1 + \cdots + z_{\mathbf{w}} = 0$ as elements of $\mathfrak{sl}(W)$ are trace-free.

Having fixed $B \subset GL(W)$, it makes sense to talk of its Lie algebra $\mathfrak{b}$ and the subalgebra $\mathfrak{n} \subset \mathfrak{b}$ of strictly upper triangular matrices. That is, $\mathfrak{n}$ is the set of linear maps $f : W \to W$ that sends $\langle e_1, \dots, e_j\rangle$ to $\langle e_1, \dots, e_{j-1}\rangle$ for all j.

Notational convention. I will distinguish Lie group actions from Lie algebra actions by using a "$\cdot$" for group actions and a "." for Lie algebra actions.

Proposition 6.8.2.1. *Let V be a $GL(W)$-module. A nonzero vector $v \in V$ is a highest weight vector if and only if $\mathfrak{n}.v = 0$. That is, $\mathfrak{n}.v = 0$ if and only if $B \cdot v \subset \langle v\rangle$.*

Proof. Let $b(t) \subset B$ be a curve with $b(0) = \mathrm{Id}$. It will be sufficient to show that $\mathfrak{n}.v = 0$ if and only if all such curves preserve $\langle v\rangle$. Assume all such curves preserve $\langle v\rangle$. That is, $v(t) := b(t) \cdot v \subset \langle v\rangle$. In particular $v'(0) = b'(0).v \subset \langle v\rangle$. Write $b'(0) = t + n$ with $t \in \mathfrak{t}$ and $n \in \mathfrak{n}$. Since T preserves $\langle v\rangle$, it is sufficient to consider n. Note that $\mathfrak{n}$ does not fix any line, so if $n.v \subset \langle v\rangle$, then $n.v = 0$. The other direction is left to the reader as Exercise 6.8.2.2. ☐

Exercise 6.8.2.2: Complete the proof of Proposition 6.8.2.1.

Exercise 6.8.2.3: Show that $\mathfrak{n}.(e_1 \wedge \cdots \wedge e_k) = 0$ for all $k \leq \mathbf{w}$.

6.8.3. Weight subspaces of $W^{\otimes d}$. Fix a basis $e_1, \dots, e_{\mathbf{w}}$ of W. This determines a maximal torus T for which the basis vectors are weight vectors. One obtains an induced basis of $W^{\otimes d}$ for which the induced basis vectors

$$e_{i_1} \otimes \cdots \otimes e_{i_d} \tag{6.8.2}$$

are weight vectors of T-weight $x_{i_1} \cdots x_{i_d}$ (or equivalently, $\mathfrak{t}$-weight $z_{i_1} + \cdots + z_{i_d}$). Since this gives us a weight basis of $W^{\otimes d}$, we see that the possible T-weights are

$$\{x_1^{i_1} \cdots x_{\mathbf{w}}^{i_{\mathbf{w}}} \mid i_1 + \cdots + i_{\mathbf{w}} = d\},$$

or equivalently, the possible $\mathfrak{t}$-weights are

$$\{i_1 z_1 + \cdots + i_{\mathbf{w}} z_{\mathbf{w}} \mid i_1 + \cdots + i_{\mathbf{w}} = d\}.$$

Define the *multinomial coefficient*

$$\binom{d}{i_1, \dots, i_{\mathbf{w}}} := \frac{d!}{i_1! i_2! \cdots i_{\mathbf{w}}!}. \tag{6.8.3}$$

For each such weight, there are the $\binom{d}{i_1,\dots,i_{\mathbf{w}}}$ basis vectors with the same weight, so the corresponding weight space is $\binom{d}{i_1,\dots,i_{\mathbf{w}}}$-dimensional. Given one such weight vector, all others are obtained by allowing $\mathfrak{S}_d$ to act on it.

6.8.4. Highest weight subspaces of $W^{\otimes d}$. As we have already seen for $W^{\otimes 2}$, the induced basis (6.8.2) does not respect the decomposition of $W^{\otimes d}$ into isotypic components. Fixing a choice of B, only some of these weights occur as highest weights.

Proposition 6.8.4.1. *A weight $x_1^{i_1} \cdots x_{\mathbf{w}}^{i_{\mathbf{w}}}$ with $i_1 + \cdots + i_{\mathbf{w}} = d$ occurs as a highest weight in $W^{\otimes d}$ if and only if $i_1 \geq i_2 \geq \cdots \geq i_{\mathbf{w}} \geq 0$. In fact, it occurs as the highest weight of the component of $S_{i_1 \dots i_{\mathbf{w}}} W$ in $W^{\otimes d}$.*

Proof. Since we have the decomposition $W^{\otimes d} = \bigoplus_{|\pi|=d} (S_\pi W)^{\dim[\pi]}$, it will suffice to show that if $\pi = (p_1, \dots, p_k)$, then the highest weight of $S_\pi W$ is $x_1^{p_1} \cdots x_k^{p_k}$. We may take any copy of $S_\pi W$, so consider the copy generated by

$$(e_1 \wedge \cdots \wedge e_{q_1}) \otimes (e_1 \wedge \cdots \wedge e_{q_2}) \otimes \cdots \otimes (e_1 \wedge \cdots \wedge e_{q_t}), \tag{6.8.4}$$

where the conjugate partition π' is $(q_1, \dots, q_t)$.

Exercise 6.8.4.2: Show that the vector (6.8.4) is indeed a highest weight vector. ⊚ □

One obtains a basis of the highest weight subspace of the isotypic component of $S_\pi W$ in $W^{\otimes d}$ by applying elements of $\mathfrak{S}_d$ to (6.8.4). To do this, it is necessary to first re-expand the wedge products into sums of tensor products. Recall $\sigma \cdot (w_1 \otimes \cdots \otimes w_d) = w_{\sigma(1)} \otimes \cdots \otimes w_{\sigma(d)}$.

6.8.5. Highest weight subspaces and Young symmetrizers. Let π be a partition of d. One may obtain $\dim[\pi]$ disjoint copies of $S_\pi W$ in $W^{\otimes d}$ (in fact bases of each copy) by using Young symmetrizers ρ_{T_π} associated to semistandard tableaux. By definition, a tableau is *semistandard* if the rows are weakly increasing and the columns strictly increasing. See, e.g., [**268**, §9.9], [**135**, §6.1,§15.5], or [**143**, §9.3]. In particular, one obtains a basis of the highest weight subspace by restricting the projection operators to the weight $x_1^{p_1} \cdots x_{\mathbf{w}}^{p_{\mathbf{w}}}$ subspace.

For example, consider $W^{\otimes 3}$. The case of $S^3 W$ is the easiest, as e_1^3 is one of our original basis vectors. The space $\Lambda^3 W$ is not much more difficult, as it has multiplicity one, so there is an essentially unique projection operator

$$e_1{\otimes}e_2{\otimes}e_3 \mapsto \sum_{\sigma\in\mathfrak{S}_3} \operatorname{sgn}(\sigma) e_{\sigma(1)}{\otimes}e_{\sigma(2)}{\otimes}e_{\sigma(3)} = 6e_1 \wedge e_2 \wedge e_3.$$

Finally consider

$$\begin{aligned}\rho_{\begin{smallmatrix}\boxed{1}\boxed{2}\\ \boxed{3}\end{smallmatrix}}(e_1{\otimes}e_1{\otimes}e_2) &= \rho_{\boxed{1}\boxed{2}}(e_1{\otimes}e_1{\otimes}e_2 - e_2{\otimes}e_1{\otimes}e_1)\\ &= 2e_1{\otimes}e_1{\otimes}e_2 - e_1{\otimes}e_2{\otimes}e_1 - e_2{\otimes}e_1{\otimes}e_1\end{aligned}$$

and

$$\begin{aligned}\rho_{\begin{smallmatrix}\boxed{1}\boxed{3}\\ \boxed{2}\end{smallmatrix}}(e_1{\otimes}e_2{\otimes}e_1) &= \rho_{\boxed{1}\boxed{2}}(e_1{\otimes}e_2{\otimes}e_1 - e_2{\otimes}e_1{\otimes}e_1)\\ &= 2e_1{\otimes}e_2{\otimes}e_1 - e_1{\otimes}e_1{\otimes}e_2 - e_2{\otimes}e_1{\otimes}e_1\end{aligned}$$

to obtain a basis of the isotypic component of $S_{21}W$. (Note that these Young symmetrizers applied to other elements of weight $x_1^2x_2$ give zero.)

6.8.6. Computing a space of highest weight vectors in $W^{\otimes d}$ via $\mathfrak{n}$. Above I described two methods to obtain a basis of the highest weight subspace of an isotypic component, via Young symmetrizers and via the $\mathfrak{S}_d$-orbit of (6.8.4). A third method is just to compute the action of $\mathfrak{n}$ on the weight subspace associated to a partition π and find a basis of the kernel.

For example, to get a description of the highest weight subspace of $S_{12}V$ in $V^{\otimes 3}$, consider an arbitrary linear combination

$$c_1e_1{\otimes}e_1{\otimes}e_2 + c_2e_1{\otimes}e_2{\otimes}e_1 + c_3e_2{\otimes}e_1{\otimes}e_1,$$

where the c_j's are constants, and the e_j's are basis vectors of V. Write e^j for the dual basis vectors. Its image under the upper-triangular endomorphism $e^2{\otimes}e_1 \in \mathfrak{n} \subset \mathfrak{gl}(V)$ is

$$(c_1 + c_2 + c_3)e_1{\otimes}e_1{\otimes}e_1.$$

Thus the highest weight space of the isotypic component of $S_{21}V$ in $V^{\otimes 3}$ is

$$\{c_1e_1{\otimes}e_1{\otimes}e_2 + c_2e_1{\otimes}e_2{\otimes}e_1 + c_3e_1{\otimes}e_1{\otimes}e_1 \mid c_1 + c_2 + c_3 = 0\}.$$

Exercise 6.8.6.1: The decomposition of $S^3(S^2V)$ as a $GL(V)$-module is

$$S^3(S^2V) = S_6V \oplus S_{42}V \oplus S_{222}V.$$

Find a highest weight vector of each component. ⊚

Exercise 6.8.6.2: Find a highest weight vector for the copy of $S_{d,d}V \subset S^dV{\otimes}S^dV$. ⊚

Remark 6.8.6.3. If one includes $\mathbb{C}^m \subset \mathbb{C}^{m+k}$ using bases such that the first m basis vectors span $\mathbb{C}^m$, then any nonzero highest weight vector for $S_{T_\pi}\mathbb{C}^m$ is also a highest weight vector for $S_{T_\pi}\mathbb{C}^{m+k}$. This observation is the key to the "inheritance" property discussed in §7.4.

6.8.7. An explicit recipe for writing down polynomials in $S_{222}A\otimes S_{222}B\otimes S_{3111}C \subset S^6(A\otimes B\otimes C)$. I follow [**20**] in this subsection.

To obtain a highest weight vector for $S_{222}A\otimes S_{222}B\otimes S_{3111}C$, first use the fillings $\begin{array}{|c|c|}\hline 1&1\\\hline 2&2\\\hline 3&3\\\hline\end{array}$, $\begin{array}{|c|c|}\hline 1&1\\\hline 2&2\\\hline 3&3\\\hline\end{array}$ and $\begin{array}{|c|c|c|}\hline 1&1&1\\\hline 2\\\cline{1-1} 3\\\cline{1-1} 4\\\cline{1-1}\end{array}$ to get a highest weight vector respectively in $S_{222}A$, $S_{222}B$, and $S_{3111}C$. We now have a vector in $A^{\otimes 6}\otimes B^{\otimes 6}\otimes C^{\otimes 6}$. Were we to take the naïve inclusion into $(A\otimes B\otimes C)^{\otimes 6}$, we would get zero when projecting to $S_{222}A\otimes S_{222}B\otimes S_{3111}C$. So instead we group the factors according to the following recipe:

We take the diagrams $\begin{array}{|c|c|}\hline 1&4\\\hline 2&5\\\hline 3&6\\\hline\end{array}$, $\begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline 5&6\\\hline\end{array}$ and $\begin{array}{|c|c|c|}\hline 1&3&6\\\hline 2\\\cline{1-1} 4\\\cline{1-1} 5\\\cline{1-1}\end{array}$; that is, in each monomial in the expression in $A^{\otimes 6}\otimes B^{\otimes 6}\otimes C^{\otimes 6}$, we group the 1's with the 1's to get our first copy of $A\otimes B\otimes C$, etc., up to the 6's with the 6's. Then we take the resulting vector and project it into $S^6(A\otimes B\otimes C)$.

6.9. Homogeneous varieties

This section brings together geometry and representation theory. Homogeneous varieties are discussed as varieties that arise in geometry problems and as the set of possible highest weight lines in a module.

6.9.1. Flag varieties.

Example 6.9.1.1. $Flag_{1,2}(V) \subset \mathbb{P}S_{21}V$. Take the realization $S_{\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}}V \subset V^{\otimes 3}$. Recall from Exercise 2.8.1(1) the exact sequence

$$0 \to S_{\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}}V \to V\otimes\Lambda^2 V \to \Lambda^3 V \to 0,$$

where the second map on decomposable elements is $u\otimes(v\wedge w) \mapsto u\wedge v\wedge w$. Thus

$$S_{\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}}V = \langle u\otimes(v\wedge w) \mid u \in \langle v, w\rangle\rangle.$$

Consider

$$Seg(\mathbb{P}V \times G(2,V)) \subset \mathbb{P}(V\otimes\Lambda^2 V).$$

Define

$$Flag_{1,2}(V) := Seg(\mathbb{P}V \times G(2,V)) \cap \mathbb{P}S_{\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}}V = \{[u\otimes v\wedge w] \mid u \in \langle v, w\rangle\},$$

the *flag variety* of lines in planes in V.

More generally, let $\pi = (p_1, \dots, p_r)$ be a partition. By Exercise 6.4.3(7),

$$S_{T_\pi} V \subset \Lambda^{q_s} V \otimes \Lambda^{q_{s-1}} V \otimes \cdots \otimes \Lambda^{q_1} V,$$

where $\pi' = (q_1, \dots, q_s)$ is the conjugate partition. For the moment assume that $q_1 > q_2 > \cdots > q_s$. Note that $Seg(G(q_s, V) \times G(q_{s-1}, V) \times \cdots \times G(q_1, V)) \subset \mathbb{P}(\Lambda^{q_s} V \otimes \Lambda^{q_{s-1}} \otimes \cdots \otimes \Lambda^{q_1} V)$. Intersecting with $S_\pi V$, as was done for $\pi = (2, 1)$, yields the conditions

$$E_{q_s} \subseteq E_{q_{s-1}} \subseteq \cdots \subseteq E_{q_1}.$$

Define
(6.9.1)

$$\begin{aligned} Flag_{q_s,\dots,q_1}(V) &:= \mathbb{P}S_\pi V \cap Seg(G(q_s, V) \times G(q_{s-1}, V) \times \cdots \times G(q_1, V)) \\ &\subset \mathbb{P}(S_\pi V) = \{(E_s, \dots, E_1) \in G(q_s, V) \times \cdots \times G(q_1, V) \mid \\ &\qquad\qquad E_s \subset E_{s-1} \subset \cdots \subset E_1\}. \end{aligned}$$

$Flag_{q_s,\dots,q_1}(V)$ is a homogeneous variety, called a *flag variety*. In the case $\pi = (\mathbf{v} - 1, \mathbf{v} - 2, \dots, 1)$, one obtains the *variety of complete flags in* V. (Exercise 6.9.2(1) below shows that it is indeed homogeneous.)

For the case that some of the q_j's are equal, recall the simple case of $\pi = (d)$, so all the $q_j = 1$, which yields $v_d(\mathbb{P}V)$. More generally, if $\pi = (d, \dots, d)$, one obtains $v_d(G(k, V))$. In the case of an arbitrary partition, one obtains the flag variety of the strictly increasing partition, with any multiplicities giving a Veronese re-embedding of the component with multiplicity.

6.9.2. Exercises on flag varieties.

(1) Show that, for $k_1 < \cdots < k_s$,

$$Flag_{k_1,\dots,k_s}(V) = GL(V) \cdot ([e_1 \wedge \cdots \wedge e_{k_1} \otimes e_1 \wedge \cdots \wedge e_{k_2} \otimes \cdots \otimes e_1 \wedge \cdots \wedge e_{k_s}]).$$

(2) Show that if $(j_1, \dots, j_t)$ is a subset of $(k_1, \dots, k_s)$, then there is a natural projection $Flag_{k_1,\dots,k_s}(V) \to Flag_{j_1,\dots,j_t}(V)$. What is the inverse image of (fiber over) a point?

(3) Show that $Flag_{1,\dim V-1}(V) = Seg(\mathbb{P}V \times \mathbb{P}V^*) \cap \{\mathrm{tr} = 0\}$, where $\{\mathrm{tr} = 0\}$ is the hyperplane of traceless endomorphisms.

6.9.3. Rational homogeneous varieties as highest weight vectors. We may view the homogeneous varieties we have seen so far as orbits of highest weight lines:

$$G(k, V) = GL(V) \cdot [e_1 \wedge \cdots \wedge e_k],$$

$$v_d(\mathbb{P}V) = GL(V) \cdot [e_1^d],$$

$$Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n) = (GL(A_1) \times \cdots \times GL(A_n)) \cdot [a_1 \otimes \cdots \otimes a_n],$$

$$Flag_{k_1,\dots,k_s}(V) = GL(V) \cdot [e_1 \wedge \cdots \wedge e_{k_1} \otimes e_1 \wedge \cdots \wedge e_{k_2} \otimes \cdots \otimes e_1 \wedge \cdots \wedge e_{k_s}].$$

Since we can move the torus in G by elements of g to get new torus, and the corresponding motion on a module will take the old highest weight line to the new one, we may think of the homogeneous variety in the projective space of a G-module W as the set of choices of highest weight lines in $\mathbb{P}W$.

(Aside for experts: in this book I deal with embedded varieties, so I do not identify isomorphic varieties in general.)

The subgroup of G that stabilizes a highest weight line in V, call it P (or P_V when we need to be explicit about V), will contain the Borel subgroup B. The subgroup P is called a *parabolic subgroup*, and we write the homogeneous variety as $G/P \subset \mathbb{P}V$.

Using these bases, for $G(k, V)$, $P_{\Lambda^k V}$ is the set of elements of $GL(V)$ of the form

$$\begin{pmatrix} * & * \\ 0 & * \end{pmatrix},$$

where the blocking is $k \times (n-k)$.

In general, the G-orbits of highest weight lines in irreducible G-modules V are exactly the rational homogeneous varieties $G/P \subset \mathbb{P}V$. (To prove that these orbits are closed and that they are the only homogeneous varieties in $\mathbb{P}V$, one can use the Borel fixed point theorem; see, e.g., [**135**, §23.3].) Since in this book the only homogeneous varieties I deal with are rational homogeneous varieties, I suppress the word "rational". (The only other type of irreducible homogeneous projective varieties are the *abelian varieties*.) Other orbits in $\mathbb{P}V$ are not closed; in fact they all contain G/P in their closures.

Exercise 6.9.3.1: Determine the group $P \subset GL(V)$ stabilizing $[e_1 \wedge \cdots \wedge e_{k_1} \otimes e_1 \wedge \cdots \wedge e_{k_2} \otimes \cdots \otimes e_1 \wedge \cdots \wedge e_{k_s}]$.

6.9.4. Tangent spaces to homogeneous varieties. Recall the notation from §4.6: for $x \in X \subset \mathbb{P}V$, $\hat{T}_x X \subset V$ denotes the affine tangent space, and $T_x X \subset T_x \mathbb{P}V$ denotes the Zariski tangent space.

If $X = G/P$ is homogeneous, then $T_{[\mathrm{Id}]}X$ has additional structure beyond that of a vector space: it is a P-module, in fact the P-module $\mathfrak{g}/\mathfrak{p}$ (where $\mathfrak{g} := T_{\mathrm{Id}}G$, $\mathfrak{p} := T_{\mathrm{Id}}P$ are respectively the *Lie algebras* associated to G, P, see §6.8.2).

Example 6.9.4.1. Let $X = G(k, V)$, so $\mathfrak{g} = \mathfrak{gl}(V) = V \otimes V^*$. Let $E = [\mathrm{Id}] \in G(k, V)$, and write $V = E \oplus V/E$, so

$$V \otimes V^* = E \otimes E^* \oplus E \otimes (V/E)^* \oplus V/E \otimes E^* \oplus V/E \otimes (V/E)^*.$$

Then $\mathfrak{p} = E^* \otimes E \oplus (V/E)^* \otimes E \oplus (V/E)^* \otimes V/E$, so $\mathfrak{g}/\mathfrak{p} = T_E G(k, V) \simeq E^* \otimes V/E$. Note that in the special case of $\mathbb{P}V = G(1, V)$, we have $T_x \mathbb{P}V \simeq \hat{x}^* \otimes V/\hat{x}$.

6.9.5. Tangent spaces to points of quasi-homogeneous varieties. If $X = \overline{G \cdot x} \subset \mathbb{P}V$ is quasi-homogeneous; then the above discussion is still valid at points of $G \cdot x$. Let $G(x) \subset G$ denote the stabilizer of x; then $T_xX \simeq \mathfrak{g}/\mathfrak{g}(x)$ as a $G(x)$-module, and for $y \in G \cdot x$, $T_yX \simeq T_xX$ as $G(x)$-modules. This assertion makes sense because $G(y)$ is conjugate to $G(x)$ in $GL(V)$.

6.10. Ideals of homogeneous varieties

In this section I describe the ideals of the homogeneous varieties in spaces of tensors, symmetric tensors and skew-symmetric tensors: $Seg(A_1 \otimes \cdots \otimes A_n)$, $v_d(\mathbb{P}W)$, and $G(k, W)$. I also explain how the ideal of any rational homogeneous variety is generated in degree two. The description of the equations will be uniform, and perhaps simpler than traditional methods (e.g., straightening laws for Grassmannians).

6.10.1. Segre case. We saw in §2.7.2 that

$$I_2(Seg(\mathbb{P}A^* \times \mathbb{P}B^*)) = \Lambda^2 A \otimes \Lambda^2 B \subset S^2(A \otimes B).$$

Another way to see this is as follows: let $\alpha_1 \otimes \beta_1 \in \widehat{Seg}(\mathbb{P}A^* \times \mathbb{P}B^*) \subset A^* \otimes B^*$, consider the induced element $(\alpha_1 \otimes \beta_1)^{\otimes 2} \in (A^* \otimes B^*)^{\otimes 2}$, and its pairing with elements of $S^2(A \otimes B) = (S^2 A \otimes S^2 B) \oplus (\Lambda^2 A \otimes \Lambda^2 B)$. Since $(\alpha_1 \otimes \beta_1)^{\otimes 2} \in S^2 A^* \otimes S^2 B^*$, it has a nondegenerate pairing with the first factor, and a zero pairing with the second. The pairing is $GL(A) \times GL(B)$-invariant and the Segre is homogeneous, so one concludes by Schur's lemma.

Similarly, using the decomposition $S^d(A \otimes B) = \bigoplus_{|\pi|=d} S_\pi A \otimes S_\pi B$, and noting that $(\alpha_1 \otimes \beta_1)^{\otimes d} \in S^d A^* \otimes S^d B^*$, we conclude that

$$I_d(Seg(\mathbb{P}A^* \times \mathbb{P}B^*)) = \bigoplus_{|\pi|=d, \pi \neq (d)} S_\pi A \otimes S_\pi B.$$

6.10.2. Exercises on the ideals of Segre varieties.

(1) Show that $I_2(Seg(\mathbb{P}A_1^* \times \cdots \times \mathbb{P}A_n^*))$ is the complement of $S^2 A_1 \otimes \cdots \otimes S^2 A_n$ in $S^2(A_1 \otimes \cdots \otimes A_n)$, and describe these modules explicitly. ⊚

(2) More generally, show that

(6.10.1)
$$I_d(Seg(\mathbb{P}A_1^* \times \cdots \times \mathbb{P}A_n^*)) = (S^d A_1^* \otimes \cdots \otimes S^d A_n^*)^\perp \subset S^d(A_1 \otimes \cdots \otimes A_n).$$

In §6.10.5, we will interpret (6.10.1) as saying that the ideal in degree d is the complement to the d-th Cartan power of $A_1 \otimes \cdots \otimes A_n$.

6.10.3. Modules of equations for Grassmannians. Let $e_1, \dots, e_n$ be a basis of V with dual basis (e^j). Consider $e^1 \wedge \cdots \wedge e^k \in \hat{G}(k, V^*)$ and the induced element $(e^1 \wedge \cdots \wedge e^k)^{\otimes 2} \in S^2(\Lambda^k V^*)$. Recall the formula of §6.7.6:

$$S^2(\Lambda^k V) = S_{2^{(k)}} V \oplus S_{2^{(k-2)},1^{(4)}} V \oplus S_{2^{(k-4)},1^{(8)}} V \oplus \cdots .$$

Proposition 6.10.3.1. $I_2(G(k, V^*)) = S_{2^{(k-2)},1^{(4)}} V \oplus S_{2^{(k-4)},1^{(8)}} V \oplus \cdots .$

Proof. $S_{2^{(k)}} V \subset S^2(\Lambda^k V)$ has highest weight vector $(e_1 \wedge \cdots \wedge e_k)^2$ and thus it pairs nondegenerately with $(e^1 \wedge \cdots \wedge e^k)^{\otimes 2}$. By Schur's lemma, $(e^1 \wedge \cdots \wedge e^k)^{\otimes 2}$ has zero pairing with all other factors. $\square$

By the same pairing argument, $I_d(G(k, V^*))$ is the complement of $S_{d^{(k)}} V$ in $S^d(\Lambda^k V)$. As explained in §6.10.5, $S_{d^{(k)}} V$ is the d-th Cartan power of $\Lambda^k V = S_{1^k} V$.

6.10.4. Modules of equations for Veronese varieties. Recalling §6.7.6,

$$S^2(S_d A) = \begin{Bmatrix} S_{2d} A \oplus S_{2d-2,2} A \oplus S_{2d-4,4} A \oplus \cdots \oplus S_{d,d} A, & d \text{ even} \\ S_{2d} A \oplus S_{2d-2,2} A \oplus S_{2d-4,4} A \oplus \cdots \oplus S_{d+1,d-1} A, & d \text{ odd} \end{Bmatrix}.$$

The same reasoning as above gives

Proposition 6.10.4.1. *The ideal of*

$$v_d(\mathbb{P}A^*) \subset \mathbb{P}(S^d A)^*$$

in degree two is

$$\begin{Bmatrix} S_{2d} A \oplus S_{2d-2,2} A \oplus S_{2d-4,4} A \oplus \cdots \oplus S_{d,d} A, & d \text{ even} \\ S_{2d} A \oplus S_{2d-2,2} A \oplus S_{2d-4,4} A \oplus \cdots \oplus S_{d+1,d-1} A, & d \text{ odd} \end{Bmatrix} \subset S^2(S^d A),$$

and more generally, $I_m(v_d(\mathbb{P}A^*)) = (S_{md} A^*)^\perp \subset S^m(S^d A)$.

Exercise 6.10.4.2: Prove the proposition.

These modules of equations may be realized geometrically and explicitly in bases by noting that $v_d(\mathbb{P}A^*) \subset Seg(\mathbb{P}A^* \times \mathbb{P}(S^{d-1} A^*))$ and considering the restriction of the modules in $\Lambda^2 A \otimes \Lambda^2(S^{d-1} A)$ to $S^2(S^d A^*)$. That is, they are the 2×2-minors of the $(\dim A) \times (\dim S^{d-1} A)$ matrix corresponding to the $(1, d-1)$ polarization (cf. §2.6.4).

6.10.5. Cartan products for $GL(V)$. Let $\pi = (p_1, \dots, p_k)$ and $\mu = (m_1, \dots, m_s)$. There is a natural submodule of $S_\pi V \otimes S_\mu V$, namely $S_\tau V$, where $\tau = (p_1 + m_1, p_2 + m_2, \dots,)$. $S_\tau V$ is called the *Cartan product* of $S_\pi V$ and $S_\mu V$. It is the irreducible submodule of $S_\pi V \otimes S_\mu V$ with highest highest weight. The Cartan product occurs with multiplicity one in the tensor product, and we call it the *Cartan component* of $S_\pi V \otimes S_\mu V$. To see this, if we realize highest weight vectors of $S_\pi V \subset V^{\otimes |\pi|}$, $S_\mu V \subset V^{\otimes |\mu|}$ respectively by images of vectors $(e_1)^{\otimes p_1} \otimes \cdots \otimes (e_k)^{\otimes p_k}$, $(e_1)^{\otimes m_1} \otimes \cdots \otimes (e_s)^{\otimes m_s}$, then a highest weight vector of $S_\tau V$ will be realized as a projection of $(e_1)^{\otimes p_1+m_1}\otimes(e_2)^{\otimes p_2+m_2}\otimes\cdots$. When we take the Cartan product of a module with itself d times, we call it the d-th *Cartan power*.

Exercise 6.10.5.1: Find the highest weight of $S^p(\Lambda^k V)\otimes S^q(\Lambda^i V)$.

Exercise 6.10.5.2: Show that $S^{a+b}V\subset S^a(S^bV)$.

Remark 6.10.5.3. There exists a partial ordering on the set of partitions (i.e., the set of irreducible $GL(V)$-modules), and the Cartan product of $S_\pi V$ and $S_\mu V$ is the (well-defined) leading term among the partitions occurring in $S_\pi V\otimes S_\mu V$.

The notion of Cartan product generalizes to irreducible modules of an arbitrary semi-simple Lie algebra as follows: after making choices in $\mathfrak{g}$, each irreducible $\mathfrak{g}$-module is uniquely determined by a highest weight, and irreducible modules are labelled by highest weights, e.g., V_λ is the (unique) module of highest weight λ. Then the Cartan product of V_λ with V_μ is $V_{\lambda+\mu}$ which is the highest weight submodule of $V_\lambda\otimes V_\mu$.

6.10.6. The ideal of $G/P\subset\mathbb{P}V_\lambda$. Refer to Chapter 16 for the necessary definitions in what follows.

If V_λ is an irreducible module for a semi-simple group G whose highest weight is λ, then S^2V_λ contains the irreducible module $V_{2\lambda}$ whose highest weight is 2λ, the Cartan square of V_λ. If V_λ is respectively $A_1\otimes\cdots\otimes A_n$, $S_{1^k}V=\Lambda^kV$, S^dA, then $V_{2\lambda}$ is respectively $S^2A_1\otimes\cdots\otimes S^2A_n$, $S_{2^k}V$, $S^{2d}A$.

Let $X=G/P=G.[v_\lambda]\subset\mathbb{P}V_\lambda$ be the orbit of a highest weight line.

Proposition 6.10.6.1 (Kostant [unpublished])**.** *Let V_λ be an irreducible G-module of highest weight λ, and let $X=G/P\subset\mathbb{P}V_\lambda$ be the orbit of a highest weight line. Then $I_d(X)=(V_{d\lambda})^\perp\subset S^dV^*$.*

The proof is the same as in the Segre, Veronese, and Grassmannian cases.

Exercise 6.10.6.2: Find the degrees of $Seg(\mathbb{P}A_1\times\mathbb{P}A_2)$ and, more generally, of $Seg(\mathbb{P}A_1\times\cdots\times\mathbb{P}A_n)$, $G(k,V)$, and $v_d(\mathbb{P}V)$. ⊚

Remark 6.10.6.3. In §13.8 explicit equations for Grassmannians and other cominuscule rational homogeneous varieties are given in terms of relations among minors.

Remark 6.10.6.4. Similarly, using the Weyl dimension formula 16.1.4.5, it is easy to determine the degree (in fact, the entire Hilbert function) of any homogeneously embedded rational homogeneous variety.

Kostant proved that, moreover, the ideal is always generated in degree two.

Theorem 6.10.6.5 (Kostant [unpublished]). *Let V_λ be an irreducible G-module of highest weight λ, and let $X = G/P \subset \mathbb{P}V_\lambda$ be the orbit of a highest weight line. Then $I(X)$ is generated in degree two by ${V_{2\lambda}}^\perp \subset S^2(V_\lambda)^*$.*

A proof of Kostant's theorem on the ideals of G/P is given in §16.2. It appeared in the unpublished PhD thesis of D. Garfinkle, and is also available in [**197, 268**].

Exercise 6.10.6.6: Prove Kostant's theorem in the case of $Seg(\mathbb{P}A^* \times \mathbb{P}B^*)$. I.e., show that the maps

$$\bigoplus_{|\pi|=d-1,\ \pi\neq(d-1)} S_\pi A{\otimes}S_\pi B{\otimes}(A{\otimes}B) \to \bigoplus_{|\mu|=d,\ \mu\neq(d)} S_\mu A{\otimes}S_\mu B$$

are surjective for all $d > 2$. ⊚

6.10.7. Explicit equations for Segre-Veronese varieties. The Cartan component of $S^2(S^{d_1}A_1{\otimes}\cdots{\otimes}\, S^{d_n}A_n)$ is $S^{2d_1}A_1{\otimes}\cdots{\otimes}\, S^{2d_n}A_n$. Kostant's Theorem 6.10.6.5 states that the set of modules of generators of the ideal of $Seg(v_{d_1}(\mathbb{P}A_1^*) \times \cdots \times v_{d_n}(A_n^*)) \subset \mathbb{P}(S^{d_1}A_1{\otimes}\cdots{\otimes}\, S^{d_n}A_n)^*$ equals the complement to the Cartan component in $S^2(S^{d_1}A_1{\otimes}\cdots{\otimes}\, S^{d_n}A_n)$. The variety $Seg(v_{d_1}(\mathbb{P}A_1^*) \times \cdots \times v_{d_n}(A_n^*)) \subset \mathbb{P}(S^{d_1}A_1{\otimes}\cdots{\otimes}\, S^{d_n}A_n)^*$ is often called a *Segre-Veronese variety.*

Consider the example of $Seg(\mathbb{P}A_1^* \times v_d(\mathbb{P}A_2^*))$ (for simplicity of exposition, assume that d is even):

$$\begin{aligned} S^2(A_1{\otimes}S^dA_2) &= S^2(A_1){\otimes}S^2(S^dA_2) \oplus \Lambda^2A_1{\otimes}\Lambda^2(S^dA_2)\\ &= S^2A_1{\otimes}(S_{2d}A_2 \oplus S_{2d-2,2}A_2 \oplus \cdots \oplus S_{d,d}A_2)\\ &\quad \oplus \Lambda^2A_1{\otimes}(S_{2d-1,1}A_2 \oplus S_{2d-3,3}A_2 \oplus \cdots \oplus S_{d+1,d-1}A_2), \end{aligned}$$

so the ideal is generated by all the factors appearing above except for $S^2A_1{\otimes}S^{2d}A_2$. The second set of factors is naturally realized as a collection of 2×2 minors of an $\mathbf{a}_1 \times \binom{\mathbf{a}_2+d-1}{d}$ matrix. In [**24**], A. Bernardi gives a geometric model for all the equations in terms of minors.

6.11. Symmetric functions

This section is only used in §§8.6.2, 13.4.2, and 13.5.3.

Recall that, giving $\mathbb{C}^{n*}$ a basis $(x_1,\dots,x_n)$, one obtains an action of $\mathfrak{S}_n$ on $\mathbb{C}^{n*}$ such that the vector $e_1 = e_1(x) = x_1 + \cdots + x_n$ is invariant. One can similarly consider the induced action of $\mathfrak{S}_n$ on polynomials $\mathbb{C}[x_1,\dots,x_n]$ and ask what are the invariant polynomials, $\mathbb{C}[x_1,\dots,x_n]^{\mathfrak{S}_n}$. This subset is in fact a subspace, even a graded subring. In degree two, there of course is

e_1^2, but also

$$p_2(x) := x_1^2 + \cdots + x_n^2,$$
$$e_2(x) := \sum_{1 \le i < j \le n} x_i x_j.$$

These three polynomials are not independent, $p_2 = e_1^2 - 2e_2$. More generally, define

$$p_d(x) := x_1^d + \cdots + x_n^d, \tag{6.11.1}$$
$$e_d(x) := \sum_{1 \le i_1 < \cdots < i_d \le n} x_{i_1} \cdots x_{i_d}, \tag{6.11.2}$$

respectively called the *power sums* and *elementary symmetric functions*. Either of these, as d runs from 1 to n, gives a set of generators of the ring $\mathbb{C}[x_1, \ldots, x_n]^{\mathfrak{S}_n}$. They can be described in terms of generating functions

$$E(t) = \sum_{j \ge 0} t^j e_j = \prod_{1 \le i \le n} (1 + tx_i), \tag{6.11.3}$$
$$P(t) = \sum t^j p_j = \sum_{r \ge 1} \Big(t^r \sum_i x_i^r \Big) = \sum_{i \ge 1} \frac{x_i}{1 - tx_i}. \tag{6.11.4}$$

Examining the two series, one sees that they are related by

$$P(-t) = \frac{E'(t)}{E(t)}. \tag{6.11.5}$$

In particular, one can convert from one to the other by formulas that are relatively cheap (polynomial cost) to implement.

Explicitly, one has the *Girard formula*
(6.11.6)
$$p_k = (-1)^k k \sum_{i_1 + 2i_2 + \cdots + di_d = k} (-1)^{i_1 + \cdots + i_d} \frac{(i_1 + \cdots + i_d - 1)!}{i_1! \cdots i_d!} e_1^{i_1} \cdots e_d^{i_d}.$$

Write this as $p_k = \mathcal{P}_k(e_1, \ldots, e_k)$. For later reference, note that e_1^k occurs in the formula with a nonzero coefficient.

Symmetric functions are closely related to the representation theory of the symmetric group. For example, there is a natural Hermitian inner product on the ring of symmetric functions in n variables and a natural isometry between the ring of characters of $\mathfrak{S}_n$ and the ring of symmetric functions; see, e.g., [**229**, Prop. 1.6.3]

Chapter 7

Tests for border rank: Equations for secant varieties

This chapter includes nearly all that is known about defining equations for secant varieties of Segre and Veronese varieties. It also discusses general techniques for finding equations. The general techniques are as follows:

Exploiting known equations. If a variety X is contained in a variety Z, then $I(Z) \subset I(X)$, and the same holds for their successive secant varieties. While this statement is obvious, exploiting it is an art. In practice, one is handed X and the art is to find varieties Z containing X whose ideals are easily described. More generally, one can try to show that $X = \bigcap_j Z_j$ with equations of Z_j known. When $X = \sigma_r(Y) \subset \mathbb{P}V$, the main use of this technique is to find linear embeddings $V \subset A{\otimes}B$ such that $Y \subset \sigma_p(Seg(\mathbb{P}A \times \mathbb{P}B))$. Then the minors $\Lambda^{rp+1}A^*{\otimes}\Lambda^{rp+1}B^*$ restricted to V give equations for $\sigma_r(Y)$.

Inheritance. In the case of homogeneous varieties coming in "series" (such as $X_n = Seg(\mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n)$), one may use modules of equations for a secant variety of a smaller variety in the series to obtain modules of equations of a secant variety of a larger variety in the series. This technique, *inheritance*, is described in detail for the special cases of secant varieties of Segre varieties and Veronese varieties in §7.4. (Inheritance is placed in a more general setting in §16.4.)

Prolongation and multiprolongation. (Multi)-Prolongation is a linear algebra construction to obtain the ideals of secant varieties (and joins) from the equations of the original variety $X \subset \mathbb{P}V$. This method always works in principle but can be difficult to implement in practice. When X is homogeneous, its usage simplifies and the state of the art for both arbitrary and homogeneous varieties is given in §7.5.

The chapter begins in §7.1 with a discussion of subspace varieties and their ideals. Subspace varieties arise naturally in applications because they govern multilinear rank (also called *Tucker rank* in the tensor literature), and they are also useful for furnishing some of the equations of secant varieties of Segre and Veronese varieties (the varieties of tensors and symmetric tensors of border rank at most r). Next, in §7.2, additional auxiliary varieties are introduced. These include varieties of flattenings and symmetric flattenings whose ideals are easily described. They are used to obtain further equations for secant varieties of Segre and Veronese varieties. Another class of auxiliary varieties introduced are those of bounded image rank which appear in several contexts, including the proof of the inadequacy of Strassen's degree-nine equations. The remaining sections are as follows: in §7.3 the utility and limits of flattenings are discussed, §7.4 and §7.5 respectively discuss inheritance and prolongation, and §7.6 discusses Strassen's equations for secant varieties of triple Segre products, applications to the study of the complexity of matrix multiplication, and numerous generalizations. Friedland's solution of the set-theoretic salmon prize problem is given in §7.7. Taking advantage of the Pieri formula, one can obtain equations for secant varieties of Veronese varieties as described in §7.8, which are called equations of *Young flattenings*.

As was said in Chapter 4, when discussing "equations" for a variety $X \subset \mathbb{P}V$, one can either be content to find a collection of polynomials such that the points of X are exactly the zero set of the collection of polynomials, which are called *set-theoretic equations* for X, or, more ambitiously, one can ask for *scheme-theoretic equations* or even *generators of the ideal of X*. For most applications, set-theoretic equations are sufficient.

7.1. Subspace varieties and multilinear rank

7.1.1. Subspace varieties in $\mathbb{P}(A_1^* \otimes \cdots \otimes A_n^*)$. Recall the subspace variety from §3.4.1:

$$
\begin{aligned}
(7.1.1)\qquad & Sub_{\mathbf{b}_1,\dots,\mathbf{b}_n}(A_1 \otimes \cdots \otimes A_n) \\
&= \mathbb{P}\{T \in A_1 \otimes \cdots \otimes A_n \mid \mathbf{R}_{\mathrm{multlin}}(T) \leq (\mathbf{b}_1, \dots, \mathbf{b}_n)\} \\
&= \mathbb{P}\{T \in A_1 \otimes \cdots \otimes A_n \mid \dim(T(A_j^*)) \leq \mathbf{b}_j \ \forall 1 \leq j \leq n\} \\
&= \bigcap_{1 \leq j \leq n} \mathrm{Zeros}(\Lambda^{\mathbf{b}_j+1} A_j^* \otimes \Lambda^{\mathbf{b}_j+1} A_{\hat{j}}^*).
\end{aligned}
$$

Exercise 7.1.1.1: Show that the ideal in degree d of $Sub_{\mathbf{b}_1,\dots,\mathbf{b}_n}$ consists of the isotypic components of all modules $S_{\pi_1}A_1 \otimes \cdots \otimes S_{\pi_n}A_n$ occurring in $S^d(A_1 \otimes \cdots \otimes A_n)$, where each π_j is a partition of d and at least one π_j has $\ell(\pi_j) > \mathbf{b}_j$.

The modules of the third line of (7.1.1) (with some redundancy) generate the ideal:

Theorem 7.1.1.2 ([**210**])**.** *The ideal of the subspace varieties $Sub_{\mathbf{b}_1,\dots,\mathbf{b}_n}$ is generated in degrees $\mathbf{b}_j + 1$ for $1 \le j \le n$ by the irreducible modules in*

$$\Lambda^{\mathbf{b}_j+1}A_j \otimes \Lambda^{\mathbf{b}_j+1}(A_1 \otimes \cdots \otimes A_{j-1} \otimes A_{j+1} \otimes \cdots \otimes A_n).$$

To eliminate redundancy, reorder so that $\mathbf{b}_1 \le \mathbf{b}_2 \le \cdots \le \mathbf{b}_n$ and take the partitions $S_{\pi_i}A_i$ that occur for $i \le j$ with $\ell(\pi_i) \le \mathbf{b}_i$, unless $\mathbf{b}_i = \mathbf{b}_j$, in which case also take partitions π_i with $\ell(\pi_i) = \mathbf{b}_i + 1$.

In particular, if all the $\mathbf{b}_i = r$, the ideal of $Sub_{r,\dots,r}$ is generated in degree $r+1$ by the irreducible modules appearing in

$$\Lambda^{r+1}A_j \otimes \Lambda^{r+1}(A_1 \otimes \cdots \otimes A_{j-1} \otimes A_{j+1} \otimes \cdots \otimes A_n)$$

for $1 \le j \le n$ (minus redundancies).

Proof. The ideal in degree d is given by Exercise 7.1.1.1. For each $1 \le j \le n$, the ideal consisting of representations $S_{\pi_1}A_1 \otimes \cdots \otimes S_{\pi_n}A_n$ occurring in $S^d(A_1 \otimes \cdots \otimes A_n)$, where $\ell(\pi_j) > \mathbf{b}_j$, is generated in degree $\mathbf{b}_j + 1$ by

$$\Lambda^{\mathbf{b}_j+1}A_j \otimes \Lambda^{\mathbf{b}_j+1}(A_1 \otimes \cdots \otimes A_{j-1} \otimes A_{j+1} \otimes \cdots \otimes A_n),$$

because it is just the ideal for rank at most r tensors in the tensor product of two vector spaces (see §6.7.4). After reordering the summands so that $\mathbf{b}_1 \le \cdots \le \mathbf{b}_n$, the assertion about which partitions $S_{\pi_i}A_i$ appear follows. $\square$

Exercises 7.1.1.3 (The three-factor GSS conjecture [**137**])**:** Prove that $I(\sigma_2(Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)))$ is generated in degree three and find the modules of generators.

7.1.2. Aside for those familiar with vector bundles: A desingularization of $Sub_{\mathbf{b}_1,\dots,\mathbf{b}_n}$. Another perspective on $Sub_{\mathbf{b}_1,\dots,\mathbf{b}_n}$ is as follows. Let $\mathcal{S}_{G(r,V)} \to G(r,V)$ denote the vector bundle whose fiber over $E \in G(r,V)$ is E. Note that $\mathcal{S}_{G(r,V)}$ is a subbundle of the trivial bundle with fiber V. $\mathcal{S}_{G(r,V)}$ is often called the *tautological subspace bundle*. Consider the product of Grassmannians $B = G(\mathbf{b}_1, A_1) \times \cdots \times G(\mathbf{b}_n, A_n)$ and the bundle $\mathcal{S} := \mathcal{S}_1 \otimes \cdots \otimes \mathcal{S}_n$, which is the tensor product of the tautological subspace bundles pulled back to B. A point of $\mathcal{S}$ is of the form $(E_1, \dots, E_n, T)$, where $E_j \subset A_j$ is a $\mathbf{b}_j$-plane, and $T \in E_1 \otimes \cdots \otimes E_n$. Consider the projection $\pi : \mathcal{S} \to A_1 \otimes \cdots \otimes A_n$, $(E_1, \dots, E_n, T) \mapsto T$. The image is $\hat{S}ub_{\mathbf{b}_1,\dots,\mathbf{b}_n}$. The map $\pi : \mathcal{S} \to \hat{S}ub_{\mathbf{b}_1,\dots,\mathbf{b}_n}$ gives a *Kempf-Weyman desingularization* of

$Sub_{\mathbf{b}_1,\dots,\mathbf{b}_n}$. Such desingularizations are useful for finding equations, minimal free resolutions, and establishing properties of singularities; see Chapter 17.

7.1.3. Subspace varieties in $\mathbb{P}S^dV$. Recall the symmetric subspace variety from Definition 3.1.3.4:

$$Sub_r(S^dV) = \mathbb{P}\{P \in S^dV \mid \exists W \subset V,\ \dim W = r,\ P \in S^dW\}.$$

$Sub_r(S^dV)$ has the geometric interpretation of the set of polynomials $P \in S^dV$ such that $\mathrm{Zeros}(P) \subset \mathbb{P}V^*$ is a cone with a $(\dim V - r)$-dimensional vertex.

Proposition 7.1.3.1. *$Sub_r(S^dV)$ is a projective variety and $\sigma_r(v_d(\mathbb{P}V) \subseteq Sub_r(S^dV)$.*

Exercise 7.1.3.2: Prove Proposition 7.1.3.1.

Thus equations for $Sub_r(S^dV)$ furnish equations for $\sigma_r(v_d(\mathbb{P}V))$.

Proposition 7.1.3.3. *Using the natural inclusion $S^dV \subset V \otimes S^{d-1}V$, the space $\Lambda^{r+1}V^* \otimes \Lambda^{r+1}S^{d-1}V^* \subset S^{r+1}(V^* \otimes S^{d-1}V^*)$ has a nonzero projection to $S^{r+1}(S^dV^*) \subset S^{r+1}(V^* \otimes S^{d-1}V^*)$ whose image gives set-theoretic defining equations for $Sub_r(S^dV)$.*

Proof. Consider $P \in S^dV \subset V \otimes S^{d-1}V$ as a linear map $P_{1,d-1} : V^* \to S^{d-1}V$. Then $P \in Sub_r(S^dV)$ if and only if $\mathrm{rank}(P_{1,d-1}) \leq r$, i.e., P is in the zero set of $\Lambda^{r+1}V^* \otimes \Lambda^{r+1}(S^{d-1}V^*)$. ☐

It is also easy to define the ideal:

Proposition 7.1.3.4. *$I_\delta(Sub_r(S^dV^*))$ consists of the isotypic components of the modules $S_\pi V$ appearing in $S^\delta(S^dV)$ such that $\ell(\pi) > r$.*

Exercise 7.1.3.5: Prove Proposition 7.1.3.4.

Theorem 7.1.3.6 ([**333**, Corollary 7.2.3])**.** *The ideal of $Sub_r(S^dV^*)$ is generated by the image of $\Lambda^{r+1}V \otimes \Lambda^{r+1}S^{d-1}V \subset S^{r+1}(V \otimes S^{d-1}V)$ in $S^{r+1}(S^dV)$.*

Proof. Consider the commutative diagram

$$\begin{tikzcd}
\Lambda^r V \otimes \Lambda^r(V^{\otimes d-1}) \otimes (V^{\otimes d}) \arrow[r] \arrow[d] & \Lambda^{r+1}V \otimes \Lambda^{r+1}V^{\otimes d-1} \arrow[d] \\
\Lambda^r V \otimes \Lambda^r(S^{d-1}V) \otimes S^dV \arrow[r] & \Lambda^{r+1}V \otimes \Lambda^{r+1}S^{d-1}V
\end{tikzcd}$$

The vertical arrows are partial symmetrization. The horizontal arrows are obtained by writing $V^{\otimes d} = V \otimes V^{\otimes d-1}$ and skew-symmetrizing. All arrows except possibly the lower horizontal arrow are known to be surjective, so the latter must be as well, which is what we were trying to prove. ☐

For those familiar with vector bundles, a desingularization of $Sub_r(S^dV)$ may be done as in the Segre case. Consider the bundle $S^d\mathcal{S} \to G(r,V)$ whose fiber over $E \in G(r,V)$ consists of points (E,P) such that $P \in S^dE$. The image of the map $\pi : S^d\mathcal{S} \to S^dV$ is $\hat{S}ub_r(S^dV)$.

7.1.4. Subspace varieties in the exterior algebra. One can similarly define $Sub_r(\Lambda^kV) \subset \mathbb{P}(\Lambda^kV)$. Again its ideal is easy to determine:

Exercise 7.1.4.1: $I_d(Sub_r(\Lambda^kV))$ consists of all isotypic components of all $S_\pi V$ appearing in $S^d(\Lambda^kV)$ with $|\pi| = dk$ and $\ell(\pi) \geq rk+1$.

On the other hand, determining generators for the ideal is more difficult.

The lowest degree for which there can be generators is $\lceil \frac{rk+1}{k} \rceil$. There are no new generators in a given degree d if and only if the map

$$I_{d-1}(Sub_r(\Lambda^kV)){\otimes}\Lambda^kV^* \to I_d(Sub_r(\Lambda^kV)) \tag{7.1.2}$$

is surjective. Generators of the ideal are not known in general; see [**333**] for the state of the art.

Exercise 7.1.4.2: The ideal in degree d for $Sub_{2t+1}(\Lambda^2V)$ is the same as that for $Sub_{2t}(\Lambda^2V)$. Why?

Note that $\sigma_t(G(k,V)) \subset Sub_{kt}(\Lambda^kV)$, so $Sub_r(\Lambda^kV)$ is useful for studying ideals of secant varieties of Grassmannians.

For those familiar with vector bundles, the variety $Sub_r(\Lambda^kV)$ may also be realized via a Kempf-Weyman desingularization. Consider the bundle $\Lambda^k\mathcal{S} \to G(rk,V)$ whose fiber over $E \in G(rk,V)$ consists of points (E,P) such that $P \in \Lambda^kE$. As above, we have a map to Λ^kV whose image is $\hat{S}ub_r(\Lambda^kV)$.

Exercise 7.1.4.3: Assume $k \leq n-k$. Show that when $k=3$, $\sigma_2(G(3,V)) = Sub_6(\Lambda^kV)$, and for all $k>3$, $\sigma_2(G(k,V)) \subsetneq Sub_{2k}(\Lambda^kV)$. Similarly show that for all $k \geq 3$ and all $t>1$, $\sigma_t(G(k,V)) \subsetneq Sub_{kt}(\Lambda^kV)$. ⊚

Remark 7.1.4.4. More generally, for any $GL(V)$-module $S_{\pi}V \subset V^{\otimes k}$ one can define the corresponding subspace varieties; see [**333**, Chap. 7].

Remark 7.1.4.5. Recently the generators of the ideal of $Sub_{\mathbf{a}',\mathbf{b}'}(A{\otimes}S^2B)$ were determined. See [**72**].

7.2. Additional auxiliary varieties

7.2.1. Varieties of flattenings. A natural generalization of the subspace varieties are the varieties of *flattenings*.

Define

$$
\begin{aligned}
(7.2.1)\qquad Flat_r &= Flat_r(A_1 \otimes \cdots \otimes A_n) \\
&= \bigcap_{\{I : |I|=r\}} \sigma_r(Seg(\mathbb{P}A_I \times \mathbb{P}A_{I^c})) \subset \mathbb{P}(A_1 \otimes \cdots \otimes A_n).
\end{aligned}
$$

Exercise 7.2.1.1: What are the generators of $I(Flat_r)$?

Remark 7.2.1.2. One can define flattenings to more than two factors. For example, Strassen's equations and their generalizations (see §7.6) can be used to obtain equations for Segre varieties of four and more factors using three-factor flattenings.

Similarly, define the variety

$$
(7.2.2)\qquad SFlat_r(S^dV^*) := \mathrm{Zeros}\Big\{ \bigcup_{a=1}^{\lfloor \frac{d}{2} \rfloor} (\Lambda^{r+1} S^a V \otimes \Lambda^{r+1} S^{d-a} V)|_{S^dV^*} \Big\},
$$

the intersection of the zero sets of the catalecticant minors.

Since the ideals of these varieties are well understood, they are useful auxiliary objects for studying the ideals of secant varieties of Segre and Veronese varieties.

7.2.2. Bounded image-rank varieties.

Definition 7.2.2.1. Given vector spaces A, B, and C, define the varieties

$$
\begin{aligned}
Rank^r_A(A \otimes B \otimes C) &= Rank^r_A \\
&:= \{ T \in A \otimes B \otimes C \mid \mathbb{P}T(A^*) \subset \sigma_r(Seg(\mathbb{P}B \times \mathbb{P}C)) \}
\end{aligned}
$$

and similarly for $Rank^r_B, Rank^r_C$.

These varieties arise naturally in the study of equations for secant varieties as follows: while Strassen's equations for $\sigma_k(Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C))$ hold automatically on $Rank^{k-2}_A \cup Rank^{k-2}_B \cup Rank^{k-2}_C$, these varieties are not necessarily contained in $\sigma_k(Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C))$; see §3.8 and §7.6.6. They also arise in Ng's study of tensors in $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$; see §10.5.

Fortunately their equations are easy to describe:

Proposition 7.2.2.2. *$Rank^r_A(A \otimes B \otimes C)$ is the zero set of the degree $r+1$ equations given by the module $S^{r+1}A^* \otimes \Lambda^{r+1}B^* \otimes \Lambda^{r+1}C^*$.*

Proof. Let $\alpha \in A^*$; then $T(\alpha)$ has rank at most r if and only if its minors of size $r+1$ all vanish, that is, if the equations $\Lambda^{r+1}B^* \otimes \Lambda^{r+1}C^* \subset S^{r+1}(B \otimes C)^*$ vanish on it. This holds if and only if $\alpha^{r+1} \otimes \Lambda^{r+1}C^* \subset S^{r+1}(A \otimes B \otimes C)^*$ vanishes on T. Since the Veronese embedding is linearly nondegenerate, we may span the full module this way. $\square$

The varieties $Rank^r_A$ are usually far from irreducible—they have many components of differing dimensions. Consider the case $\mathbf{b} = \mathbf{c} = 3$. The linear subspaces of $\mathbb{C}^3 \otimes \mathbb{C}^3$ of bounded rank two admit normal forms [**10, 120**]. There are two such:

$$\begin{pmatrix} 0 & x & y \\ -x & 0 & z \\ -y & -z & 0 \end{pmatrix}, \quad \begin{pmatrix} s & t & u \\ v & 0 & 0 \\ w & 0 & 0 \end{pmatrix}. \tag{7.2.3}$$

These give rise to different components.

7.3. Flattenings

Recall the flattenings $A_1 \otimes \cdots \otimes A_n \subset A_I \otimes A_{I^c}$ from §3.4 and §7.2.1, where $A_I = A_{i_1} \otimes \cdots \otimes A_{i_{|I|}}$, and $I \cup I^c = \{1, \ldots, n\}$. The modules $\Lambda^{r+1} A_I^* \otimes \Lambda^{r+1} A_{I^c}^*$ furnish some of the equations for $\sigma_r(Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n))$. Similarly the minors of symmetric flattenings discussed in §3.5.1, $S^d V \subset S^k V \otimes S^{d-k} V$, give equations for $\sigma_r(v_d(\mathbb{P}V))$. In this section I discuss situations where flattenings and symmetric flattenings provide enough equations.

7.3.1. Equations by "catching-up" to a known variety. A classical technique for showing that a secant variety of a variety $X \subset \mathbb{P}V$ is degenerate is to find a variety $Y \subset \mathbb{P}V$, with Y containing X and $\sigma_k(Y)$ very degenerate.

Proposition 7.3.1.1 ([**81**]). *Let $X \subset Y \subset \mathbb{P}V$. If the secant varieties of X "catch up" to those of Y, i.e., if there exists r such that $\sigma_r(X) = \sigma_r(Y)$, then $\sigma_t(X) = \sigma_t(Y)$ for all $t > r$ as well.*

Proof. First note that for $t = mr$ the proposition is immediate by Exercise 5.1.1.6. Now note that for $u < r$,

$$\sigma_r(X) = J(\sigma_{r-u}(X), \sigma_u(X)) \subseteq J(\sigma_{r-u}(Y), \sigma_u(X)) \subseteq \sigma_r(Y),$$

so $\sigma_r(Y) = J(\sigma_{r-u}(Y), \sigma_u(X))$. Write $t = mr + u$,

$$\begin{aligned} \sigma_t(X) &= J(\sigma_{mr}(X), \sigma_u(X)) \\ &= J(\sigma_{(m-1)r}(Y), \sigma_u(Y), J(\sigma_{r-u}(Y), \sigma_u(X))) \\ &= \sigma_{mr+u}(Y). \end{aligned}$$

$\square$

Since $\sigma_r(Seg(\mathbb{P}A \times \mathbb{P}B))$ is very degenerate, if an n-factor case is "unbalanced" in the sense that one space is much larger than the others, it can catch up to a corresponding two-factor case.

Exercise 7.3.1.2: Show that

$$\sigma_2(Seg(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^3)) = \sigma_2(Seg(\mathbb{P}(\mathbb{C}^2 \otimes \mathbb{C}^2) \times \mathbb{P}^3)). \circledcirc$$

The following generalizes Exercise 7.3.1.2 (which is the case of $X = Seg(\mathbb{P}^1 \times \mathbb{P}^1)$).

Lemma 7.3.1.3 (Terracini [**307**]). *Let $X^n \subset \mathbb{P}^N = \mathbb{P}V$ be a variety not contained in a hyperplane. Then $\sigma_r(Seg(X \times \mathbb{P}W)) = \sigma_r(Seg(\mathbb{P}V \times \mathbb{P}W))$ for all $r > N - n$.*

Proof. By Proposition 7.3.1.1, it is sufficient to prove the case $r = N - n + 1$, and since $\sigma_r(Seg(X \times \mathbb{P}W)) \subseteq \sigma_r(Seg(\mathbb{P}V \times \mathbb{P}W))$, it is sufficient to prove that a general point of $\sigma_{N-n+1}(Seg(\mathbb{P}V \times \mathbb{P}W))$ is contained in $\sigma_{N-n+1}(Seg(X \times \mathbb{P}W))$. Write $p = v_1 \otimes w_1 + \cdots + v_{N-n+1} \otimes w_{N-n+1}$, so $[p] \in \sigma_{N-n+1}(Seg(\mathbb{P}V \times \mathbb{P}W))$. Since $[p]$ is a general point, $v_1, \ldots, v_{N-n+1}$ span a $\mathbb{P}^{N-n}$ which must intersect X in $\deg(X)$ points (counting multiplicity, but by our general point assumption we may assume we have distinct points). Since (see §4.9.4) $\deg(X) \geq N - n$, there exist $[x_1], \ldots, [x_{N-n+1}] \in X$ such that $\langle x_1, \ldots, x_{N-n+1} \rangle = \langle v_1, \ldots, v_{N-n+1} \rangle$. Now re-express p with respect to the x_j's. $\square$

Consider Segre varieties $Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n)$ with $\mathbf{a}_n >> \mathbf{a}_i$ for $1 \leq i \leq n-1$. To apply Lemma 7.3.1.3 to $X = Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_{n-1})$, one must have $r \geq \prod \mathbf{a}_i - \sum \mathbf{a}_i - n + 1$. On the other hand, to have a nontrivial secant variety of $Seg(\mathbb{P}(A_1 \otimes \cdots \otimes A_{n-1}) \times \mathbb{P}A_n)$, one needs $r < \min\{\prod \mathbf{a}_i, \mathbf{a}_n\}$. This shows half of:

Theorem 7.3.1.4 ([**81**, Thm. 2.4.2], also see [**76**]). *Consider Segre varieties $Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n)$ where $\dim A_s = \mathbf{a}_s$, $1 \leq s \leq n$. Assume $\mathbf{a}_n \geq \prod_{i=1}^{n-1} \mathbf{a}_i - \sum_{i=1}^{n-1} \mathbf{a}_i - n + 1$.*

(1) *If $r \leq \prod_{i=1}^{n-1} \mathbf{a}_i - \sum_{i=1}^{n-1} \mathbf{a}_i - n + 1$, then $\sigma_r(Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n))$ has the expected dimension $r(\mathbf{a}_1 + \cdots + \mathbf{a}_n - n + 1) - 1$.*

(2) *If $\mathbf{a}_n > r \geq \prod_{i=1}^{n-1} \mathbf{a}_i - \sum_{i=1}^{n-1} \mathbf{a}_i - n + 1$, then*

$$\sigma_r(Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n)) = \sigma_r(Seg(\mathbb{P}(A_1 \otimes \cdots \otimes A_{n-1}) \times \mathbb{P}A_n).$$

In particular, in this situation, $I(\sigma_r(Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n)))$ is generated in degree $r+1$ by $\Lambda^{r+1}(A_1 \otimes \cdots \otimes A_{n-1})^* \otimes \Lambda^{r+1} A_n^*$, and $\sigma_r(Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n)$ has defect $\delta_r = r^2 - r(\prod \mathbf{a}_j - \sum \mathbf{a}_j + n - 1)$.

(3) *If $r \geq \min\{\mathbf{a}_n, \mathbf{a}_1 \cdots \mathbf{a}_{n-1}\}$, then $\sigma_r(Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n)) = \mathbb{P}(A_1 \otimes \cdots \otimes A_n)$.*

Proof. It remains to prove (1) and the statement about the defect. Note that $\dim \sigma_r(Seg(\mathbb{P}(A_1 \otimes \cdots \otimes A_{n-1}) \times \mathbb{P}A_n) = r(\prod \mathbf{a}_i + \mathbf{a}_n - r) - 1$ (see Proposition 5.3.1.4) and the expected dimension of $\sigma_r(Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n))$ is $r(\sum \mathbf{a}_i + \mathbf{a}_n - n + 1) - 1$, which proves the statement about the defect. But the defect is zero when $r = \mathbf{a}_1 \cdots \mathbf{a}_{n-1} - \sum_{i=1}^{n-1} \mathbf{a}_i - n + 1$; thus for all $r' < r$, $\sigma_{r'}(Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n))$ must also be of the expected dimension (see Exercise 5.3.2(4)). $\square$

7.3.2. The ideal-theoretic GSS conjecture. The precise version of the conjecture mentioned in §3.9.1 was that the ideal of $\sigma_2(Seg(\mathbb{P}A_1\times\cdots\times\mathbb{P}A_n))$ is generated in degree three by the minors of flattenings. This problem was recently solved by C. Raicu [**271**]. An outline of the proof, which uses prolongation, is given in §7.5.6.

7.3.3. Equations for secant varieties of Veronese varieties via symmetric flattenings. Here is a sharper version of Corollary 3.5.1.5.

Corollary 7.3.3.1 (Clebsch [**91**]). *The varieties $\sigma_{\binom{n+2}{2}-1}(v_4(\mathbb{P}^n))$ for $2\leq n\leq 4$ are degenerate because any $\phi\in\sigma_{\binom{n+2}{2}-1}(v_4(\mathbb{P}^n))$ satisfies the equation*

$$\det(\phi_{2,2})=0. \tag{7.3.1}$$

This equation is of degree $\binom{n+2}{2}$ and corresponds to the trivial $SL(V)$-module $S_{2(n+2),\dots,2(n+2)}V\subset S^{\binom{n+2}{2}}(S^4V)$.

Moreover, $\sigma_{\binom{n+2}{2}-1}(v_4(\mathbb{P}^n))$ is a hypersurface, so equation (7.3.1) *generates its ideal.*

The case $\sigma_5(v_4(\mathbb{P}^2))$ is called the *Clebsch hypersurface* of degree 6, defined by the equation obtained by restricting $\Lambda^6(S^2\mathbb{C}^{3*})\otimes\Lambda^6(S^2\mathbb{C}^{3*})$ to $S^6(S^4\mathbb{C}^3)$.

The only assertion above that remains to be proved is that the equation is the generator of the ideal (and not some power of it), which follows, e.g., from Exercise 3.7.2.1.

The following result dates back to the work of Sylvester and Gundelfinger.

Theorem 7.3.3.2. *The ideal of $\sigma_r(v_d(\mathbb{P}^1))$ is generated in degree $r+1$ by the size $r+1$ minors of $\phi_{u,d-u}$ for any $r\leq u\leq d-r$, i.e., by any of the modules $\Lambda^{r+1}S^u\mathbb{C}^2\otimes\Lambda^{r+1}S^{d-u}\mathbb{C}^2$. The variety $\sigma_r(v_d(\mathbb{P}^1))$ is projectively normal and arithmetically Cohen-Macaulay, its singular locus is $\sigma_{r-1}(v_d(\mathbb{P}^1))$, and its degree is $\binom{d-r+1}{r}$.*

Proof. Here is the set-theoretic result, which is apparently due to Sylvester. For the other assertions, see, e.g., [**172**, Thm. 1.56].

Let $V=\mathbb{C}^2$. Consider $\phi_{r,d-r}: S^rV^*\to S^{d-r}V$. The source has dimension $r+1$. Assume that $\operatorname{rank}(\phi_{r,d-r})=r$; we need to show that $\phi\in\sigma_r(v_d(\mathbb{P}V))$. Let $g\in\ker\phi_{r,d-r}$ be a basis of the kernel.

Any homogeneous polynomial on $\mathbb{C}^2$ decomposes into a product of linear factors, so write $g=\ell_1\cdots\ell_r$ with $\ell_j\in V^*$. Since one only needs to prove the result for general ϕ, assume that the ℓ_j are all distinct without loss of generality. Consider the contraction map $g\lrcorner : S^dV\to S^{d-r}V$. I claim its kernel is exactly r-dimensional. To see this, factor it as maps $\ell_1\lrcorner\, S^dV\to S^{d-1}V$, which is clearly surjective, then $\ell_2\lrcorner\, S^{d-1}V\to S^{d-2}V$, etc., showing that $\dim(\ker(g\lrcorner)) \leq r$. On the other hand, letting $v_i \in V$ be such that $\ell_i(v_i) = 0$, $\langle v_1^d, \dots, v_r^d \rangle \subset \ker(g\lrcorner)$, which is an r-dimensional space.

Now $g \cdot V^* \subseteq \ker \phi_{r+1,d-r-1}$, so $\dim(\ker \phi_{r+1,d-r-1}) \geq 2$. Similarly, $g \cdot S^k V^* \subseteq \ker \phi_{r+k,d-r-k}$ implies $\dim(\ker \phi_{r+k,d-r-k}) \geq k+1$, and if equality holds at any one step, it must hold at all previous steps. Now $\phi_{d-r,r}$ is the transpose of $\phi_{r,d-r}$; in particular, they have the same rank. This implies all the inequalities are equalities, showing that the different flattenings in the statement of the theorem give equivalent equations. □

Theorem 7.3.3.3 ([**172**, Thms. 4.10A, 4.5A]). *Let $\mathbf{v} \geq 2$, and let $V = \mathbb{C}^{\mathbf{v}}$. Let $\delta = \lfloor \frac{d}{2} \rfloor$. If $r \leq \binom{\delta+\mathbf{v}}{\mathbf{v}-1}$, then $\sigma_r(v_d(\mathbb{P}V))$ is an irreducible component of the size $(r+1)$ minors of the $(\delta, d-\delta)$-flattening. In other words, $\sigma_r(v_d(\mathbb{P}V))$ is an irreducible component of $Rank^r_{(\delta,d-\delta)}(S^dV)$.*

For example, when $\mathbf{v} = 4$, and say d is even, $\sigma_r(v_d(\mathbb{P}^3))$ is an irreducible component of the zero set of the flattenings for $r \leq \binom{\delta+2}{3}$, the flattenings give some equations up to $r \leq \binom{\delta+3}{3}$, and there are nontrivial equations up to $r \leq \frac{1}{4}\binom{2\delta+3}{3}$.

Proof. Let $[\phi] \in \sigma_r(v_d(\mathbb{P}V))$ be a general point. It will be sufficient to show the inclusion of conormal spaces $\hat{N}^*_{[\phi]} Rank^r_{(a,d-a)}(S^dV) \subseteq \hat{N}^*_{[\phi]}\sigma_r(v_d(\mathbb{P}V))$.

Write $\phi = y_1^d + \cdots + y_r^d$. Recall from Proposition 5.3.3.1 that

$$\hat{N}^*_{[\phi]} Rank^r_{(a,d-a)}(S^dV) = \ker(\phi_{a,d-a}) \circ \operatorname{image}(\phi_{a,d-a})^\perp \subset S^dV^*,$$

and from Proposition 5.3.3.5 that

$$\begin{aligned} \hat{N}^*_{[\phi]}\sigma_r(v_d(\mathbb{P}V)) &= (S^{d-2}V^* \circ S^2 {y_1}^\perp) \cap \cdots \cap (S^{d-2}V^* \circ S^2 {y_r}^\perp) \\ &= S^2 {y_1}^\perp \circ \cdots \circ S^2 {y_r}^\perp \circ S^{d-2r}V^* \\ &= \{P \in S^dV^* \mid \operatorname{Zeros}(P) \text{ is singular at } [y_1], \dots, [y_r]\}. \end{aligned}$$

Now

$$\begin{aligned} \ker(\phi_{a,d-a}) &= \Big(\bigcap {y_j}^\perp\Big) \circ S^{a-1}V^* = {y_1}^\perp \circ \cdots \circ {y_r}^\perp \circ S^{a-r}V^*, \\ (\operatorname{image}(\phi_{a,d-a}))^\perp &= \Big(\bigcap {y_j}^\perp\Big) \circ S^{d-a-1}V^* = {y_1}^\perp \circ \cdots \circ {y_r}^\perp \circ S^{d-a-r}V^*, \end{aligned}$$

and thus

$$\ker(\phi_{a,d-a}) \circ (\operatorname{image}(\phi_{a,d-a}))^\perp = S^2 {y_1}^\perp \circ \cdots \circ S^2 {y_r}^\perp \circ S^{d-2r}V^*,$$

and the result follows. □

7.3.4. Advanced results on symmetric flattenings. Naïvely, one might think that for $\phi \in S^dV$, the flattening $\phi_{\lfloor \frac{d}{2} \rfloor, d-\lfloor \frac{d}{2} \rfloor}$ would give all the information of all the flattenings, but this is not always the case, which motivates the following definition.

Definition 7.3.4.1 ([**215, 172**])**.** Fix a sequence $\vec{r} := (r_1, \dots, r_{\lfloor \frac{d}{2} \rfloor})$ and let

$$\begin{aligned} SFlat_{\vec{r}}(S^dW) :=& \Big\{\phi \in S^dW \mid \operatorname{rank}(\phi_{j,d-j}) \leq r_j,\ j = 1, \dots, \lfloor \frac{d}{2} \rfloor \Big\} \\ =& \Big[\bigcap_{j=1}^{\lfloor \frac{d}{2} \rfloor} \sigma_{r_j}(Seg(\mathbb{P}S^jW \times \mathbb{P}S^{d-j}W)) \Big] \cap S^dW. \end{aligned}$$

Call a sequence $\vec{r}$ *admissible* if there exists $\phi \in S^dW$ such that $\operatorname{rank}(\phi_{j,d-j}) = r_j$ for all $j = 1, \dots, \lfloor \frac{d}{2} \rfloor$. It is sufficient to consider the $\vec{r}$ that are admissible because if $\vec{r}$ is nonadmissible, the zero set of $SFlat_{\vec{r}}$ will be contained in the union of admissible $SFlat$'s associated to smaller $\vec{r}$'s in the natural partial order. Note that $SFlat_{\vec{r}}(S^dW) \subseteq Sub_{r_1}(S^dW)$.

Even if $\vec{r}$ is admissible, it still can be the case that $SFlat_{\vec{r}}(S^dW)$ is reducible. For example, when $\dim W = 3$ and $d \geq 6$, the zero set of the size 5 minors of the $(2, d-2)$-flattening has two irreducible components; one of them is $\sigma_4(v_d(\mathbb{P}^2))$ and the other one has dimension $d + 6$ [**172**, Ex. 3.6].

To remedy this, let $\vec{r}$ be admissible. Define

$$SFlat^0_{\vec{r}}(S^dW) := \Big\{\phi \in S^dW \mid \operatorname{rank}(\phi_{j,d-j}) = r_j,\ j = 1, \dots, \lfloor \frac{d}{2} \rfloor \Big\},$$

and

$$Gor(T(\vec{r})) := \overline{SFlat^0_{\vec{r}}(S^dW)}.$$

Remark 7.3.4.2. The choice of notation comes from the commutative algebra literature (e.g. [**114, 172**]), there $T(\vec{r}) = (1, r_1, \dots, r_{\lfloor \frac{d}{2} \rfloor}, r_{\lfloor \frac{d}{2} \rfloor}, \dots, r_1, 1)$, and "Gor" is short for Gorenstein; see [**172**, Def. 1.11] for a history.

Unfortunately, defining equations for $Gor(T(\vec{r}))$ are not known. One can test for membership of $SFlat^0_{\vec{r}}(S^dW)$ by checking the required vanishing and nonvanishing of minors.

Theorem 7.3.4.3 ([**114**, Thm. 1.1])**.** *If* $\dim W = 3$, *and* $\vec{r}$ *is admissible, then* $Gor(T(\vec{r}))$ *is irreducible.*

Theorem 7.3.4.3 combined with Theorem 7.3.3.3 allows one to extend the set of secant varieties of Veronese varieties defined by flattenings:

Theorem 7.3.4.4 ([**215**])**.** *The following varieties are defined scheme-theoretically by minors of flattenings.*

(1) *Let* $d \geq 4$. *The variety* $\sigma_3(v_d(\mathbb{P}^n))$ *is defined scheme-theoretically by the* 4×4 *minors of the* $(1, d-1)$, *and* $(\lfloor \frac{d}{2} \rfloor, d - \lfloor \frac{d}{2} \rfloor)$*-flattenings.*

(2) *For* $d \geq 4$ *the variety* $\sigma_4(v_d(\mathbb{P}^2))$ *is defined scheme-theoretically by the* 5×5 *minors of the* $(\lfloor \frac{d}{2} \rfloor, d - \lfloor \frac{d}{2} \rfloor)$*-flattenings.*

(3) *For* $d \geq 6$ *the variety* $\sigma_5(v_d(\mathbb{P}^2))$ *is defined scheme-theoretically by the* 6×6 *minors of the* $(\lfloor \frac{d}{2} \rfloor, d - \lfloor \frac{d}{2} \rfloor)$*-flattenings.*

(4) *Let* $d \geq 6$. *The variety* $\sigma_6(v_d(\mathbb{P}^2))$ *is defined scheme-theoretically by the* 7×7 *minors of the* $(\lfloor \frac{d}{2} \rfloor, d - \lfloor \frac{d}{2} \rfloor)$*-flattenings.*

For the proof, see [**215**]. The idea is simply to show that the scheme defined by the flattenings in the hypotheses coincides with the scheme of some $Gor(T(\vec{r}))$ with $\vec{r}$ admissible, and then Theorem 7.3.3.3 combined with Theorem 7.3.4.3 implies the result.

Remark 7.3.4.5. For $\sigma_4(v_6(\mathbb{P}^4))$ the $(2,4)$-flattenings are not sufficient: $\sigma_4(v_6(\mathbb{P}^2))$ is just one of the two components given by the size 5 minors of $\phi_{2,4}$; see [**172**, Ex. 3.6]. Case (2) when $d = 4$ was proved in [**283**, Thm. 2.3].

7.4. Inheritance

Inheritance is a general technique for studying equations of G-varieties that come in series. In this section I explain it in detail for secant varieties of Segre and Veronese varieties.

Recall from §6.4 the notation $S_{\overline{\pi}}V$ for a specific realization of $S_\pi V$ in $V^{\otimes|\pi|}$. If $V \subset W$, then $S_{\overline{\pi}}V$ induces a module $S_{\overline{\pi}}W$.

7.4.1. Inheritance for secant varieties of Segre varieties.

Proposition 7.4.1.1 ([**205**, Prop. 4.4]). *For all vector spaces* B_j *with* $\dim B_j = \mathbf{b}_j \geq \dim A_j = \mathbf{a}_j \geq r$, *a module* $S_{\overline{\mu}_1}B_1 \otimes \cdots \otimes S_{\overline{\mu}_n}B_n$ *such that* $\ell(\mu_j) \leq \mathbf{a}_j$ *for all* j, *is in* $I_d(\sigma_r(Seg(\mathbb{P}B_1^* \times \cdots \times \mathbb{P}B_n^*)))$ *if and only if* $S_{\overline{\mu}_1}A_1 \otimes \cdots \otimes S_{\overline{\mu}_n}A_n$ *is in* $I_d(\sigma_r(Seg(\mathbb{P}A_1^* \times \cdots \times \mathbb{P}A_n^*)))$.

Proof. Let $Z \subset \mathbb{P}V$ be a G-variety. Recall (i) that a G-module $M \subset S^dV^*$ is in $I_d(Z)$ if and only if its highest weight vector is and (ii) that the $GL(V)$-highest weight vector of any realization of $S_\pi V$ is the image of $v_1^{\otimes p_1} \otimes \cdots \otimes v_{\mathbf{v}}^{\otimes p_{\mathbf{v}}}$ under a $GL(V)$-module projection map.

Write

$$G_B = \prod_j GL(B_j),\ G_A = \prod_j GL(A_j),$$
$$\sigma_A = \sigma_r(Seg(\mathbb{P}A_1^* \times \cdots \times \mathbb{P}A_n^*)),\ \sigma_B = \sigma_r(Seg(\mathbb{P}B_1^* \times \cdots \times \mathbb{P}B_n^*)),$$
$$M_A = S_{\overline{\mu}_1}A_1 \otimes \cdots \otimes S_{\overline{\mu}_n}A_n,\ M_B = S_{\overline{\mu}_1}B_1 \otimes \cdots \otimes S_{\overline{\mu}_n}B_n.$$

Choose inclusions $A_j \subset B_j$ and ordered bases for B_j such that the first $\mathbf{a}_j$ basis vectors form a basis of A_j. The inclusions of vector spaces imply:

- $\sigma_A \subset \sigma_B$.
- $M_B = G_B \cdot M_A$.
- $\sigma_B = G_B \cdot \sigma_A$.
- A highest weight vector for M_B is also a highest weight vector for M_A as long as $\ell(\mu_j) \leq \mathbf{a}_j$, $1 \leq j \leq n$.

It remains to show that $P \in M_A$ and $P \in I_{\sigma_A}$ imply $G_B \cdot P \subset I_{\sigma_B}$.

If a homogeneous polynomial P vanishes on σ_B, it vanishes on the subvariety σ_A, so $M_B \subset I_{\sigma_B}$ implies $M_A \subset I_{\sigma_A}$.

Say $P \in I_{\sigma_A}$. Since $r \leq \mathbf{a}_j$, $\sigma_B \subseteq Sub_{r,\dots,r}(B_1^* \otimes \cdots \otimes B_n^*)$, if $x \in \sigma_B$, there exists $g \in G_B$ such that $g \cdot x \in \sigma_A$. In fact one can choose $g = (g_1, \dots, g_n)$ such that $g_i \in U_i \subset GL(B_i)$, where U_i is the subgroup that is the identity on A_i and preserves A_i. In terms of matrices,

$$U_i = \begin{pmatrix} Id_{\mathbf{a}_i} & * \\ 0 & * \end{pmatrix}.$$

Note that $h \cdot P = P$ for all $h \in U_1 \times \cdots \times U_n$. Now given $x \in \sigma_B$,

$$\begin{aligned} P(x) &= P(g^{-1} \cdot g \cdot x) \\ &= (g^{-1} \cdot P)(g \cdot x) \\ &= P(g \cdot x) \\ &= 0. \end{aligned}$$

Similarly $P(g \cdot x) \neq 0$ implies $P(x) \neq 0$. □

Thus if $\dim A_j = r$ and $\ell(\mu_j) \leq r$, then

$$S_{\overline{\mu}_1} A_1 \otimes \cdots \otimes S_{\overline{\mu}_1} A_n \in I(\sigma_r(Seg(\mathbb{P}A_1^* \times \cdots \times \mathbb{P}A_n^*)))$$

if and only if

$$S_{\overline{\mu}_1} \mathbb{C}^{\ell(\mu_1)} \otimes \cdots \otimes S_{\overline{\mu}_1} \mathbb{C}^{\ell(\mu_n)} \in I(\sigma_r(Seg(\mathbb{P}^{\ell(\mu_1)-1} \times \cdots \times \mathbb{P}^{\ell(\mu_n)-1}))).$$

In summary:

Corollary 7.4.1.2 (**[205, 8]**). *Let $\dim A_j \geq r$, $1 \leq j \leq n$. The ideal of $\sigma_r(Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n))$ is generated by the modules inherited from the ideal of $\sigma_r(Seg(\mathbb{P}^{r-1} \times \cdots \times \mathbb{P}^{r-1}))$ and the modules generating the ideal of $Sub_{r,\dots,r}$. The analogous scheme and set-theoretic results hold as well.*

See Chapter 16 for generalizations of inheritance to "series" of rational homogeneous varieties.

7.4.2. Inheritance for secant varieties of Veronese varieties. The same arguments as above show:

Proposition 7.4.2.1. *Let $r, \ell(\pi) \leq \dim V \leq \dim W$. A module $S_{\overline{\pi}}W \subset I_\delta(\sigma_r(v_d(\mathbb{P}W^*))$ if and only if $S_{\overline{\pi}}V \subset I_\delta(\sigma_r(v_d(\mathbb{P}V^*))$.*

Exercise 7.4.2.2: Write out the proof.

Corollary 7.4.2.3. *If $\mathbf{v} = \dim V > r$, then $I(\sigma_r(v_d(\mathbb{P}V^*)))$ is generated by the modules inherited from the ideal of $\sigma_r(v_d(\mathbb{P}^{r-1}))$ and the modules generating the ideal of $Sub_r(S^dV^*)$. Set-theoretic (resp. scheme-theoretic, resp. ideal-theoretic) defining equations for $\sigma_r(v_d(\mathbb{P}V^*))$ may be obtained from the set-theoretic (resp. scheme-theoretic, resp. ideal-theoretic) defining equations for $\sigma_r(v_d(\mathbb{P}^{r-1}))$ and the modules generating the ideal of $Sub_r(S^dV^*)$.*

Corollary 7.4.2.4 (Kanev [**184**]). *The ideal of $\sigma_2(v_d(\mathbb{P}^{\mathbf{v}-1})) = \sigma_2(v_d(\mathbb{P}V^*))$ is generated in degree three by the 3×3 minors of $\phi_{1,d-1}$ and $\phi_{2,d-2}$, i.e., by the modules $\Lambda^3 V \otimes \Lambda^3(S^{d-1}V)$ and $\Lambda^3(S^2V) \otimes \Lambda^3(S^{d-2}V)$.*

A. Geramita [**142**] conjectured that the minors of $\phi_{1,d-1}$ are redundant in the result above, which was recently proven by C. Raicu:

Theorem 7.4.2.5 ([**270**]). *The ideal of $\sigma_2(v_d(\mathbb{P}^{\mathbf{v}-1})) = \sigma_2(v_d(\mathbb{P}V^*))$ is generated in degree three by the 3×3 minors of $\phi_{s,d-s}$ for any $2 \leq s \leq d-2$.*

For $d \geq 4$, the 3×3 minors of $\phi_{1,d-1}$ form a proper submodule of these equations.

The proof is similar in spirit to his proof of the GSS conjecture discussed below. Namely, Proposition 7.5.5.1 is applied to the weight zero subspace, where one can take advantage of the Weyl group $\mathfrak{S}_{\mathbf{v}}$ action.

7.5. Prolongation and multiprolongation

Here is a more systematic and general study of prolongation, which was introduced for secant varieties of Segre and Veronese varieties in §3.7.2.

For clarity of exposition in this section, I write $P \in S^dV^*$ when it is viewed as a polynomial, and the same element is written $\overline{P}$ when regarded as a d-linear form on V.

7.5.1. Prolongation and secant varieties. Consider a variety $X \subset \mathbb{P}V$, not contained in a hyperplane. Recall that Proposition 3.7.2.1 showed that there are no quadrics in the ideal of $\sigma_2(X)$.

Say $P \in S^3V^*$, $P \in I_3(\sigma_2(X))$ if and only if $P(x+ty) = 0$ for all $x, y \in \hat{X}$, and $t \in \mathbb{C}$. Now

$$\begin{aligned} P(x+ty) &= \overline{P}(x+ty, x+ty, x+ty) \\ &= \overline{P}(x,x,x) + 3t\overline{P}(x,x,y) + 3t^2\overline{P}(x,y,y) + t^3\overline{P}(y,y,y). \end{aligned}$$

Each of these terms must vanish. Thus $P \in I_3(\sigma_2(X))$ implies that $P \in I_3(X)$ (which we knew already) and that $P(x,\cdot,\cdot) = \frac{\partial P}{\partial x} \in I_2(X)$ for all $x \in \hat{X}$. Since X is not contained in a hyperplane, one can obtain a basis of V from elements of $\hat{X}$, so one concludes by linearity that:

$P \in S^3V^*$ is in $I_3(\sigma_2(X))$ if and only if $\frac{\partial P}{\partial v} \in I_2(X)$ for all $v \in V$.

Note that this implies $I_3(\sigma_2(X)) = (I_2(X) \otimes V^*) \cap S^3V^*$.

Example 7.5.1.1. Let $X = Seg(\mathbb{P}A^* \times \mathbb{P}B^*)$. Choosing bases and thinking of $I_2(X) = \Lambda^2 A \otimes \Lambda^2 B$ as the set of two by two minors of an $\mathbf{a} \times \mathbf{b}$ matrix with variable entries, we see that all the partial derivatives of a three by three minor are linear combinations of two by two minors, i.e., $\Lambda^3 A \otimes \Lambda^3 B \subseteq (\Lambda^2 A \otimes \Lambda^2 B)^{(1)}$.

The same argument shows:

Proposition 7.5.1.2 (Multiprolongation)**.** *Given a variety $X \subset \mathbb{P}V$, a polynomial $P \in S^dV^*$ is in $I_d(\sigma_r(X))$ if and only if for any sequence of nonnegative integers $m_1, \dots, m_r$, with $m_1 + \dots + m_r = d$,*

$$\overline{P}(v_1, \dots, v_1, v_2, \dots, v_2, \dots, v_r, \dots, v_r) = 0 \tag{7.5.1}$$

for all $v_i \in \hat{X}$, where the number of v_j's appearing in the formula is m_j. In particular, $I_d(\sigma_r(X)) = 0$ for $d \leq r$ unless X is contained in a hyperplane.

Corollary 7.5.1.3. *Given $X \subset \mathbb{P}V^*$, $I_{r+1}(\sigma_r(X)) = (I_2(X) \otimes S^{r-1}V) \cap S^{r+1}V$.*

Exercise 7.5.1.4: Prove Corollary 7.5.1.3. ⊚

Definition 7.5.1.5. For $A \subset S^dV$ define

$$A^{(p)} = (A \otimes S^pV) \cap S^{p+d}V,$$

the *p-th prolongation of A*.

In coordinates, this translates to:

$$A^{(p)} = \Big\{ f \in S^{d+p}V^* \mid \frac{\partial^p f}{\partial x^\beta} \in A \ \forall \beta \in \mathbb{N}^{\mathbf{v}} \text{ with } |\beta| = p \Big\}. \tag{7.5.2}$$

A more algebraic perspective on multiprolongation was given by Raicu: Let $X \subset \mathbb{P}V$ be a variety. Consider the addition map $V \times \dots \times V \to V$, $(v_1, \dots, v_r) \mapsto v_1 + \dots + v_r$. Restrict the left hand side to $\hat{X} \times \dots \times \hat{X}$;

the image is $\hat{\sigma}_r(X)$. By polarizing and quotienting (recall that $\mathbb{C}[\hat{X}]_m = S^m V^*/I_m(X)$), we obtain a map

$$S^{\bullet}V^* \to \mathbb{C}[\hat{X}] \otimes \cdots \otimes \mathbb{C}[\hat{X}]$$

whose image is $\mathbb{C}[\sigma_r(X)]$ and kernel is $I(\sigma_r(X))$. The map respects degree, so one obtains a map

$$\begin{aligned} S^d V^* &\to \sum_{\substack{m_1 \leq \cdots \leq m_r \\ m_1 + \cdots + m_r = d}} \mathbb{C}[\hat{X}]_{m_1} \otimes \cdots \otimes \mathbb{C}[\hat{X}]_{m_r} \\ P &\mapsto \overline{P}_{m_1,\dots,m_r}|_{\hat{X}_1 \times \cdots \times \hat{X}_r} \end{aligned}$$

whose kernel is $I_d(\sigma_r(X))$, rephrasing Proposition 7.5.1.2.

Remark 7.5.1.6. Prolongations are essential to the area of *exterior differential systems* (see [**180, 43**]). They are used to determine the space of local solutions to systems of partial differential equations by checking the compatibility of differential equations under differentiation (i.e., checking that mixed partial derivatives commute). There one works with $A \subset S^p V^* \otimes W$ and the prolongation is $A^{(k)} = (A \otimes S^k V^*) \cap S^{p+k} V^* \otimes W$.

7.5.2. Exercises on prolongation.

(1) Determine $I_4(\sigma_2(X))$ in terms of $I(X)$.

(2) Let X consist of four points in $\mathbb{P}^2$ with no three colinear. Determine a generator of the ideal of $\sigma_2(X)$.

(3) Show that a general complete intersection (cf. §5.1.3) of quadrics will not have any cubics in the ideal of its secant variety.

(4) Show that (7.5.2) indeed agrees with Definition 7.5.1.5.

(5) Show that for $A \subset S^d V^*$,

$$A^{(p)} = \{ f \in S^{d+p} V^* \mid f_{p,d}(S^p V) \subset A \},$$

where $f_{p,d} : S^p V \to S^d V^*$ is the polarization of f.

(6) Show that $A^{(1)} = \ker \delta|_{A \otimes V^*}$, where $\delta : S^d V^* \otimes V^* \to S^{d-1} V^* \otimes \Lambda^2 V^*$ is the map $p \otimes \alpha \mapsto dp \wedge \alpha$ with $dp \in S^{d-1} V^* \otimes V^*$ the exterior derivative, or if the reader prefers, $dp = p_{d-1,1}$. (In linear coordinates x^i on V, $dp = \sum_d \frac{\partial p}{\partial x^i} \otimes dx^i$, identifying dx^i as an element of V^*.)

(7) What is the image of $\delta : S^d V^* \otimes V^* \to S^{d-1} V^* \otimes \Lambda^2 V^*$ as a $GL(V)$-module?

Example 7.5.2.1 (Example 7.5.1.1 continued)**.** One often uses prolongations combined with representation theory. As modules,

$$\begin{aligned}(\Lambda^2 A\otimes\Lambda^2 B)&\otimes(A\otimes B)\\ &= (\Lambda^3 A\otimes\Lambda^3 B)\oplus(S_{21}A\otimes S_{21}B)\oplus(S_{21}A\otimes\Lambda^3 B)\\ &\quad\oplus(\Lambda^3 A\otimes S_{21}B)\oplus(\Lambda^3 A\otimes S_{21}B).\end{aligned}$$

Only the first two occur in $S^3(A\otimes B)$, but the module description does not tell us if the copy of $(S_{21}A\otimes S_{21}B)$ that arises is the copy that lies in $S^3(A\otimes B)$. Examining the map δ of Exercise 7.5.2(6), one sees that the copy of $(S_{21}A\otimes S_{21}B)$ does not lie in $S^3(A\otimes B)$ and the prolongation is exactly $\Lambda^3 A\otimes\Lambda^3 B$.

7.5.3. Ideals of secant varieties via prolongation. The following result apparently first appeared in [**293**] (with a significantly longer proof):

Proposition 7.5.3.1. *Let $X, Y\subset \mathbb{P}V$ be subvarieties and assume that $I_\delta(X)=0$ for $\delta<d_1$ and $I_\delta(Y)=0$ for $\delta<d_2$. Then $I_\delta(J(X,Y))=0$ for $\delta\leq d_1+d_2-2$.*

Proof. Say $P\in I_\delta(J(X,Y))$; then $\overline{P}(x^s,y^{\delta-s})=0$ for all $x\in\hat{X}$, $y\in\hat{Y}$, and all $0\leq s\leq\delta$. If $I_{\delta-s}(Y)=0$, then there exists $y\in\hat{Y}$ such that $P(\cdot,y^{\delta-s})\in S^sV^*$ is not identically zero, and thus $P(x^s,y^{\delta-s})=0$ for all $x\in\hat{X}$, i.e., $P(\cdot,y^{\delta-s})\in I_s(X)$. Now if $\delta=(d_1-1)+(d_2-1)$, taking $s=d_1-1$, we would either obtain a nonzero element of $I_{d_1-1}(X)$ or of $I_{d_2-1}(Y)$, a contradiction. $\square$

Corollary 7.5.3.2. *Let $X_1,\dots,X_r\subset\mathbb{P}V$ be varieties such that $I_\delta(X_j)=0$ for $\delta<d_j$. Then $I_\delta(J(X_1,\dots,X_r))=0$ for $\delta\leq d_1+\cdots+d_r-r$.*

In particular, if $X\subset\mathbb{P}V$ is a variety with $I_\delta(X)=0$ for $\delta<d$, then $I_\delta(\sigma_r(X))=0$ for $\delta\leq r(d-1)$.

Exercise 7.5.3.3: Prove Corollary 7.5.3.2.

The following theorem generalizes and makes explicit results of [**204**]:

Theorem 7.5.3.4 ([**291**])**.** *Suppose $X\subset\mathbb{P}V$ is a variety with $I_{d-1}(X)=0$. Then $I_d(X)^{((r-1)(d-1))}=I_{r(d-1)+1}(\sigma_r(X))$, and $I_{r(d-1)}(\sigma_r(X))=0$.*

The second assertion has already been proven.

Exercise 7.5.3.5: Prove that $I_d(X)^{((r-1)(d-1))}\subseteq I_{r(d-1)+1}(\sigma_r(X))$.

For the proof of equality, see [**291**].

7.5.4. Prolongation and cones. Let $Z=J(X,L)\subset\mathbb{P}V$ be a cone, where $X\subset\mathbb{P}V$ is a variety and $L\subset\mathbb{P}V$ is a linear space.

Exercise 7.5.4.1: Show that $I_d(Z)=I_d(X)\cap S^d\hat{L}^\perp$.

Here the situation is similar to the subspace varieties in the exterior algebra: assuming one has generators of the ideal of X, one easily determines the ideal of Z in all degrees, but how to determine generators of the ideal of Z?

Example 7.5.4.2. A case that arises in applications is $X = \sigma_k(v_2(\mathbb{P}A))$, $\mathbf{a} = \dim A$, and $\hat{L} \subset S^2A$ is $\mathbf{a}$-dimensional and diagonalizable. More precisely, one begins in coordinates with $\hat{L}$ the set of diagonal matrices. The object that arises in applications is called the *set of covariance matrices for the Gaussian k-factor model with* $\mathbf{a}$ *observed variables*. It is defined in affine space over $\mathbb{R}$, and one adds the requirement that the entries of $\hat{L}$ are positive; see [**117**].

I continue nevertheless to work over $\mathbb{C}$. Let $F_{r,\mathbf{a}} = J(\sigma_r(v_2(\mathbb{P}A)), \mathbb{P}D)$, where $D \subset S^2A$ is a choice of diagonal subspace. Note that $F_{r,\mathbf{a}}$ is a $\mathfrak{S}_{\mathbf{a}}$-variety. In [**117**] it is shown that the ideal of $F_{1,\mathbf{a}}$ is generated in degree two, and that the ideal of $F_{2,5}$ is generated in degree five. ($F_{2,5}$ is a hypersurface.) Then in [**42**] the authors show that for $\mathbf{a} > 5$ the ideal of $F_{2,\mathbf{a}}$ is generated in degrees 3 and 5 by the 3×3 minors that do not involve diagonal elements and the equations inherited from the equation of $F_{2,5}$.

At this writing it is an open problem to find generators of the ideal of $F_{r,\mathbf{a}}$ in the general case.

7.5.5. Prolongation and secant varieties of homogeneous varieties. For homogeneous varieties $X = G/P \subset \mathbb{P}V_\lambda^*$ the above results can be sharpened. For a G-module $W = W_\lambda$, let $W^j = W_{j\lambda}$ denote the j-th Cartan power of W (see §6.10.5). When $G = GL(A_1) \times \cdots \times GL(A_n)$ and $W = A_1 \otimes \cdots \otimes A_n$, then $W^j = S^jA_1 \otimes \cdots \otimes S^jA_n$. When $G = GL(V)$ and $W = S^dV$, then $W^j = S^{jd}V$.

Proposition 7.5.5.1 ([**205**]). *Let $X = G/P \subset \mathbb{P}V^*$ be a homogeneously embedded rational homogenous variety. Then $I_d(\sigma_r(X))$ is the kernel of the G-module map (here $a_j \geq 0$)*

$$S^dV^* \to \bigoplus_{\substack{a_1+2a_2+\cdots+ra_r=d \\ a_1+\cdots+a_r=r}} S^{a_1}(V)^* \otimes S^{a_2}(V^2)^* \otimes \cdots \otimes S^{a_r}(V^r)^*. \tag{7.5.3}$$

Corollary 7.5.5.2 ([**205**]). *Let $X = G/P \subset \mathbb{P}V^*$ be a rational homogeneous variety. Then for all $d > 0$,*

(1) $I_d(\sigma_d(X)) = 0$;

(2) $I_{d+1}(\sigma_d(X))$ *is the complement of the image of the contraction map* $(V^2)^* \otimes S^{d+1}V \to S^{d-1}V$;

(3) *let W be an irreducible component of S^dV, and suppose that for all integers $(a_1, \ldots, a_p)$ such that $a_1 + 2a_2 + \cdots + pa_p = d$*

and $a_1 + \cdots + a_p = k$*,* W *is not an irreducible component of* $S^{a_1}(V)\otimes S^{a_2}(V^2)\otimes\cdots\otimes S^{a_p}(V^p)$*. Then* $W \subset I_d(\sigma_k(X))$*.*

Proof. (1) is Proposition 3.7.2.1. (2) follows from the remarks about the ideals of homogeneous varieties and Theorem 7.5.3.4. (3) follows from (7.5.3) and Schur's lemma. □

Aside 7.5.5.3. One can construct the *cominuscule varieties*, which include the Grassmannians, Lagrangian Grassmannians, and spinor varieties via prolongation. The maps in §13.8 can be interpreted as mappings given by the polynomials $I_2(Y), I_3(\sigma_2(Y)), I_4(\sigma_3(Y)), \ldots$, where Y is a smaller homogeneous variety. For example, to construct the Grassmannian $G(\mathbf{a}, \mathbf{a}+\mathbf{b})$, one takes $Y = Seg(\mathbb{P}^{\mathbf{a}-1} \times \mathbb{P}^{\mathbf{b}-1})$, and the ideals are the various minors. In particular, one can construct the Cayley plane E_6/P_1 and the E_7-variety E_7/P_7 in this manner, where in the last case the ideals correspond to the "minors" of the 3×3 octonionic Hermitian matrices; see [**206**].

Here is an example illustrating Corollary 7.5.5.2:

Proposition 7.5.5.4 ([**205**])**.** *Let* $X = Seg(\mathbb{P}A^* \times \mathbb{P}B^* \times \mathbb{P}C^*)$*, with* $\mathbf{a}, \mathbf{b}, \mathbf{c} \geq 3$*. The space of quartic equations of* $\sigma_3(X)$ *is*

$$I_4(\sigma_3(X)) = S_{211}S_{211}S_{211} \oplus S_4S_{1111}S_{1111} \oplus S_{31}S_{211}S_{1111} \oplus S_{22}S_{22}S_{1111}.$$

Here $S_\lambda S_\mu S_\nu$ *is to be read as* $S_\lambda A\otimes S_\mu B\otimes S_\nu C$ *plus permutations giving rise to distinct modules.*

If $X = Seg(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2)$*, only the first module consists of nonzero and this space has dimension* 27*.*

Remark 7.5.5.5. In Proposition 7.5.5.4 the first module consists of Strassen's equations and all the others come from flattenings.

Proof. Recall the decomposition of Proposition 6.6.2.8:

$$\begin{aligned} S^4(A\otimes B\otimes C) = {} & S_4S_4S_4 \oplus S_4S_{31}S_{31} \oplus S_4S_{22}S_{22} \\ & \oplus S_4S_{211}S_{211} \oplus S_4S_{1111}S_{1111} \oplus S_{31}S_{31}S_{31} \oplus S_{31}S_{31}S_{22} \\ & \oplus S_{31}S_{31}S_{211} \oplus S_{31}S_{22}S_{211} \oplus S_{31}S_{211}S_{211} \oplus S_{31}S_{211}S_{1111} \\ & \oplus S_{22}S_{22}S_{22} \oplus S_{22}S_{22}S_{1111} \oplus S_{22}S_{211}S_{211} \oplus S_{211}S_{211}S_{211}. \end{aligned}$$

By inheritance (Proposition 7.4.1.1) any module containing a partition of length greater than three is in the ideal, and it remains to study the case $\mathbf{a}, \mathbf{b}, \mathbf{c} = 3$. But also by inheritance, since $\sigma_3(Seg(\mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^2)) = \mathbb{P}(\mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3)$ by Exercise 5.3.2(2), we may assume that $\ell(\pi_j) \geq 3$ for each j.

Thus it remains to consider the terms $S_{\pi_1}A\otimes S_{\pi_2}B\otimes S_{\pi_3}C$ with each $\ell(\pi_j)=3$. Examining the decomposition of $S^4(A\otimes B\otimes C)$, the only possible term is $S_{211}A\otimes S_{211}B\otimes S_{211}C$, which occurs with multiplicity one.

Now apply Corollary 7.5.5.2. That is, check that $W = S_{211}A\otimes S_{211}B\otimes S_{211}C$ is not contained in $V^2\otimes S^2V$, where $V = A\otimes B\otimes C$ and $V^2 = S^2A\otimes S^2B\otimes S^2C$. Each term in S^2V must have at least one symmetric power, say S^2A. Tensoring by the other S^2A, coming from V^2, cannot yield the $S_{211}A$ term of W, e.g., by the Pieri formula of Proposition 6.7.2.1. □

Thus Strassen's equations in §3.8.2 for $\sigma_3(Seg(\mathbb{P}^2\times\mathbb{P}^2\times\mathbb{P}^2))$ must form a subspace of the module of equations $S_{211}A\otimes S_{211}B\otimes S_{211}C$. In §7.6 I present Strassen's equations from an invariant perspective to show that they span the module.

7.5.6. Raicu's proof of the GSS conjecture. Let $X = G/P \subset \mathbb{P}V_\lambda^*$ be homogeneous. Say, having fixed d, that the $S^{a_i}V_{i\lambda} = S^{a_i}V^i$ appearing in Proposition 7.5.5.1 all have nonzero weight zero spaces. Then one can study which modules are in the image and kernel of the map (7.5.3) by restricting to weight zero subspaces and considering the corresponding maps of $\mathcal{W}_G$-modules, where $\mathcal{W}_G$ is the Weyl group of G. In the case of $Seg(\mathbb{P}A_1^*\times\cdots\times\mathbb{P}A_n^*)$, this group is the product of permutation groups, $\mathfrak{S}_{\mathbf{a}_1}\times\cdots\times\mathfrak{S}_{\mathbf{a}_n}$. For the GSS conjecture, it is sufficient to consider the modules $\lambda=(\pi_1,\dots,\pi_n)$, where each π_j is a partition of d with at most two parts. Now were one to consider only the case $\mathbf{a}_j = 2$ for all j, there would be no weight zero subspace, so Raicu uses inheritance in reverse, requiring $\mathbf{a}_j = d$ to insure a nonzero weight zero subspace in S^dV. Since the coordinate ring is much smaller than the ideal asymptotically, he compares the coordinate ring of the variety whose ideal is generated by the flattenings and the coordinate ring of $\sigma_2(Seg(\mathbb{P}A_1^*\times\cdots\times\mathbb{P}A_n^*))$ and shows they are equal. The proof is combinatorial in nature and is aided by the use of auxiliary graphs he associates to each λ.

7.6. Strassen's equations, applications and generalizations

In §3.8.2, a geometric method of finding equations for $\sigma_3(Seg(\mathbb{P}A\times\mathbb{P}B\times\mathbb{P}C))$ was given. The method depended on making choices of elements in A^*, which are eliminated below.

7.6.1. Proof of Theorem 3.8.2.4. Recall Theorem 3.8.2.4:

Theorem 7.6.1.1 ([**300**]). *Let $\mathbf{a}=3$, $\mathbf{b}=\mathbf{c}\geq 3$, and let $T\in A\otimes B\otimes C$ be concise. Let $\alpha\in A^*$ be such that $T_\alpha := T(\alpha): C^*\to B$ is invertible. For $\alpha_j\in A^*$, write $T_{\alpha,\alpha_j} := T_\alpha{}^{-1}T_{\alpha_j}: B\to B$. Then for all $\alpha_1,\alpha_2\in A^*$,*

$$\operatorname{rank}[T_{\alpha,\alpha_1}, T_{\alpha,\alpha_2}] \leq 2(\underline{\mathbf{R}}(T)-\mathbf{b}).$$

Proof. (Following [**207**].) Fix an auxiliary vector space $D \simeq \mathbb{C}^r$ and write $T_\alpha : C^* \to B$ as a composition of maps

$$C^* \xrightarrow{i} D \xrightarrow{\delta_\alpha} D \xrightarrow{p} B.$$

To see this explicitly, write $T_\alpha = \sum_{j=1}^{\mathbf{b}} b_j \otimes c_j$. Assume that $b_1, \dots, b_{\mathbf{b}}$, $c_1, \dots, c_{\mathbf{b}}$ are bases of B, C. Then letting $d_1, \dots, d_r$ be a basis of D,

$$\begin{aligned} i(\eta) &= \sum_{j=1}^{r} \eta(c_j) d_j, \\ \delta_\alpha(d_j) &= \alpha(a_j) d_j, \\ p(d_s) &= b_s, \quad 1 \le s \le \mathbf{b}, \end{aligned}$$

and for $\mathbf{b}+1 \le x \le r$, writing $b_x = \xi_x^s b_s$, one has $p(d_x) = \xi_x^s b_s$. Let $D' = \langle d_1, \dots, d_{\mathbf{b}} \rangle$. By rescaling our vectors, we may assume that $\delta_\alpha|_{D'} = \mathrm{Id}$.

Write $i' : C^* \to D'$ and set $p_\alpha := p|_{D'}$, so $p_\alpha : D' \to B$ is a linear isomorphism. Then ${T_\alpha}^{-1} = (i')^{-1} {p_\alpha}^{-1}$.

Note that $\operatorname{rank}[T_{\alpha,\alpha_1}, T_{\alpha,\alpha_2}] = \operatorname{rank}(T_{\alpha_1} {T_\alpha}^{-1} T_{\alpha_2} - T_{\alpha_2} {T_\alpha}^{-1} T_{\alpha_1})$ because T_α is invertible. We have

$$\begin{aligned} &T_{\alpha_1} {T_\alpha}^{-1} T_{\alpha_2} - T_{\alpha_2} {T_\alpha}^{-1} T_{\alpha_1} \\ &\quad = (p\delta_{\alpha_1} i')((i')^{-1-1} {p_\alpha}^{-1})(p\delta_{\alpha_2} i') - (p\delta_{\alpha_2} i')((i')^{-1} {p_\alpha}^{-1})(p\delta_{\alpha_1} i') \\ &\quad = p[\delta_{\alpha_1} {p_\alpha}^{-1} p \delta_{\alpha_2} - \delta_{\alpha_2} {p_\alpha}^{-1} p \delta_{\alpha_1}] i'. \end{aligned}$$

Now ${p_\alpha}^{-1} p\, |_{D'} = \mathrm{Id}$, so write $D = D' \oplus D''$, where D'' is any complement to D' in D. Thus $\dim D'' = r - \mathbf{b}$ and ${p_\alpha}^{-1} p = \mathrm{Id}_{D'} + f$ for some map $f : D'' \to D$ of rank at most $r - \mathbf{b}$. Thus

$$T_{\alpha_1} {T_\alpha}^{-1} T_{\alpha_2} - T_{\alpha_2} {T_\alpha}^{-1} T_{\alpha_1} = p(\delta_{\alpha_1} f \delta_{\alpha_2} - \delta_{\alpha_2} f \delta_{\alpha_1}) i'$$

and is therefore of rank at most $2(r - \mathbf{b})$. □

7.6.2. Invariant description of Strassen's equations. I follow [**207**], eliminating the choices of $\alpha, \alpha^1, \alpha^2$. Tensors will replace endomorphisms, composition of endomorphisms will correspond to contractions of tensors, and the commutator of two endomorphisms will correspond to contracting a tensor in two different ways and taking the difference of the two results.

The punch line is:

Strassen's polynomials of degree $\mathbf{b}+1$ *are obtained by the composition of the inclusion*

$$\Lambda^2 A \otimes S^{\mathbf{b}-1} A \otimes \Lambda^{\mathbf{b}} B \otimes B \otimes \Lambda^{\mathbf{b}} C \otimes C \to (A \otimes B \otimes C)^{\otimes \mathbf{b}+1}$$

with the projection

$$(A \otimes B \otimes C)^{\otimes \mathbf{b}+1} \to S^{\mathbf{b}+1}(A \otimes B \otimes C).$$

Here are the details: Strassen's commutator is

$$T(\alpha^1)T(\alpha)^{-1}T(\alpha^2)T(\alpha)^{-1} - T(\alpha^2)T(\alpha)^{-1}T(\alpha^1)T(\alpha)^{-1},$$

and we have already seen that $T(\alpha)^{-1}$ can be removed from the right hand side of each term, and to avoid taking inverses, one can work with $T(\alpha)^{\wedge(\mathbf{b}-1)}$ instead of $T(\alpha)^{-1}$.

Given $T \in A{\otimes}B{\otimes}C$ and $\alpha \in A^*$, write $T_\alpha \in B{\otimes}C$. Consider $T_\alpha^{\wedge \mathbf{b}-1} \in \Lambda^{\mathbf{b}-1}B{\otimes}\Lambda^{\mathbf{b}-1}C = \Lambda^{\mathbf{b}-1}B{\otimes}C^*{\otimes}\Lambda^{\mathbf{b}}C$ and

$$T_\alpha^{\wedge \mathbf{b}-1}{\otimes}T_{\alpha^1}{\otimes}T_{\alpha^2} \in \Lambda^{\mathbf{b}-1}B{\otimes}C^*{\otimes}\Lambda^{\mathbf{b}}C{\otimes}(B{\otimes}C){\otimes}(B{\otimes}C).$$

Contract the $\Lambda^{\mathbf{b}-1}B$ with the first B and the C^* with the second C to get an element of $\Lambda^{\mathbf{b}}B{\otimes}C{\otimes}B$, then contract the $\Lambda^{\mathbf{b}-1}B$ with the second B and the C^* with the first C to get a second element of $\Lambda^{\mathbf{b}}B{\otimes}C{\otimes}B$. Take their difference. Strassen's theorem is that the resulting endomorphism $C^* \to B{\otimes}\Lambda^{\mathbf{b}}B$ has rank at most $2(\underline{\mathbf{R}}(T) - \mathbf{b})$.

To eliminate the choices of $\alpha, \alpha^1, \alpha^2$, consider the tensor $T_\alpha^{\wedge \mathbf{b}-1}$ without having chosen α as $T_{(\cdot)}^{\wedge \mathbf{b}-1} \in S^{\mathbf{b}-1}A{\otimes}\Lambda^{\mathbf{b}-1}B{\otimes}\Lambda^{\mathbf{b}-1}C$, which is obtained as the projection of $(A{\otimes}B{\otimes}C)^{\otimes \mathbf{b}-1}$ to the subspace $S^{\mathbf{b}-1}A{\otimes}\Lambda^{\mathbf{b}-1}B{\otimes}\Lambda^{\mathbf{b}-1}C$. (Recall that $S^{\mathbf{b}-1}A$ may be thought of as the space of $(\mathbf{b}-1)$-linear forms on A^* that are symmetric in each argument. In this case the form eats α $\mathbf{b}-1$ times.) Similarly, $T_{\alpha^j} \in B{\otimes}C$ may be thought of as $T_{(\cdot)} \in A{\otimes}B{\otimes}C$. Now do the same contractions letting the A-factors go along for the ride.

7.6.3. A wiring diagram for Strassen's commutator. A wiring diagram for this approach to Strassen's equations is depicted in Figure 7.6.1.

7.6.4. Strassen's equations as modules. It remains to determine which modules in $\Lambda^2A{\otimes}S^{\mathbf{b}-1}A{\otimes}\Lambda^{\mathbf{b}}B{\otimes}B{\otimes}C{\otimes}\Lambda^{\mathbf{b}}C$ map nontrivially into $S^{\mathbf{b}+1}(A{\otimes}B{\otimes}C)$, when the inclusion $\Lambda^2A{\otimes}S^{\mathbf{b}-1}A{\otimes}\Lambda^{\mathbf{b}}B{\otimes}B{\otimes}C{\otimes}\Lambda^{\mathbf{b}}C \subset (A{\otimes}B{\otimes}C)^{\otimes \mathbf{b}+1}$ is composed with the projection $(A{\otimes}B{\otimes}C)^{\otimes \mathbf{b}+1} \to S^{\mathbf{b}+1}(A{\otimes}B{\otimes}C)$, as in §7.6.2. Since $\mathbf{b} = \dim B = \dim C$,

$$\Lambda^2A{\otimes}S^{\mathbf{b}-1}A{\otimes}\Lambda^{\mathbf{b}}B{\otimes}B{\otimes}C{\otimes}\Lambda^{\mathbf{b}}C = (S_{\mathbf{b},1}A{\oplus}S_{\mathbf{b}-1,1,1}A){\otimes}S_{2,1^{\mathbf{b}-1}}B{\otimes}S_{2,1^{\mathbf{b}-1}}C,$$

so there are two possible modules. Were the first mapped nontrivially, then one would be able to get equations in the case $\dim A = 2$, but $\sigma_3(Seg(\mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^2)) = \mathbb{P}(A{\otimes}B{\otimes}C)$, so only the second can occur and we have already seen that some module must occur.

In summary:

Proposition 7.6.4.1 ([**207**]). *Strassen's equations for $\sigma_{\mathbf{b}}(Seg(\mathbb{P}^2 \times \mathbb{P}^{\mathbf{b}-1} \times \mathbb{P}^{\mathbf{b}-1})) = \sigma_{\mathbf{b}}(Seg(\mathbb{P}A^* \times \mathbb{P}B^* \times \mathbb{P}C^*))$ are of degree $\mathbf{b}+1$. They are the modules $S_{\mathbf{b}-1,1,1}A{\otimes}S_{2,1^{\mathbf{b}-1}}B{\otimes}S_{2,1^{\mathbf{b}-1}}C$.*

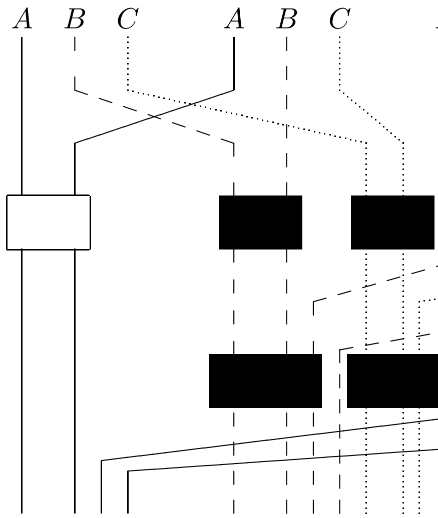


Figure 7.6.1. Strassen diagram 1 for $\dim B = \dim C = 3$, the output of which is an element of $(S^2A{\otimes}A{\otimes}A){\otimes}(\Lambda^3B{\otimes}B){\otimes}(\Lambda^3C{\otimes}C)$. Diagram 2 is the same, but with the roles of the third and fourth copies of $A{\otimes}B{\otimes}C$ reversed. The equations are the output of diagram 1 minus that of diagram 2.

In particular, when $\mathbf{b} = 3$, one recovers $S_{211}A{\otimes}S_{211}B{\otimes}S_{211}C$ of Proposition 7.5.5.4 and sees that Strassen's equations generate the whole module. Thus despite the apparently different role of A from B, C, in this case, exchanging the role of A with B or C yields the same space of equations. (In this situation $\mathbf{b} = 3$ is the only case with redundancies for Proposition 7.6.4.1, although redundancies do occur in other situations, e.g., for Strassen's degree nine equations.)

Proposition 7.6.4.2. *Let $\lfloor \frac{k}{2} \rfloor \leq \mathbf{b}$. Strassen's equations for*

$$\sigma_{\lfloor \frac{k}{2} \rfloor}(Seg(\mathbb{P}^2 \times \mathbb{P}^{\mathbf{b}-1} \times \mathbb{P}^{\mathbf{b}-1}))$$

are of degree $k+1$.

Exercise 7.6.4.3: Prove Proposition 7.6.4.2. ⊚

Proposition 7.6.4.4. *$\sigma_4(Seg(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2))$ is a hypersurface of degree nine.*

Exercise 7.6.4.5: Prove Proposition 7.6.4.4. ⊚

7.6.5. The varieties $Comm_{\mathbf{a},\mathbf{b}}$ and $Diag_{\mathbf{a},\mathbf{b}}$. Assume $\mathbf{a} \leq \mathbf{b} = \mathbf{c}$. Below when we choose $\alpha \in A^*$ such that $T(\alpha)$ is of full rank, we use it to identify

$B^* \simeq C$. Let

$$Comm_{\mathbf{a},\mathbf{b}} := \overline{\left\{[T] \in \mathbb{P}(A\otimes B\otimes C) \;\middle|\; \begin{array}{c}\exists \alpha \in A^*,\ \mathrm{rank}(T(\alpha)) = \mathbf{b},\ T_\alpha(A^*) \text{ is}\\ \text{an abelian subalgebra of } \mathrm{End}(C)\end{array}\right\}},$$

$$Diag_{\mathbf{a},\mathbf{b}} := \overline{\left\{[T] \in \mathbb{P}(A\otimes B\otimes C) \;\middle|\; \begin{array}{c}\exists \alpha \in A^*,\ \mathrm{rank}(T(\alpha)) = \mathbf{b},\ T_\alpha(A^*) \text{ is a}\\ \text{diagonalizable subalgebra of } \mathrm{End}(C)\end{array}\right\}}.$$

If T is not in a subspace variety and $T \in Diag_{\mathbf{a},\mathbf{b}}$, then $\underline{\mathbf{R}}(T) = \mathbf{b}$ by the same arguments as above. Unfortunately, the defining equations for $Diag_{\mathbf{a},\mathbf{b}}$ are not known in general. Equations for $Comm_{\mathbf{a},\mathbf{b}}$ are natural generalizations of Strassen's equations and are discussed in §7.6.7 below. However, when $\mathbf{a} = 3$, an abelian subalgebra is diagonalizable or a limit of diagonalizable subalgebras—and this is also true when $\mathbf{a} = 4$; see [**176**]. Theorem 3.8.2.5 follows. It is known that there exist abelian subalgebras that are not limits of diagonalizable subalgebras when $\mathbf{a} \geq 6$. Thus the equations, while useful, will not give a complete set of defining equations.

7.6.6. Strassen's equations are enough for $\sigma_3(Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C))$.

Theorem 7.6.6.1 ([**207, 130**]). *$\sigma_3(Seg(\mathbb{P}A^* \times \mathbb{P}B^* \times \mathbb{P}C^*))$ is the zero set of the size four minors of flattenings and Strassen's degree four equations, i.e., $\Lambda^4 A\otimes\Lambda^4(B\otimes C)$ plus permutations and $S_{211}A\otimes S_{211}B\otimes S_{211}C$.*

Remark 7.6.6.2. In [**207**] there was a gap in the proof, which was first observed by S. Friedland, who also furnished a correction. What follows is a variant of his argument.

Let $Strassen^3$ denote the zero set of the modules in the theorem. Recall the variety $Rank_A^k$ from §7.2.2. The discussion above implies that $Strassen^3 = \sigma_3(Seg(\mathbb{P}A^* \times \mathbb{P}B^* \times \mathbb{P}C^*)) \cup (Rank_A^2 \cap Strassen^3)$. So it remains to prove:

Lemma 7.6.6.3. $(Rank_A^2 \cap Strassen^3) \subset \sigma_3(Seg(\mathbb{P}A^* \times \mathbb{P}B^* \times \mathbb{P}C^*))$.

Proof. Assume that $\mathbf{a} = \mathbf{b} = \mathbf{c} = 3$ and that $T(A) \subset B^*\otimes C^*$ is of bounded rank two.

Three-dimensional spaces of matrices of rank at most two have been classified; see [**120, 10**]. There are two normal forms,

$$\begin{pmatrix} x_0^0 & x_1 & \cdots & x_r \\ x^1 & 0 & \cdots & 0 \\ \vdots & 0 & \cdots & 0 \\ x^r & 0 & \cdots & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & x & y \\ -x & 0 & z \\ -y & -z & 0 \end{pmatrix},$$

where in the first case we assume that each $x = x(s,t,u)$ is linear in s, t, u, and in the second case one can place the matrix as, e.g., the upper right hand corner of a matrix filled with zeros.

Assume that $\mathbf{a} = \mathbf{b} = \mathbf{c} = 3$ and that $T(A) \subset B^* \otimes C^*$ is of bounded rank two. The second case never satisfies the equations; in fact, the corresponding matrix is easily seen to be of maximal rank in the Ottaviani model.

The first yields one equation. (This is easier to see using Strassen's model.) Writing $T = a_1 \otimes m + a_2 \otimes n + a_3 \otimes p$, choosing bases, the equation is:

$$\begin{aligned} &- n_{2,1}p_{1,2}m_{1,3}m_{3,1} + n_{2,1}p_{1,3}m_{1,2}m_{3,1} + n_{3,1}p_{1,2}m_{1,3}m_{2,1} \\ &- n_{3,1}p_{1,3}m_{1,2}m_{2,1} + p_{2,1}n_{1,2}m_{1,3}m_{3,1} - p_{2,1}n_{1,3}m_{1,2}m_{3,1} \\ &- p_{3,1}n_{1,2}m_{1,3}m_{2,1} + p_{3,1}n_{1,3}m_{1,2}m_{2,1}. \end{aligned}$$

First note that if $p^i_j = 0$ for all $(i,j) \neq (1,1)$, then Strassen's equations are satisfied. In this case one does have a point of σ_3. Explicitly (see Theorem 10.10.2.1), it may be obtained by taking a point on a $\mathbb{P}^2$ that is a limit of three points limiting to colinear points, say $a_1 \otimes b_1 \otimes c_1, a_2 \otimes b_1 \otimes c_1$, $(a_1 + a_2) \otimes b_1 \otimes c_1$. Then the limiting $\mathbb{P}^2$ can be made to contain any point in $\hat{T}_{[a_1 \otimes b_1 \otimes c_1]} Seg(\mathbb{P}A^* \times \mathbb{P}B^* \times \mathbb{P}C^*) + \hat{T}_{[a_2 \otimes b_1 \otimes c_1]} Seg(\mathbb{P}A^* \times \mathbb{P}B^* \times \mathbb{P}C^*)$. Then p corresponds to a point $a_3 \otimes b_1 \otimes c_1$ (which occurs in both tangent spaces), m to a point in the first tangent space, and n to a point in the second.

It remains to show that, up to rechoosing bases, this is the only way Strassen's equations can be satisfied. First use $SL(B) \times SL(C)$ to make $p_{1,3} = p_{3,1} = 0$ and $p_{1,2} = p_{2,1}$. Then add an appropriate multiple of p to n, m to make $n_{1,2} = -n_{2,1}$, and then add appropriate multiples of p, n to m to obtain $m_{1,2} = m_{2,1} = 0$. Finally add an appropriate multiple of m to n to obtain $n_{1,3} = 0$. Strassen's equation reduces to

$$-2p_{1,2}m_{1,3}m_{3,1}n_{1,2},$$

which vanishes if and only if one of the three matrices has rank one. If it is not p, just relabel bases. $\square$

7.6.7. Generalizations of Strassen's conditions. The key point in the discussion of §7.6.2 was that contracting $T \in \sigma_r(Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C))$ in two different ways yielded equivalent tensors.

Consider, for s, t such that $s + t \leq \mathbf{b}$ and $\alpha, \alpha_j \in A^*$, the tensors

$$T^{\wedge s}_{\alpha_j} \in \Lambda^s B \otimes \Lambda^s C, \qquad T^{\wedge t}_{\alpha} \in \Lambda^t B \otimes \Lambda^t C$$

(the previous case was $s = 1$, $t = \mathbf{b} - 1$). Contract $T^{\wedge t}_{\alpha} \otimes T^{\wedge s}_{\alpha_1} \otimes T^{\wedge s}_{\alpha_2}$ to obtain elements of $\Lambda^{s+t} B \otimes \Lambda^{s+t} C \otimes \Lambda^s B \otimes \Lambda^s C$ in two different ways, and call these contractions $\psi^{s,t}_{\alpha,\alpha_1,\alpha_2}(T)$ and $\psi^{s,t}_{\alpha,\alpha_2,\alpha_1}(T)$.

Now say $\mathbf{R}(T) = r$ so it may be written as $T = a_1 \otimes b_1 \otimes c_1 + \cdots + a_r \otimes b_r \otimes c_r$ for elements $a_i \in A$, $b_i \in B$, $c_i \in C$. Then

$$\psi^{s,t}_{\alpha,\alpha_1,\alpha_2}(T) = \sum_{|I|=s,|J|=t,|K|=s} \langle \tilde{a}_I, \alpha_1^s \rangle \langle \tilde{a}_J, \alpha^t \rangle \langle \tilde{a}_K, \alpha_2^s \rangle (b_{I+J} \otimes b_K) \otimes (c_I \otimes c_{J+K}),$$

where $b_I = b_{i_1} \wedge \cdots \wedge b_{i_s} \in \Lambda^s B$, $\tilde{a}_I = a_{i_1} \cdots a_{i_s}$, and $b_{I+J} = b_I \wedge b_J$, etc. For $\psi^{s,t}_{\alpha,\alpha_1,\alpha_2}(T)$ to be nonzero, I and J must be disjoint subsets of $\{1, \ldots, r\}$. Similarly, I and K must be disjoint. If $s + t = r$, this implies $J = K$. In summary:

Proposition 7.6.7.1 ([**207**]). *If $T \in \sigma_{s+t}(Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C))$, then for all $\alpha, \alpha_1, \alpha_2 \in A^*$,*

$$\psi^{s,t}_{\alpha,\alpha^1,\alpha^2}(T) - \psi^{s,t}_{\alpha,\alpha^2,\alpha^1}(T) = 0.$$

Consider the bilinear map

$$(\Lambda^2(S^s A) \otimes S^t A)^* \times (A \otimes B \otimes C)^{\otimes 2s+t} \to \Lambda^{s+t} B \otimes \Lambda^{s+t} C \otimes \Lambda^s B \otimes \Lambda^s C,$$

whose image is $\psi^{s,t}_{\alpha,\alpha^1,\alpha^2}(T) - \psi^{s,t}_{\alpha,\alpha^2,\alpha^1}(T)$. To avoid choosing elements in A^*, rewrite it as a polynomial map

$$\Psi^{s,t} : A \otimes B \otimes C \to (\Lambda^2(S^s A) \otimes S^t A) \otimes \Lambda^{s+t} B \otimes \Lambda^{s+t} C \otimes \Lambda^s B \otimes \Lambda^s C.$$

The only problem is that we do not know whether or not $\Psi^{s,t}(T)$ is identically zero for all tensors T. This is actually the most difficult step, but in [**207**] many of the $\Psi^{s,t}$ are shown to be nonzero and independent $GL(A) \times GL(B) \times GL(C)$-modules in the ideal of $\sigma_{s+t}(Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C))$. In particular:

Theorem 7.6.7.2 ([**207**]). *Let s be odd. For each r and $s \leq \frac{r}{2}$ if r is even and $s \leq \frac{r}{3}$ if r is odd, the module*

$$S_{r-s,s,s} A \otimes S_{2^s,1^{r-s}} B \otimes S_{2^s,1^{r-s}} C \subset S^{r+s}(A \otimes B \otimes C)$$

is in $I_{r+s}(\sigma_r(Seg(\mathbb{P}A^* \times \mathbb{P}B^* \times \mathbb{P}C^*)))$. Moreover each of these modules is independent of the others in $I(\sigma_r(Seg(\mathbb{P}A^* \times \mathbb{P}B^* \times \mathbb{P}C^*)))$.

Theorem 7.6.7.2 indicates that there may be many modules of generators for the ideals of secant varieties of triple (and higher) Segre products.

The above discussion indicates many possible generalizations of Strassen's equations. For example, consider a collection of elements $\alpha, \alpha_1, \ldots, \alpha_p$ and ask that the space of tensors T_{α,α_j} forms an abelian subspaces of $C^* \otimes C$. Here are some specific examples with the bounds on border rank they induce:

Proposition 7.6.7.3. [**207**] *Let $T \in \sigma_r(Seg(\mathbb{P}A^* \times \mathbb{P}B^* \times \mathbb{P}C^*))$ be a general point and $\alpha_0, \alpha_1 \in A$ such that $T(\alpha_0)$ and $T(\alpha_1)$ have maximal rank. Then*

$$\text{rank}\ [T_{\alpha_0,\alpha}, T_{\alpha_1,\alpha'}] \leq 3(r - \mathbf{b}) \qquad \forall \alpha, \alpha' \in A.$$

Another generalization involving more operators is the following:

Proposition 7.6.7.4 ([**207**]). *Let $T \in \sigma_r(Seg(\mathbb{P}A^* \times \mathbb{P}B^* \times \mathbb{P}C^*))$ be a general point and $\alpha_0 \in A$ such that $T(\alpha_0)$ has maximal rank. Then for any permutation $\sigma \in \mathfrak{S}_k$, and for any $\alpha_1, \ldots, \alpha_k \in A$,*

$$\text{rank}\ \left(T_{\alpha_0,\alpha_1} \cdots T_{\alpha_0,\alpha_k} - T_{\alpha_0,\alpha_{\sigma(1)}} \cdots T_{\alpha_0,\alpha_{\sigma(k)}}\right) \leq 2(k-1)(r - \mathbf{b}).$$

Another variant is obtained by using products of $T_{\alpha,\alpha'}$ with different α's and permuted α''s. One can even get new equations when $\mathbf{b} \neq \mathbf{c}$. For more details on these generalizations, see [**207**].

Remark 7.6.7.5. Note that the modules of equations appearing in this subsection are only for relatively small secant varieties. Their mutual independence illustrates how it might be very difficult to determine defining equations in general.

7.7. Equations for $\sigma_4(Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C))$

7.7.1. Reduction to $\sigma_4(Seg(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^3))$. Recall Proposition 7.6.4.4 that $\sigma_3(Seg(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2))$ is a hypersurface of degree nine.

Theorem 7.7.1.1 ([**207, 130**]). *$\sigma_4(Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C))$ is the zero set of the size five minors of flattenings, Strassen's degree five and nine equations, and equations inherited from $\sigma_4(Seg(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^3))$.*

Remark 7.7.1.2. Theorem 7.7.1.1 was stated in [**207**], but there was a gap in the proof that was observed and fixed by S. Friedland in [**130**].

Let

$$\begin{aligned} Strassen^4_A := \mathrm{Zeros}\{&S_{311}A \otimes S_{2111}B \otimes S_{2111}C, \Lambda^5 A \otimes \Lambda^5(B \otimes C), \\ &\Lambda^5 A \otimes \Lambda^5(B \otimes C), \Lambda^5 A \otimes \Lambda^5(B \otimes C),\ S_{333}A \otimes S_{333}B \otimes S_{333}C\}. \end{aligned}$$

The first and last modules occur with multiplicity one respectively in $S^5(A \otimes B \otimes C)$ and $S^9(A \otimes B \otimes C)$, so the description is unambiguous. These equations are all of degree five except the last which is of degree nine.

Theorem 7.7.1.1 is a consequence of the following two propositions:

Proposition 7.7.1.3 ([**207**]). *For* $\mathbf{a} = 4$,

$$\begin{aligned} Strassen^4_A = \sigma_4(Seg(\mathbb{P}A^* \times \mathbb{P}B^* \times \mathbb{P}C^*)) \cup Sub_{3,3,4} \\ \cup Sub_{3,4,3} \cup (Rank^3_A \cap Strassen^4_A). \end{aligned}$$

Exercise 7.7.1.4: Prove Proposition 7.7.1.3.

Proposition 7.7.1.5 ([**130**]).

$$\begin{aligned} Rank^3_A \cap Rank^3_B \cap Rank^3_C \cap Strassen^4_A \cap Strassen^4_B \cap Strassen^4_C \\ \subset \sigma_4(Seg(\mathbb{P}A^* \times \mathbb{P}B^* \times \mathbb{P}C^*)) \cup Sub_{3,3,4} \cup Sub_{3,4,3} \cup Sub_{4,3,3}. \end{aligned}$$

The idea of the proof of Proposition 7.7.1.5 is as follows: first eliminate the case of tensors $T \in Rank^2_A \cap Rank^2_B \cap Rank^2_C$, which is very restrictive. Then, for $T \in Rank^3_A \cap Rank^3_B \cap Rank^3_C$, one observes that for $\alpha \in A^*$, the cofactor matrix of $T(\alpha)$ has rank one, so it distinguishes lines in B, C. One then analyzes how these lines move infinitesimally as one varies α. The arguments of [**130**] can be simplified by using the normal forms for systems of bounded rank four.

7.7.2. Friedland's example. In this subsection I explain Friedland's example of a tensor satisfying the degree nine equations but of border rank five.

Say $T \in A{\otimes}B{\otimes}C$ satisfies the degree nine equations for $\sigma_4(Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C))$, i.e., the linear map $T_A^{\wedge} : A{\otimes}B^* \to \Lambda^2 A{\otimes}C$ has a kernel. Let $\psi \in \ker T_A^{\wedge}$, so $\psi : B \to A$.

Consider $\psi(T) \in A{\otimes}A{\otimes}C$. Since $\psi \in \ker T_A^{\wedge}$, we actually have $\psi(T) \in S^2A{\otimes}C$. If ψ is injective, then $\psi(T)$ is an equivalent tensor to T. By Proposition 5.5.3.2, $S^2\mathbb{C}^3{\otimes}\mathbb{C}^4 = \hat{\sigma}_4(Seg(v_2(\mathbb{P}^2) \times \mathbb{P}^3)) \subset \hat{\sigma}_4(Seg(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^3))$. Thus if ψ is injective, the degree nine equations are enough!

So we look for a tensor T where $T_A^{\wedge}$ has a kernel that does not contain an injective linear map.

Example 7.7.2.1 (Friedland [**130**])**.** Let a_j be a basis of A, b_j a basis of B, and c_s a basis of C, $1 \leq j \leq 3$, $1 \leq s \leq 4$. Consider

$$\begin{aligned} X = {} & (a_1{\otimes}b_1 + a_2{\otimes}b_2){\otimes}c_1 + (a_1{\otimes}b_1 + a_2{\otimes}b_3){\otimes}c_2 \\ & + (a_1{\otimes}b_1 + a_3{\otimes}b_2){\otimes}c_3 + (a_1{\otimes}b_1 + a_3{\otimes}b_3){\otimes}c_4. \end{aligned}$$

In matrices,

$$X(C^*) = \left\{ \begin{pmatrix} \alpha+\beta+\gamma+\delta & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & \gamma & \delta \end{pmatrix} \;\middle|\; \alpha, \beta, \gamma, \delta \in \mathbb{C} \right\}.$$

It is straightforward to verify that $\det(X_A^{\wedge}) = \det(X_B^{\wedge}) = 0$, so the degree nine equations are satisfied. The calculations below show that $\underline{\mathbf{R}}(X) \geq 5$.

In order to eliminate such tensors, one needs further equations. As remarked earlier, the degree six equations will suffice, but since we lack a geometric model from them, I describe Friedland's degree sixteen equations which have nice presentation.

7.7.3. Friedland's equations. Write $\psi = \psi_{AB} : B \to A$ and consider the analogous $\psi_{BA} : A \to B$ by reversing the roles of A and B. I claim that if T satisfies the degree nine equations and $\operatorname{rank}(\psi_{AB}) = \operatorname{rank}(\psi_{BA}) = 3$, then $\psi_{AB}\psi_{BA} = \lambda\,\mathrm{Id}$. To see this, use the normal form for a general point of $\hat{\sigma}_4$, namely, $T = a_1{\otimes}b_1{\otimes}c_1 + a_2{\otimes}b_2{\otimes}c_2 + a_3{\otimes}b_3{\otimes}c_3 + (a_1+a_2+a_3){\otimes}(b_1 + b_2 + b_3){\otimes}c_4$. (See §10.7 for justification of this normal form.)

Thus if $T \in \hat{\sigma}_4$,

$$\operatorname{proj}_{\mathfrak{sl}(A)}(\psi_{AB}\psi_{BA}) = 0, \qquad \operatorname{proj}_{\mathfrak{sl}(B)}(\psi_{BA}\psi_{AB}) = 0. \tag{7.7.1}$$

These are additional equations of degree 16 that X of Example 7.7.2.1 fails to satisfy, showing that $\underline{\mathbf{R}}(X) \geq 5$.

Theorem 7.7.3.1 ([**130**])**.** *The variety $\sigma_4(Seg(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^3))$ is the zero set of the degree nine equations $\det(T_A^\wedge) = 0$ and the degree sixteen equations* (7.7.1).

Corollary 7.7.3.2 ([**130**])**.** *Let A, B, C be vector spaces; then $\sigma_4(Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C))$ is the zero set of the inherited degree nine and the degree sixteen equations plus the equations of flattenings.*

Proof of the theorem. We have seen that these modules are in the ideal of $\sigma_4(Seg(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^3))$; it remains to show that their common zero set is $\sigma_4(Seg(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^3))$.

If $\dim T(C^*) < 4$, we are reduced to $\sigma_4(Seg(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2))$, which is handled by the degree nine Strassen equations. If $\ker T_A^\wedge$ or $\ker T_B^\wedge$ contains an invertible map, we are done by the above discussion.

Consider the case where $\ker T_A^\wedge$ contains a map of rank two. Write $\psi_{AB} = \beta^1 \otimes a_1 + \beta^2 \otimes a_2$ for some independent elements $\beta^1, \beta^2 \in B^*$, $a_1, a_2 \in A$. Extend these to bases, take the corresponding dual bases, and choose a basis for C. Write $T = \sum_{i,j,s} T^{ijs} a_i \otimes b_j \otimes c_s$. Then $\psi_{AB}(T) = \sum_{i,s}(T^{i1s} a_i \otimes a_1 \otimes c_s + T^{i2s} a_i \otimes a_2 \otimes c_s)$. The requirement that $\psi_{AB}(T)$ is symmetric implies $T^{31s} = T^{32s} = 0$ and $T^{12s} = T^{21s}$ for all s.

Consider the subcase where ψ_{BA} has rank one and write $\psi_{BA} = \alpha \otimes b$. Then $0 = \psi_{AB}\psi_{BA} = \alpha(a_1)\beta^1 \otimes b + \alpha(a_2)\beta^2 \otimes b$, and we conclude that $\alpha(a_1) = \alpha(a_2) = 0$, i.e., up to scale, $\alpha = \alpha_3$. Similarly, computing $0 = \psi_{BA}\psi_{AB} = b^1(b)\alpha \otimes a_1 + b^2(b)\alpha \otimes a_2$ shows that $b = b_3$ up to scale.

The requirement that $\psi_{BA}(T)$ be symmetric implies $T^{31s} = T^{32s} = 0$ for all s. In terms of matrices,

$$T(c^s) = \begin{pmatrix} T^{11s} & T^{12s} & 0 \\ T^{12s} & T^{22s} & 0 \\ 0 & 0 & T^{33s} \end{pmatrix},$$

but this is a 4-dimensional subspace of $A \otimes B$ spanned by rank one elements, proving this case.

Now say ψ_{BA} also has rank two; write it as $\alpha \otimes b + \alpha' \otimes b'$, with α, α' and b, b' independent vectors. Then

$$0 = \psi_{AB}\psi_{BA} = b^1 \otimes [\alpha(a_1)b + \alpha'(a_1)b'] + b^2 \otimes [\alpha(a_2)b + \alpha'(a_2)b'].$$

Since the four vectors $b^1 \otimes b$, $b^2 \otimes b$, $b^1 \otimes b'$, $b^2 \otimes b'$ are linearly independent, all the coefficients must vanish. This implies that α, α' are both a multiple of α_3—a contradiction since we assumed that they were linearly independent.

Finally, say both $\ker T_A^\wedge$, $\ker T_B^\wedge$ consist of rank one elements. Write $\psi_{AB} = \beta \otimes a$, $\psi_{BA} = \alpha \otimes b$. The trace conditions imply that $\alpha(a) = \beta(b) = 0$. Write $a = a_1$, $b = b_1$ and extend to bases, so α is a linear combination of a^2, a^3 and we choose bases such that $\alpha = a^3$, $\beta = b^3$. Write $T = \sum_{ijs} T^{ijs} a_i \otimes b_j \otimes c_s$. Then $\psi_{AB}(T) = \sum_{i,s} T^{i3s} a_i \otimes a_1 \otimes c_s$, and $\psi_{BA}(T) = \sum_{j,s} T^{3js} b_1 \otimes b_j \otimes c_s$. The symmetry conditions imply that $T^{23s} = T^{33s} = 0$ and $T^{32s} = T^{33s} = 0$. In these bases we may write

$$T(c_s) = \begin{pmatrix} T^{11s} & T^{12s} & T^{13s} \\ T^{21s} & T^{22s} & 0 \\ T^{31s} & 0 & 0 \end{pmatrix}.$$

To see that this 4-dimensional subspace is spanned by rank one elements, first note that any 4-plane must intersect any 3-plane in this 6-dimensional space;, in particular it must intersect the three-plane spanned by the first column. Let $v_1 \in A \otimes b_1 \cap T(C^*)$, with $v_1 \cap \langle a_3 \otimes b_1 \rangle \neq 0$ (if this is not possible, one is already done). Choose a complement to v_1 contained in $\langle a_1, a_2 \rangle \otimes B$ (in fact, the complement must lie in the hyperplane inside this space given by $a^2 \otimes b^3 = 0$). Since this is a 3-plane in $\mathbb{C}^2 \otimes \mathbb{C}^3$, it is spanned by rank one elements. $\square$

7.8. Young flattenings

In this section I derive equations for $\sigma_r(v_d(\mathbb{P}V))$ with the help of representation theory.

7.8.1. Preliminaries. In what follows, V will be endowed with a volume form and thus $S_{(p_1,\dots,p_{\mathbf{v}})}V$ will be identified with $S_{(p_1-p_{\mathbf{v}},p_2-p_{\mathbf{v}},\dots,p_{\mathbf{v}-1}-p_{\mathbf{v}},0)}V$ (as $SL(V)$-modules). Call $(p_1 - p_{\mathbf{v}}, p_2 - p_{\mathbf{v}}, \dots, p_{\mathbf{v}-1} - p_{\mathbf{v}}, 0)$ the *reduced partition* associated to $(p_1, \dots, p_{\mathbf{v}})$.

Recall from Exercise 6.4.3(9) that the dual $SL(V)$-module to $S_\pi V$ is obtained by considering the complement to π in the $\ell(\pi) \times \mathbf{v}$ rectangle and rotating it to give a Young diagram with associated partition π^*.

Also recall that the *Pieri formula*, Theorem 6.7.2.1, states that $S_\pi V^* \subset S_\nu V^* \otimes S^d V^*$ if and only if the Young diagram of π is obtained by adding d boxes to the Young diagram of ν, with no two boxes added to the same column. Moreover, if this occurs, the multiplicity of $S_\pi V^*$ in $S_\nu V^* \otimes S^d V^*$ is one.

Say $S_\pi V^* \subset S_\nu V \otimes S^d V^*$ and consider the map $S^d V \to S_\pi V \otimes S_\nu V^*$. Let $S_\mu V = S_\nu V^*$, where μ is the reduced partition with this property, to obtain an inclusion $S^d V \to S_\pi V \otimes S_\mu V$.

Given $\phi \in S^dV$, let $\phi_{\pi,\mu} \in S_\pi V \otimes S_\mu V$ denote the corresponding element. If $S_\mu V = S_\nu V^*$ as an $SL(V)$-module, write $\phi_{\pi,\nu^*} = \phi_{\pi,\mu}$ when it is considered as a linear map $S_\nu V \to S_\pi V$.

Proposition 7.8.1.1 ([**215**])**.** *Rank conditions on $\phi_{\pi,\mu}$ provide equations for the secant varieties of $v_d(\mathbb{P}V)$ as follows. Say $\operatorname{rank}(x^d_{\pi,\mu}) = t$ for some (and hence all) $[x^d] \in v_d(\mathbb{P}V)$. If $[\phi] \in \sigma_r(v_d(\mathbb{P}V))$, then $\operatorname{rank}(\phi_{\pi,\mu}) \leq rt$. Thus if $rt+1 \leq \min\{\dim S_\pi V, \dim S_\mu V\}$, the $(rt+1)\times(rt+1)$ minors of $\phi_{\pi,\mu}$ provide equations for $\sigma_r(v_d(\mathbb{P}V))$, i.e.,*

$$\Lambda^{rt+1}(S_\pi V^*) \otimes \Lambda^{rt+1}(S_\nu V^*) \subset I_{rt+1}(\sigma_r(v_d(\mathbb{P}V))).$$

Let

$$YFlat^s_{\pi,\mu}(S^dV) := \overline{\mathbb{P}\{\phi \in S^dV \mid \operatorname{rank}(\phi_{\pi,\mu}) \leq s\}} = \sigma_s(Seg(\mathbb{P}S_\pi V \times \mathbb{P}S_\mu V)) \cap \mathbb{P}S^dV. \tag{7.8.1}$$

We have $\sigma_r(v_d(\mathbb{P}V)) \subseteq YFlat^{rt}_{\pi,\mu}(S^dV)$. If $S_\pi V \simeq S_\mu V$ as $SL(V)$-modules and $\phi_{\pi,\mu}$ is symmetric, then

$$YFlat^s_{\pi,\mu}(S^dV) = \sigma_s(v_2(\mathbb{P}S_\pi V)) \cap \mathbb{P}S^dV,$$

and if $\phi_{\pi,\mu}$ is skew-symmetric, then

$$YFlat^s_{\pi,\mu}(S^dV) = \sigma_s(G(2, S_\pi V)) \cap \mathbb{P}S^dV.$$

Remark 7.8.1.2. Recall $\hat{Y}F^r_{d,n}$ from §3.10.2 whose description had redundancies. We can now give an irredundant description of its defining equations:

$$YF^r_{d,n} = YFlat^{\binom{\mathbf{v}}{\lfloor \frac{\mathbf{v}}{2} \rfloor} r}_{((\delta+1)^a,\delta^{n-a}),(\delta+1,1^a)}(S^dV).$$

7.8.2. The surface case, $\sigma_r(v_d(\mathbb{P}^2))$. Fix $\dim V = 3$ and a volume form Ω on V.

Lemma 7.8.2.1 ([**215**])**.** *Let $a \geq b$. Write $d = \alpha+\beta+\gamma$ with $\alpha \leq b$, $\beta \leq a-b$, so $S_{(a+\gamma-\alpha,b+\beta-\alpha)}V \subset S_{a,b}V \otimes S^dV$. For $\phi \in S^dV$, consider the induced map*

$$\phi_{(a,b),(a+\gamma-\alpha,b+\beta-\alpha)} : S_{a,b}V^* \to S_{(a+\gamma-\alpha,b+\beta-\alpha)}V. \tag{7.8.2}$$

Let $x \in V$; then

$$\operatorname{rank}((x^d)_{(a,b),(a+\gamma-\alpha,b+\beta-\alpha)}) = \frac{1}{2}(b-\alpha+1)(a-b-\beta+1)(a+\beta-\alpha+2) =: R. \tag{7.8.3}$$

Thus in this situation $\Lambda^{pR+1}(S_{ab}V^*) \otimes \Lambda^{pR+1}(S_{a+\gamma-\alpha,b+\beta-\alpha}V^*)$ gives nontrivial degree $pR+1$ equations for $\sigma_p(v_d(\mathbb{P}V))$.

For the proof, see [**215**].

Remark 7.8.2.2. The right hand sides of the equations in Exercise 6.4.3(5) and (7.8.3) are the same when $\alpha = \beta = 0$. To get useful equations, one wants R small with respect to $\dim S_{a,b}\mathbb{C}^3$.

Particularly interesting are the cases where $(a, b) = (a+\gamma-\alpha, b+\beta-\alpha)$. Then

$$\alpha = \gamma = \frac{1}{3}(d + 2b - a), \tag{7.8.4}$$

$$\beta = \frac{1}{3}(d - 4b + 2a). \tag{7.8.5}$$

Plugging into the conclusion of Lemma 7.8.2.1, the rank of the image of a d-th power in this situation is

$$\frac{1}{9}(a + b - d + 3)^2(a - b + 1).$$

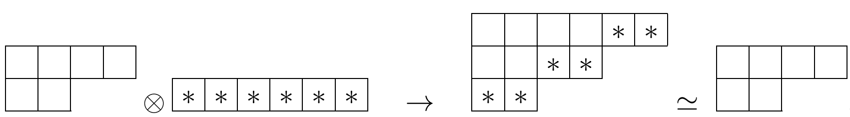
To keep this small, it is convenient to take $d = a+b$, so the rank is $a-b+1$. One can then fix this number and let a, b grow to study series of cases.

If (7.8.3) has rank one when $d = 2p$, one just recovers the usual symmetric flattenings as $S_{(p,p)}V = S_pV^*$. Recall that $(p + q, p)^* = (p + q, q)$.

If $d = 2p+2$ is even, and one requires both modules to be the same, the smallest possible $\text{rank}((x^d)_{\pi,\mu})$ is three, which is obtained with $\phi_{(p+2,p),(p+2,p)}$.

Proposition 7.8.2.3 ([**215**]). *Let $d = 2p + 2$. The Young flattening $\phi_{(p+2,2),(p+2,2)} \in S_{p+2,2}V\otimes S_{p+2,2}V$ is symmetric. It is of rank three for $\phi \in v_d(\mathbb{P}^2)$ and gives degree $3(k + 1)$ equations for $\sigma_r(v_{2p+2}(\mathbb{P}^2))$ for $r \leq \frac{1}{2}(p^2 + 5p + 4) - 1$. A convenient model for the equations is given in the proof.*

A pictorial description when $p = 2$ is as follows:



Proof. Let $\tilde{\Omega} \in \Lambda^3V$ be dual to the volume form Ω. To prove the symmetry, for $\phi = x^{2p+2}$, consider the map,

$$\begin{aligned} M_{x^{2p+2}} : S^pV^*\otimes S^2(\Lambda^2V^*) &\to S^pV\otimes S^2(\Lambda^2V), \\ \alpha_1 \cdots \alpha_p\otimes(\gamma_1 \wedge \delta_1)(\gamma_2 \wedge \delta_2) &\mapsto \alpha_1(x) \cdots \alpha_p(x)x^p \\ &\qquad\qquad \otimes\tilde{\Omega}(x\lrcorner\gamma_1 \wedge \delta_1)\tilde{\Omega}(x\lrcorner\gamma_2 \wedge \delta_2) \end{aligned}$$

and define M_ϕ for arbitrary $\phi \in S^{2p+2}V$ by linearity and polarization. Take bases of $S^pV\otimes S^2(\Lambda^2V)$ with indices $((i_1, \ldots, i_p), (kl), (k'l'))$. Most of the matrix of $M_{e_1^{2p+2}}$ is zero. Consider the upper right 6×6 block, where $(i_1, \ldots, i_p) = (1, \ldots, 1)$ in both rows and columns and order the other indices

$$((12),(12)),((13),(13)),((12),(13)),((12),(23)),((13),(23)),((23),(23)).$$

It is

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

showing the symmetry. Now

$$\begin{aligned}(S^pV \otimes S^2(\Lambda^2 V))^{\otimes 2} &= (S^pV \otimes S_{22}V)^{\otimes 2} \\ &= S_{p+2,2}V^{\otimes 2} \oplus \mathit{stuff}\end{aligned}$$

where all the terms in *stuff* have partitions with at least three parts. The image is the first factor and $M_\phi \in S^2(S_{p+2,2}V)$. $\square$

The usual symmetric flattenings give nontrivial equations for $\sigma_r(v_{2p+2}(\mathbb{P}^2))$ for $r \leq \frac{1}{2}(p^2 + 5p + 6) - 1$, a larger range than in Proposition 7.8.2.3. In [**215**] it is shown that the symmetric flattenings alone are not enough to cut out $\sigma_7(v_6(\mathbb{P}^2))$, but the $((p+1,2),(p+1,2))$-Young flattening plus the symmetric flattenings do cut out $\sigma_7(v_6(\mathbb{P}^2))$.

Here is a more general Young flattening:

Proposition 7.8.2.4 ([**215**]). *Let $d = p + 4q - 1$. The Young flattening*

$$\phi_{(p+2q,2q-1),(p+2q,2q-1)} \in S_{(p+2q,2q-1)}V \otimes S_{(p+2q,2q-1)}V$$

is skew-symmetric if p is even, and it is symmetric if p is odd.

Since it has rank p for $\phi \in v_d(\mathbb{P}^2)$, if p is even (resp. odd), the size $rp+2$ sub-Pfaffians (resp. size $rp+1$ minors) of $\phi_{(p+2q,2q-1),(p+2q,2q-1)}$ give degree $\frac{rp}{2}+1$ (resp. $rp+1$) equations for $\sigma_r(v_{p+4q-1}(\mathbb{P}^2))$ for

$$r \leq \frac{q(p+2q+2)(p+2)}{p}.$$

Proof. Consider $M_\phi : S^{p-1}V^* \otimes S^q(\Lambda^2 V^*) \to S^pV \otimes S^q(\Lambda^2 V)$ given for $\phi = x^{p+4q-1}$ by

$$\alpha_1 \cdots \alpha_{p-1} \otimes \beta_1 \wedge \gamma_1 \cdots \beta_q \wedge \gamma_q \mapsto \prod_j (\alpha_j(x)) x^{p-1} \otimes \tilde{\Omega}(x \lrcorner \beta_1 \wedge \gamma_1) \cdots \tilde{\Omega}(x \lrcorner \beta_q \wedge \gamma_q)$$

and argue as above. $\square$

Here the usual flattenings give degree $r+1$ equations for $\sigma_r(v_d(\mathbb{P}^2))$ in the generally larger range $r \leq \frac{1}{8}(p+4q+2)(p+4q)-1$.

Recall that the ideals of $\sigma_r(v_d(\mathbb{P}^2))$ for $d \leq 4$ were determined in Theorem 7.3.4.4.

Here is what is known beyond that:

Theorem 7.8.2.5. *Classical results and results from* [**215**]*:*

(1) *The variety* $\sigma_k(v_5(\mathbb{P}^2))$ *for* $k \leq 5$ *is an irreducible component of* $YFlat^{2k}_{31,31}(S^5\mathbb{C}^3)$*, the variety given by the principal size* $2k+2$ *Pfaffians of the* $[(31),(31)]$*-Young flattenings.*

(2) *The principal size* 14 *Pfaffians of the* $[(31),(31)]$*-Young flattenings are scheme-theoretic defining equations for* $\sigma_6(v_5(\mathbb{P}^2))$*, i.e., as schemes,* $\sigma_6(v_5(\mathbb{P}^2)) = YFlat^{12}_{31,31}(S^5\mathbb{C}^3)$*.*

(3) $\sigma_7(v_5(\mathbb{P}^2))$ *is the ambient space.*

(4) *As schemes,* $\sigma_6(v_6(\mathbb{P}^2)) = Rank^6_{3,3}(S^6\mathbb{C}^3)$*, i.e., the size* 7 *minors of* $\phi_{3,3}$ *cut out* $\sigma_6(v_6(\mathbb{P}^2))$ *scheme-theoretically.*

(5) $\sigma_7(v_6(\mathbb{P}^2))$ *is an irreducible component of* $Rank^7_{3,3}(S^6\mathbb{C}^3) \cap YFlat^{21}_{42,42}(S^6\mathbb{C}^3)$*, i.e., of the variety defined by the size* 8 *minors of the symmetric flattening* $\phi_{3,3}$ *and by the size* 22 *minors of the* $[(42),(42)]$*-Young flattenings.*

(6) $\sigma_8(v_6(\mathbb{P}^2))$ *is an irreducible component of* $Rank^8_{3,3}(S^6\mathbb{C}^3) \cap YFlat^{24}_{42,42}(S^6\mathbb{C}^3)$*, i.e., of the variety defined by the size* 9 *minors of the symmetric flattening* $\phi_{3,3}$ *and by the size* 25 *minors of the* $[(42),(42)]$*-Young flattenings.*

(7) $\sigma_9(v_6(\mathbb{P}^2))$ *is the hypersurface of degree* 10 *defined by* $\det(\phi_{3,3})$*.*

(8) *For* $k \leq 10$*, the variety* $\sigma_k(v_7(\mathbb{P}^2))$ *is an irreducible component of* $YFlat^{2k}_{41,41}(S^7\mathbb{C}^3)$*, which is defined by the size* $(2k+2)$ *sub-Pfaffians of of* $\phi_{41,41}$*.*

(9) $\sigma_{11}(v_7(\mathbb{P}^2))$ *has codimension* 3 *and it is contained in the hypersurface* $YFlat^{22}_{41,41}(S^7\mathbb{C}^3)$ *of degree* 12 *defined by* $\mathrm{Pf}(\phi_{41,41})$*.*

Remark 7.8.2.6. $\sigma_7(v_6(\mathbb{P}^2))$ is the first example where the known equations are not of minimal possible degree.

Remark 7.8.2.7. At this writing, no equations for $\sigma_k(v_9(\mathbb{P}^2))$ are known when $k = 17, 18$. The $k = 18$ case is particularly interesting because $\sigma_{18}(v_9(\mathbb{P}^2))$ is a hypersurface.

Chapter 8

Additional varieties useful for spaces of tensors

This chapter introduces additional varieties that play a role in the study of tensors. Most of these varieties can be defined in the general situation of natural auxiliary varieties constructed from a given projective variety $X \subset \mathbb{P}W$.

To motivate the chapter, consider the set of tensors (resp. symmetric tensors) of border rank at most two: $\sigma_2(Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n)) \subset \mathbb{P}(A_1 \otimes \cdots \otimes A_n)$ (resp. $\sigma_2(v_d(\mathbb{P}V)) \subset \mathbb{P}S^dV$). The points in these varieties having rank less than two are $Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n)$ and $v_d(\mathbb{P}V)$. What are the points of rank greater than two? All points of rank greater than two lie on a proper subvariety, called the tangential variety, respectively denoted $\tau(Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n))$ and $\tau(v_d(\mathbb{P}V))$. Informally, for $X \subset \mathbb{P}W$, $\tau(X) \subset \mathbb{P}W$ is the set of points on some embedded tangent line to X. In the case of symmetric tensors, all the points on $\tau(v_d(\mathbb{P}V))\backslash v_d(\mathbb{P}V)$ have rank exactly d (see §9.2.2). So for symmetric tensors, equations for the tangential variety provide a test for a symmetric tensor of border rank two to have rank greater than two (in fact d). In the case of tensors, the situation is more subtle; see §10.10. Tangential varieties are discussed in §8.1.

Another question that arises in applications is as follows. Over $\mathbb{C}$, there is a unique typical rank for a tensor or symmetric tensor; i.e., for any linear measure on $A_1 \otimes \cdots \otimes A_n$ or S^dV, there is an r_0 such that if one selects a tensor at random, the tensor will have rank r_0 with probability one. Over

$\mathbb{R}$, there can be several typical ranks, that is, several numbers $r_0, \ldots, r_k$ such that if one selects a tensor at random, there is a nonzero probability of the tensor having rank r_j, $0 \leq j \leq k$. The smallest of these values is always the typical rank over $\mathbb{C}$. The other values are not well understood. In the cases they are understood there is a hypersurface (or hypersurfaces) whose equation(s) separates the different regions. When $W = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ (resp. $S^3\mathbb{C}^2$), this hypersurface is best interpreted as the dual variety of $Seg(\mathbb{P}^{1*} \times \mathbb{P}^{1*} \times \mathbb{P}^{1*})$ (resp. $v_3(\mathbb{P}^{1*})$). Dual varieties are discussed in §8.2. Dual varieties can be used to stratify spaces, in particular spaces of tensors. This stratification is explained in §8.5.

A cousin of the dual variety of the two-factor Segre, the zero set of the Pascal determinant, is briefly mentioned in §8.3. It arises in quantum information theory and complexity theory.

Invariants from local differential geometry to study projective varieties are presented in §8.4. These are the Fubini forms, including the second fundamental form.

In §8.6, the Chow variety of polynomials that decompose into a product of linear factors is discussed, along with its equations. Chow varieties play an important role in applications. I present a derivation of Brill's equations that includes E. Briand's filling of a gap in earlier presentations.

Finally, in §8.7, the Fano variety of lines on a projective variety is defined. It arises in many situations, and its study generalizes the classical linear algebra problem of finding spaces of matrices of bounded rank. Results on matrices of bounded and constant rank are summarized for easy reference.

As the study of tensors becomes more sophisticated, I expect that additional G-varieties will play a role in the study. Such varieties will be natural varieties in the sense that they can be constructed geometrically from the original variety $Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n)$ in the case of tensors and $v_d(\mathbb{P}V)$ in the case of symmetric tensors. Throughout this chapter I mention several additional varieties that one can construct naturally from a projective variety $X \subset \mathbb{P}W$, which will be G-varieties if X itself is.

8.1. Tangential varieties

8.1.1. Definition of the tangential variety. Recall the definition of the affine tangent space at a smooth point x of a variety $X \subset \mathbb{P}V$, $\hat{T}_x X \subset V$ from §4.6. Let $X^n \subset \mathbb{P}V = \mathbb{P}^{n+a}$ be a smooth variety. Define $\tau(X) \subset \mathbb{P}V$ by

$$\tau(X) := \bigcup_{x \in X} \mathbb{P}\hat{T}_x X,$$

the *tangential variety* of X. Note that the name requires justification—to justify it, observe that $\tau(X)$ is the image of the bundle $\mathbb{P}(\hat{T}X)$ under the map $(x,[v]) \mapsto [v]$, which shows that it is indeed a variety, and in fact, an irreducible variety.

It is possible to define the tangential variety of an algebraic variety that is not necessarily smooth. While there are several possible definitions, the best one appears to be the union of points on *tangent stars* $T^\star_x X$. Intuitively, $T^\star_x X$ is the limit of secant lines. More precisely, let $x \in X$. Then $\mathbb{P}^1_*$ is a line in $T^\star_x X$ if there exist smooth curves $p(t), q(t)$ on X such that $p(0) = q(0) = x$ and $\mathbb{P}^1_* = \lim_{t\to 0} \mathbb{P}^1_{p(t)q(t)}$, where $\mathbb{P}^1_{pq}$ denotes the projective line through p and q. (One takes the limit in $\mathbb{G}(\mathbb{P}^1, \mathbb{P}V) = G(2,V)$.) $T^\star_x X$ is the union of all points on all $\mathbb{P}^1_*$'s at x. In general $\overline{\tau(X_{smooth})} \subseteq \tau(X)$, and strict containment is possible.

An important theorem due to Fulton and Hansen [**134**] implies that if $X^n \subset \mathbb{P}V$ is any variety, then either $\dim \sigma(X) = 2n+1$ and $\dim \tau(X) = 2n$ or $\sigma(X) = \tau(X)$. This is an application of their famous connectedness theorem [**134**].

8.1.2. Exercises on tangential varieties.

(1) Show that $\dim \tau(X) \leq \min\{2n, n+a\}$ and that one expects equality to hold in general.

(2) Let $C \subset \mathbb{P}V$ be a smooth curve that is not a union of lines. Show that $\tau(C)$ has dimension two.

(3) Let X be a variety and let $x \in X_{smooth}$. Show that $T^\star_x X = \mathbb{P}\hat{T}_x X$.

(4) Show that tangential varieties of smooth irreducible varieties are irreducible.

8.1.3. Ranks of points on $\tau(X)$. Let $X \subset \mathbb{P}V$ be a variety. Then, essentially by definition, $\{x \in \sigma_2(X) \mid \mathbf{R}_X(x) > 2\} \subseteq \tau(X)\backslash X$.

The rank of a point on $\tau(X)$ can already be the maximum, as is the case for points on a rational normal curve $v_d(\mathbb{P}^1)$ (see §9.2.2), where the rank of a point on $\tau(v_d(\mathbb{P}^1))$ is the maximum d. For the n-factor Segre, the rank of a point of $\tau(X)\backslash X$ can be anywhere from 2 to $n-1$.

8.1.4. Equations for tangential varieties. For $Seg(\mathbb{P}A \times \mathbb{P}B)$, $v_2(\mathbb{P}V)$ and $G(2,V)$, the Fulton-Hansen theorem implies the tangential varieties are just the secant varieties, so we have already seen their equations.

Just as with secant varieties, inheritance (see §7.4) holds for tangential varieties of Segre and Veronese varieties, so we can reduce the study of equations to "primitive" cases.

Theorem 8.1.4.1. *Let $d > 3$. Set-theoretic defining equations for $\tau(v_d(\mathbb{P}V))$ are given by (any of) the modules $S_{d-2k,2k}V^* \subset S^2(S^dV^*)$ for $k > 1$ and the images of the modules $\Lambda^3V^*\otimes\Lambda^3(S^{d-1}V^*)$ and $\Lambda^3(S^2V^*)\otimes\Lambda^3(S^{d-2}V^*)$ in $S^3(S^dV^*)$.*

Proof. The second set of equations reduces to the case of $\tau(v_d(\mathbb{P}^1))$. The third reduces the problem further to a subvariety of $\sigma_2(v_d(\mathbb{P}^1))$ by Theorem 7.3.3.2. Finally, $\tau(v_d(\mathbb{P}^1)$ is the only GL_2-subvariety of $\sigma_2(v_d(\mathbb{P}^1))$ not equal to $v_d(\mathbb{P}^1)$, so any additional module of equations vanishing on it in degree two will be enough to cut it out set-theoretically. (Of course to generate the ideal, one needs at least all such modules in degree two.) $\square$

Theorem 8.1.4.2. *The ideal of $\tau(Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)) \subset \mathbb{P}(A\otimes B\otimes C)$ is generated in degrees 3 and 4 by the modules $\Lambda^3A^*\otimes\Lambda^3(B\otimes C)^*$, $\Lambda^3B^*\otimes\Lambda^3(A\otimes C)^*$, $\Lambda^3C^*\otimes\Lambda^3(A\otimes B)^*$ (minus redundancies), and $S_{22}A^*\otimes S_{22}B^*\otimes S_{22}C^*$.*

Proof. For the case $\mathbf{a} = \mathbf{b} = \mathbf{c} = 2$, $\tau(Seg(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1))$ is a hypersurface. The first possible (trivial) module is $S_{22}A^*\otimes S_{22}B^*\otimes S_{22}C^* \subset S^4(A\otimes B\otimes C)^*$, as there is no trivial representation in lower degree. The variety $\tau(Seg(\mathbb{P}A\times \mathbb{P}B\times\mathbb{P}C))$ is quasi-homogeneous, it is the orbit closure of, e.g., $[a_1\otimes b_1\otimes c_1 + a_1\otimes b_1\otimes c_2 + a_1\otimes b_2\otimes c_1 + a_2\otimes b_1\otimes c_1]$. But this vector raised to the fourth power will have no term of weight zero, and thus it will pair to be zero with a weight zero vector. The general case follows by inheritance and Theorem 7.1.1.2. $\square$

In [**209**, Conj. 7.6] some generators of $I(\tau(Seg(PA_1 \times \cdots \times PA_n))$ are found: the quadrics in $S^2(A_1\otimes \cdots \otimes A_m)^*$ which have at least four Λ^2 factors, the cubics with four $S_{2,1}$ factors and all other factors $S_{3,0}$, and the quartics with three $S_{2,2}$'s and all other factors $S_{4,0}$. It is also shown that the ideal is generated in degrees less than or equal to six. It was conjectured that these modules generate the ideal. This was shown to hold set-theoretically by L. Oeding in [**252**].

8.1.5. Higher tangential varieties.

Just as $\tau(X) \subset \sigma_2(X)$, one can define varieties of different types of limiting curves inside higher secant varieties. The simplest of them is as follows. Let $X \subset \mathbb{P}V$ be a variety, and let $x_t \subset \hat{X}$ be a smooth curve.

Exercise 8.1.5.1: Show that $x_0 + x_0' + x_0'' + \cdots + x_0^{(k)} \in \hat{\sigma}_{k+1}(X)$.

Define for a smooth variety $X \subset \mathbb{P}V$,

$$\tau_r(X) := \overline{\{x_0 + x_0' + x_0'' + \cdots + x_0^{(r-1)} \mid x_t \subset \hat{X}\backslash 0 \text{ is a smooth curve}\}}.$$

Note that $\tau_r(X)$ is not to be confused with the osculating variety $\tau^{(r)}(X)$ of §8.4.3, which is usually of much larger dimension and $\tau_r(X) \subset \tau^{(r)}(X)$.

8.2. Dual varieties

8.2.1. Singular hypersurfaces and the dual of the Veronese. Recall the Veronese variety $X = v_d(\mathbb{P}V) \subset \mathbb{P}(S^dV)$. Let $v_d(\mathbb{P}V)^\vee \subset \mathbb{P}(S^dV^*)$ denote the set of polynomials whose zero sets $Z_P \subset \mathbb{P}V$ are singular varieties. $v_d(\mathbb{P}V)^\vee$ is sometimes called the *discriminant hypersurface*. I now interpret $v_d(\mathbb{P}V)^\vee$ in a way that will lead to the definition of a dual variety of an arbitrary variety.

Let $P \in S^dV^*$, and let $Z_P \subset \mathbb{P}V$ be the corresponding hypersurface. By definition, $[x] \in Z_P$ if and only if $P(x) = 0$, i.e., $\overline{P}(x, \dots, x) = \overline{P}(x^d) = 0$ considering $\overline{P}$ as a multilinear form. Similarly, $[x] \in (Z_P)_{sing}$ if and only if $P(x) = 0$, and $dP_x = 0$, i.e., for all $y \in W$, $0 = dP_x(y) = \overline{P}(x, \dots, x, y) = \overline{P}(x^{d-1}y)$. Recall that $\hat{T}_{[x]}v_d(\mathbb{P}V) = \{x^{d-1}y \mid y \in V\}$. Thus

$$v_d(\mathbb{P}V)^\vee = \{[P] \in \mathbb{P}S^dV^* \mid \exists x \in v_d(\mathbb{P}V),\ \hat{T}_x v_d(\mathbb{P}V) \subset P^\perp\},$$

where $P^\perp \subset S^dV$ is the hyperplane annihilating the vector P. (Here P is considered as a linear form on S^dV.)

8.2.2. Dual varieties in general. For a smooth variety $X \subset \mathbb{P}V$, define

$$X^\vee := \{H \in \mathbb{P}V^* \mid \exists x \in X,\ \mathbb{P}\hat{T}_xX \subseteq H\}.$$

Here I abuse notation by using H to refer both to a point in $\mathbb{P}V^*$ and a hyperplane $\mathbb{P}(\ker \hat{H}) \subset \mathbb{P}V$. $X^\vee$ is called the *dual variety* of X. It can be defined even when X is singular by

$$\begin{aligned} X^\vee :&= \overline{\{H \in \mathbb{P}V^* \mid \exists x \in X_{smooth},\ \mathbb{P}\hat{T}_xX \subseteq H\}} \\ &= \{H \in \mathbb{P}V^* \mid X \cap H \text{ is not smooth}\}. \end{aligned}$$

Consider $\mathbb{P}N_x^*X \subset \mathbb{P}V^*$ as $\mathbb{P}(\hat{T}_xX^\perp)$, which gives rise to the geometric interpretation

$$\mathbb{P}N_x^*X = \{H \in \mathbb{P}V^* \mid\ \mathbb{P}\hat{T}_xX \subseteq H\}.$$

Remark 8.2.2.1. The dual variety of $Seg(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)$ was studied by Cayley. Its equation is a quartic polynomial called the *hyperdeterminant*, which we will encounter in §8.2.7. See [**141**] for an extensive discussion of the hyperdeterminant.

Remark 8.2.2.2. Dual varieties are related to studying typical ranks over $\mathbb{R}$; see [**100**].

8.2.3. Exercises on dual varieties.

(1) Let $X = Seg(\mathbb{P}A \times \mathbb{P}B) \subset \mathbb{P}(A \otimes B)$. Show that $X^\vee$ admits the geometric interpretation of the linear maps $A^* \to B$ not of maximal rank. (In particular, if $\mathbf{a} = \mathbf{b}$, $X^\vee$ is the hypersurface given by the determinant being zero.)

(2) Show that if X has codimension s, then $X^\vee$ is uniruled by $\mathbb{P}^{s-1}$'s. (A variety Z is *uniruled* by $\mathbb{P}^k$'s if for all $z \in Z$ there exists some $\mathbb{P}^k$ with $z \in \mathbb{P}^k \subset Z$.)

(3) Consider the set

$$\mathcal{I} := \overline{\{(x, H) \in X_{smooth} \times \mathbb{P}V^* \mid \mathbb{P}\hat{T}_x X \subseteq H\}} \subset \mathbb{P}V \times \mathbb{P}V^*$$

and note that its images under the two projections are respectively X and $X^\vee$. Show that

$$\mathcal{I} = \overline{\{(x, H) \in \mathbb{P}V \times (X^\vee)_{smooth} \mid \mathbb{P}\hat{T}_H X^\vee \subseteq x\}} \subset \mathbb{P}V \times \mathbb{P}V^*$$

and thus $(X^\vee)^\vee = X$. (This is called the *reflexivity theorem* and dates back to C. Segre.)

8.2.4. B. Segre's dimension formula. Let W be a complex vector space of dimension N, and $P \in S^d W^*$ a homogeneous polynomial of degree d. Let $Z(P) \subset \mathbb{P}W$ denote the hypersurface defined by P. If P is irreducible, then $Z(P)$ and its dual variety $Z(P)^\vee$ are both irreducible. The Segre dimension formula [**286**] states that

$$\dim Z(P)^\vee = \operatorname{rank}(P_{d-2,2}(w^{d-2})) - 2,$$

where w is a general point of the affine cone over $Z(P)$. Here $(P_{d-2,2}(w^{d-2})) \in S^2 W^*$.

Segre's formula implies that $Z(P)^\vee$ has dimension less or equal to k if and only if, for any $w \in W$ such that $P(w) = 0$, and any $(k+3)$-dimensional subspace F of W,

$$\det(P_{d-2,2}(w^{d-2})|_F) = 0.$$

Equivalently (assuming P is irreducible), for any such subspace F, the polynomial P must divide $\det(P_{d-2,2}|_F) \in S^{(k+3)(d-2)} W^*$, where det is evaluated on the $S^2 W^*$ factor in $S^2 W^* \otimes S^{d-2} W^*$.

Note that for polynomials in $N' < N$ variables, the maximum rank of $P_{d-2,2}$ (considered as an element of $S^2 W^*$) is N'; so, in particular, $\det(P_{d-2,2}|_F)$ will vanish on any F of dimension $N' + 1$.

These observations give rise to explicit equations for the variety of hypersurfaces of degree d with degenerate duals; see [**213**] for details.

8.2.5. Application: The group preserving the polynomial $\det_n$**.** By Exercises 8.2.3, the dual variety of the hypersurface $\{\det_n = 0\} \subset \mathbb{P}^{n^2-1}$ is the Segre variety. Thus if $g \in GL_{n^2} = GL(A{\otimes}B)$ preserves $\{\det_n = 0\}$, it must also preserve $Seg(\mathbb{P}A \times \mathbb{P}B)$. This fact enables one to determine the connected component of the identity of the group preserving $\det_n$.

If X is a G-variety, $[x] \in X$, and $v \in \mathfrak{g}$, then $v.x \in \hat{T}_x X$. One can use this to determine the Lie algebra of the subgroup of $GL(A{\otimes}B)$ preserving $Seg(\mathbb{P}A \times \mathbb{P}B)$. Since $GL(A) \times GL(B)$ preserves the variety, it remains to show that no larger connected group does. Let e_i be a basis of A, with dual basis e^i, write $e^i_j := e^i{\otimes}e_j$, and similarly let f_j be a basis of B. A basis of $\mathfrak{gl}(A{\otimes}B) = (A{\otimes}B){\otimes}(A^*{\otimes}B^*) = (A{\otimes}A^*){\otimes}(B{\otimes}B^*)$ is given by $e^i_j{\otimes}f^s_t$, $1 \leq i, j, s, t \leq n$. The action of $e^i_j{\otimes}f^s_t$ on $e_k{\otimes}f_u$ is

$$(e^i_j{\otimes}f^s_t).e_k{\otimes}f_u = \delta^i_k\delta^s_u e_j{\otimes}f_t.$$

Recall that $\hat{T}_{[e_i \otimes f_s]}Seg(\mathbb{P}A \times \mathbb{P}B) = e_i{\otimes}B + A{\otimes}f_s$. Let $v = c^{jt}_{is}e^i_j{\otimes}f^s_t \in \mathfrak{gl}(A{\otimes}B)$ be an arbitrary element. Calculate

$$v.e_k{\otimes}f_u = c^{jt}_{ku}e_j{\otimes}e_t.$$

Exercise 8.2.5.1: Show that in order that $v.(e_k{\otimes}f_u) \in \hat{T}_{[e_k \otimes f_u]}Seg(\mathbb{P}A \times \mathbb{P}B)$, $v \in \mathfrak{gl}(A){\otimes}\operatorname{Id}_B + \operatorname{Id}_A{\otimes}\mathfrak{gl}(B)$. Here $\operatorname{Id}_A = \sum_i e^i_i$ and similarly for B.

Exercise 8.2.5.1 shows that the connected component of the identity preserving $\{\det_n = 0\} \subset \mathbb{P}(A{\otimes}B)$ is the image of $GL(A) \times GL(B)$ in $GL(A \otimes B)$.

Exercise 8.2.5.2: Show that this image is $SL(A) \times SL(B)/\mu_n$, where μ_n is the group of n-th roots of unity.

The precise subgroup of $GL(A{\otimes}B)$ preserving $\det_n$ is $(SL(A) \times SL(B))/\mu_n \ltimes \mathbb{Z}_2$, where $\mathbb{Z}_2$ acts as $A \mapsto A^T$, which is due to Frobenius [**132**]. See [**154**] for a more modern treatment.

8.2.6. Dual varieties and projections. Recall from §4.9.1 that a projection $\hat{\pi}_W : V \to V/W$ gives rise to a rational map $\pi_W : \mathbb{P}V \dashrightarrow \mathbb{P}(V/W)$ defined on $\mathbb{P}V \backslash \mathbb{P}W$.

Proposition 8.2.6.1. *Let $X \subset \mathbb{P}V$ be a variety and let $W \subset V$ be a linear subspace. Assume that $X \not\subset \mathbb{P}W$. Then*

$$\pi_W(X)^\vee \subseteq \mathbb{P}W^\perp \cap X^\vee.$$

Equality holds if $\pi_W(X) \simeq X$.

Proof. I give the proof in the case when X is smooth. Let $x \in X$ be a point where $\pi_W(x)$ is defined; then

$$\hat{T}_{\pi_W(x)}\pi_W(X) = \hat{T}_x X \bmod W.$$

So $\hat{T}_x X \subset \hat{H}$ implies that $\hat{T}_{\pi_W(x)}\pi_W(X) \subset \hat{H} \bmod W$, proving the proposition for hyperplanes tangent to x. But now π and its differential are defined on a Zariski open subset of X, so the other points can be accounted for by taking limits. In these limits one may gain extra hyperplanes, but if $\pi_W(X) \simeq X$, then π and its differential is defined at all points of X, so there is no need to take limits. For the case where X is singular, one must study the projection of limiting planes as well; see, e.g., [**141**, p. 31]. $\square$

8.2.7. Hyperdeterminants. An often studied example is $Seg(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1))^\vee = \tau(Seg(\mathbb{P}^{1*} \times \mathbb{P}^{1*} \times \mathbb{P}^{1*}))$. It is a hypersurface that is the zero set of $S_{22}\mathbb{C}^2 \otimes S_{22}\mathbb{C}^2 \otimes S_{22}\mathbb{C}^2 \subset S^4(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2)$, which is (up to scale) *Cayley's hyperdeterminant*; see [**141**].

In general, the *hyperdeterminant* hypersurface is defined to be $Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n)^\vee \subset \mathbb{P}(A_1^* \otimes \cdots \otimes A_n^*)$ whenever this is a hypersurface. It is a hypersurface if and only if $\mathbf{a}_j \leq \sum_{i \neq j} \mathbf{a}_i$ for all $1 \leq j \leq n$; see [**141**, p. 466, Thm. 1.3]. Its equation has a degree that grows very fast. For example, let $\dim A_i = \mathbf{a}$ [**141**, p. 456, Cor. 2.9]; then

$$\deg(Seg(\mathbb{P}A_1 \times \mathbb{P}A_2 \times \mathbb{P}A_3))^\vee = \sum_{0 \leq j \leq (\mathbf{a}-1)/2} \frac{(j+\mathbf{a})! 2^{\mathbf{a}-1-2j}}{(j!)^3 (\mathbf{a}-1-2j)!}.$$

For $Seg(\mathbb{P}^1 \times \cdots \times \mathbb{P}^1)^\vee$ the equation already has a rapidly growing degree; in particular, for four factors the degree is 24, whereas the lowest degree of an invariant polynomial is two.

8.3. The Pascal determinant

In light of the high degree of the hyperdeterminant, one might want to study the $GL(A_1) \times \cdots \times GL(A_n)$-invariant hypersurface of lowest degree. This is easiest when $n = 2p$ is even and all vector spaces have the same dimension.

Let $A_1, \ldots, A_{2p}$ be vector spaces of dimension $\mathbf{a}$. Then

$$\Lambda^{\mathbf{a}} A_1 \otimes \cdots \otimes \Lambda^{\mathbf{a}} A_{2p} \subset S^{\mathbf{a}}(A_1 \otimes \cdots \otimes A_{2p}).$$

By choosing a basis element of this one-dimensional vector space, we obtain a degree $\mathbf{a}$ polynomial Pasdet on $(\mathbb{C}^{\mathbf{a}})^{\otimes 2p}$ that is invariant under the action of $SL(A_1) \times \cdots \times SL(A_{2p}) \times \mathfrak{S}_{2p}$, called the *$2p$-dimensional Pascal determinant*. The case $p = 1$ is the usual determinant. The case $p = 2$ is particularly interesting because L. Gurvits [**152**] has shown that the corresponding sequence of polynomials is **VNP**-complete.

If $a^j_1, \dots, a^j_{\mathbf{a}}$ is a basis of A_j and we write $X = x^{i_1,\dots,i_n} a^1_{i_1} \otimes \cdots \otimes a^n_{i_n}$, then
(8.3.1)
$$\mathrm{Pasdet}(X) = \sum \mathrm{sgn}(\sigma_2) \cdots \mathrm{sgn}(\sigma_n) x^{1,\sigma_2(1),\dots,\sigma_n(1)} \cdots x^{\mathbf{a},\sigma_2(\mathbf{a}),\dots,\sigma_n(\mathbf{a})}.$$

The Pascal determinant and other invariant hypersurfaces are studied in [**46**]. The structure of the ring of invariant polynomials is related to the action of the symmetric group. The Pascal detreminant was evidently first defined in 1901 in [**263**].

8.4. Differential invariants of projective varieties

A basic goal in differential geometry is to take derivatives in a geometrically meaningful way. For example, if one differentiates in a coordinate system, one would like to isolate quantities constructed from the derivatives that are invariant under changes in coordinates. For first derivatives this is easy even in the category of manifolds. In projective geometry, taking a second derivative is still straightforward, and the geometrically meaningful information in the second derivatives is contained in the *projective second fundamental form* defined in §8.4.1. Higher derivatives are more complicated—they are encoded in the *Fubini forms* described in §8.4.3. For proofs and more details on the material in this section see [**180**, Chap. 3].

8.4.1. The Gauss map and the projective second fundamental form.

Let $X^n \subset \mathbb{P}V$ be an n-dimensional subvariety or a complex manifold. The *Gauss map* is defined by

$$\begin{aligned} \gamma : X_{smooth} &\to G(n+1, V), \\ x &\mapsto \hat{T}_x X. \end{aligned}$$

Here $\hat{T}_x X \subset V$ is the affine tangent space to X at x, defined in §4.6. Note that if X is a hypersurface, then $\overline{\gamma(X_{smooth})} = X^\vee$.

Let $x \in X_{smooth}$ and consider the differential

$$d\gamma_x : T_x X \to T_{\hat{T}_x X} G(n+1, V) \simeq (\hat{T}_x X)^* \otimes (V/\hat{T}_x X).$$

For all $v \in T_x X$, $\hat{x} \subset \ker d\gamma_x(v)$, where $d\gamma_x(v) : \hat{T}_x X \to V/\hat{T}_x X$. Quotient by $\hat{x}$ to obtain

$$d\underline{\gamma}_x \in T^*_x X \otimes (\hat{T}_x X/\hat{x})^* \otimes V/(\hat{T}_x X) = (T^*_x X)^{\otimes 2} \otimes N_x X.$$

In fact, essentially because mixed partial derivatives commute,

$$d\underline{\gamma}_x \in S^2 T^*_x X \otimes N_x X.$$

Define $II_x = d\underline{\gamma}_x$, the *projective second fundamental form* of X at x. II_x describes how X is moving away from its embedded tangent space to first order at x.

The second fundamental form can be used to calculate dimensions of auxiliary varieties:

Proposition 8.4.1.1 ([**146**]). *If $X \subset \mathbb{P}V$ is a smooth variety, $x \in X_{general}$ and $v \in T_xX$ is a generic tangent vector, then*

$$\dim \tau(X) = n + \dim II(v, T_xX). \tag{8.4.1}$$

For a proof see [**146**] or [**180**, Chap. 3].

The linear map $II_x : N_x^*X \to S^2T_x^*X$ gives rise to a rational map

$$ii : \mathbb{P}(N_x^*X) \dashrightarrow \mathbb{P}(S^2T_x^*X),$$

which is defined off $\mathbb{P}(\ker {}^tII_x)$. The map ii may be thought of as follows. Consider $\mathbb{P}N_x^*X \subset \mathbb{P}V^*$ as $N_x^*X = \hat{x}\otimes\hat{T}_xX^\perp \subset \hat{x}\otimes V^*$. By projectivizing, one may ignore the $\hat{x}$ factor. Now

$$\mathbb{P}N_x^*X = \{H \in \mathbb{P}V^* \mid \hat{T}_xX \subset \hat{H}\} = \{H \in \mathbb{P}V^* \mid (X \cap H) \text{ is singular at } x\}.$$

The map ii takes H to the quadratic part of the singularity of $X \cap H$ at x.

Proposition 8.4.1.2. *Let $H \in X^\vee$ be a general point and let $x \in \mathbb{P}N_H^*X^\vee$ be a general point. Then*

$$N_H^*X^\vee = \ker II_{X,x}(\overline{H}),$$

where $\overline{H} \in N_x^*X$ is a representative of H, and $II_{X,x}(\overline{H}) : T_xX \to T_x^*X$ is considered as a linear map.

Exercise 8.4.1.3: Prove Proposition 8.4.1.2.

Proposition 8.4.1.2 implies:

Proposition 8.4.1.4. *Let $X^n \subset \mathbb{P}V$ be a variety. If $x \in X_{general}$, then $\dim X^\vee = \mathrm{codim}(X) - 1 + \max\mathrm{rank}(q)$, where $q \in II(N_x^*X)$.*

8.4.2. Exercises on dual varieties.

(1) Show that if X is smooth, then one expects $X^\vee$ to be a hypersurface. Let $\delta_*(X) = \dim \mathbb{P}V - 1 - \dim X^\vee$, the *dual defect* of X. In particular, if $X^\vee$ is degenerate with defect δ_*, then $II(N_x^*X) \subset S^2T_x^*X$ is a subspace of bounded rank $n - \delta_*$.

(2) Show that if X is a curve, then $X^\vee$ is a hypersurface.

(3) Use Exercise 8.4.2(1) to show that if X is a smooth variety with degenerate dual variety, then $H \in (X^\vee)_{general}$ and $II_H(N_H^*X^\vee) \subset S^2T_H^*X^\vee$ is a subspace of *constant rank*. See [**174**].

(4) Show that if $X \subset \mathbb{P}V$ is a G-variety, then $X^\vee \subset \mathbb{P}V^*$ is as well.

8.4.3. The Fubini forms. There is a well-defined sequence of ideals defined on the tangent space T_xX given by the *relative differential invariants* F_k, which are equivalence classes of elements of $S^kT^*_xX \otimes N_xX$. Here F_k is an equivalence class of vector spaces of homogeneous polynomials of degree k on T_xX parametrized by the conormal space N^*_xX. A coordinate definition of these invariants is as follows. Take adapted local coordinates (w^α, z^μ), $1 \leq \alpha \leq n$, $n+1 \leq \mu \leq \dim \mathbb{P}V$, on $\mathbb{P}V$ such that $[x] = (0,0)$ and $T_{[x]}X$ is spanned by the first n coordinates $(1 \leq \alpha \leq n)$. Then locally X is given by equations

$$z^\mu = f^\mu(w^\alpha)\,, \tag{8.4.2}$$

and at $(0,0)$ we define

$$F_k\left(\frac{\partial}{\partial w^{i_1}}, \ldots, \frac{\partial}{\partial w^{i_k}}\right) = (-1)^k \sum_\mu \frac{\partial^k f^\mu}{\partial w^{i_1} \cdots \partial w^{i_k}} \frac{\partial}{\partial z^\mu} \operatorname{mod} \textit{stuff}.$$

The invariant F_2 is the second fundamental form II, which is a well-defined tensor. For the higher order invariants, different choices (e.g., of a complement to T_xX in $T_x\mathbb{P}V$) will yield different systems of polynomials, but the new higher degree polynomials will be the old polynomials, plus polynomials in the ideal generated by the lower degree forms (see [**180**], §3.5), which is what I mean by "*stuff*". Let $|F_k| = F_k(N^*_xX) \subseteq S^kT^*_xX$. The ideals $I_{\mathcal{C}_{k,x}}$ in $Sym(T^*_xX)$ generated by $\{|F_2|, \ldots, |F_k|\}$ are well defined. Define the variety $\mathcal{C}_{k,x}$ to be the zero set of $I_{\mathcal{C}_{k,x}}$; it has the geometric interpretation as the *tangent directions to lines in* $\mathbb{P}V$ *having contact to order* k *with* X *at* x.

The quotient of the Fubini form $F_k : S^kT^*_xX \to N_xX$ defined by $\mathbb{FF}_k : S^kT^* \to N_xX/F_{k-1}(S^{k-1}T_xX)$ is well defined as a tensor over X and is called the *k-th fundamental form.* One can refine the fundamental forms by fixing a tangent vector; for example, $\mathbb{F}_{3,v} : T_xX \to N_xX/II(v, T_xX)$, $w \mapsto F_3(v,v,w) \operatorname{mod} II(v, T_xX)$ is well defined.

Fixing $x \in X$, the *osculating sequence*

$$\hat{x} \subset \hat{T}_xX \subset \hat{T}_x{}^{(2)}X \subset \cdots \subset \hat{T}_x{}^{(f)}X = V$$

is defined by $\hat{T}_x{}^{(k)}X = \hat{T}_x{}^{(k-1)}X + \operatorname{image} \mathbb{FF}_{k,x}$.

Methods to compute the dimension of joins and secant varieties in general were discussed in §5.3. Here is a formula for the dimension of $\sigma_2(X)$ in terms of three derivatives at a general point.

Proposition 8.4.3.1 ([**146**]). *Let $X^n \subset \mathbb{P}V$ be a smooth variety, fix $x \in X_{general}$ and $v \in T_xX$ a generic tangent vector. Then*

$$\dim \sigma_2(X) = \begin{cases} n + \dim II(v, T_xX) & \text{if } \mathbb{F}_{3,v}(v) = 0, \\ 2n+1 & \text{if } \mathbb{F}_{3,v}(v) \neq 0. \end{cases} \tag{8.4.3}$$

This dimension test can fail if X is singular, for example $X = \tau(v_d(\mathbb{P}^1))$ has a nondegenerate secant variety when $d > 4$ but $\dim II(v, T_xX) < 2$ for all $(x, v) \in TX$.

Exercise 8.4.3.2: Show that if $X^n \subset \mathbb{P}V$ is a variety such that $\mathbb{FF}_{3,X,x} \neq 0$, then $\dim \sigma_2(X) = 2n + 1$.

Exercise 8.4.3.3: Show that if $\operatorname{codim}(X) > \binom{n+1}{2}$ and X is not contained in a hyperplane, then $\dim \sigma_2(X) = 2n + 1$.

I record the following for reference.

Proposition 8.4.3.4 ([**146, 180, 204**]). (1) *Let $X = G/P \subset \mathbb{P}V$ be a generalized cominuscule variety (see* [**204**]*). (Segre varieties, Veronese varieties, and Grassmannians are all cominuscule.) Then the only nonzero Fubini forms are the fundamental forms.*

(2) $\mathbb{FF}_{k,v_d(\mathbb{P}W),x}(N_x^* v_d(\mathbb{P}W)) = S^k T_x^* v_d(\mathbb{P}W)$ *for $k \leq d$ and is zero for $k > d$.*

(3) $\mathbb{FF}_{k,G(p,W),x}(N_x^* G(p, W)) = \Lambda^k x \otimes \Lambda^k (W/x)^*$ *for $k, \mathbf{w} - k \leq p$ and is zero for larger k.*

(4) *If X is a cominuscule variety in its minimal homogeneous embedding, then the fundamental forms are the successive prolongations (see Definition* 7.5.1.5*) of the second, and $II_{X,x}(N_x^* X) = I_2(\mathcal{C}_x)$, where $\mathcal{C}_x$ denotes the tangent directions to the lines on X through x.*

(5) *For $X = Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n) \subset \mathbb{P}(A_1 \otimes \cdots \otimes A_n)$, $\hat{T}_x X = \hat{x} \oplus A_1' \oplus \cdots \oplus A_n'$, where $x = [a_1 \otimes \cdots \otimes a_n]$, $A_j' = a_1 \otimes a_2 \otimes \cdots \otimes a_{j-1} \otimes (A_j / \langle a_j \rangle) \otimes a_{j+1} \otimes \cdots \otimes a_n$ (direct sum is a slight abuse of notation as the spaces to the right of $\hat{x}$ are only defined modulo $\hat{x}$), and $II(N_x^* X) = \bigoplus_{i \neq j} (A_i')^* \otimes (A_j')^*$.*

(6) *If $X = \overline{G/H} \subset \mathbb{P}V$ is quasi-homogeneous and one takes a point in the open orbit, then the fundamental forms correspond to the action of the universal enveloping algebra of $\mathfrak{g}$ on V. For example, if $Y_1, Y_2 \in \mathfrak{g}$ and so $Y_1.v, Y_2.v \in \hat{T}_{[v]} X$, then $II(Y_1.v, Y_2.v) = Y_1 Y_2.v \bmod \hat{T}_{[v]} X$, where I abuse notation identifying $Y_j.v$ with a corresponding element of $T_{[v]} X$.*

Exercise 8.4.3.5: Let $X = Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n) \subset \mathbb{P}(A_1 \otimes \cdots \otimes A_n)$. What is $III(N_x^* X)$? Conclude that if $n > 2$, then $\sigma_2(X)$ is nondegenerate.

The tangential variety generalizes to the *osculating varieties*

$$\tau^{(k)}(X) := \overline{\bigcup_{x \in X_{smooth}} \hat{T}_x^{(k)} X}.$$

8.4.4. A normal form for curves. In this subsection I give an explicit description of curves on submanifolds of a projective space. This description will be used to describe possible limits of secant 3-planes in Theorem 10.8.1.3.

Let $X^n \subset \mathbb{P}^N = \mathbb{P}V$ be a submanifold and take adapted coordinates as in (8.4.2). Let $x(t) \subset X$ be a curve with $x(0) = x$. Expand it out in a Taylor series

$$x(t) = x_0 + tx_1 + t^2\tilde{x}_2 + t^3\tilde{x}_3 + \cdots .$$

Lemma 8.4.4.1 ([**201**]). *The terms in the Taylor series satisfy*

$$\begin{aligned}
&x_1 \in \hat{T}_x X,\\
&\tilde{x}_2 = II(x_1, x_1) + x_2 \quad \text{with } x_2 \in \hat{T}_x X,\\
&\tilde{x}_3 = F_3(x_1, x_1, x_1) + 3II(x_1, x_2) + x_3 \quad \text{with } x_3 \in \hat{T}_x X,\\
&\vdots\\
&\tilde{x}_k = \sum_{s=0}^{k} \Big(\frac{k!}{s_1! \cdots s_k!}\Big) F_s(x_1^{s_1}, x_2^{s_2}, \ldots, x_k^{s_k}),
\end{aligned}$$

where in the last equation $s_1 + 2s_2 + \cdots + ks_k = k$, $\sum s_i = s$, $x_k \in \hat{T}_x X$, and with the convention $F_1(x_k) = x_k$ and $F_2 = II$.

The proof proceeds by expanding (8.4.2) in Taylor series and restricting to the curve.

8.5. Stratifications of $\mathbb{P}V^*$ via dual varieties

In addition to the stratification of spaces of tensors and polynomials by border rank, and the classification of tensors and polynomials by their rank, one can organize spaces of tensors and polynomials respectively using dual varieties of Segre and Veronese varieties. A multistratification of $\mathbb{P}V^*$ may be obtained from the singularities of $X^\vee$. When $X = v_d(\mathbb{P}W)$, this is the classical subject of studying singularities of hypersurfaces. Weyman and Zelevinsky [**332**] initiated a study of this stratification in the case $X = Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n)$. In what follows, assume that $X^\vee$ is a hypersurface.

8.5.1. Stratifications for any variety. Define

$$X^\vee_{cusp} := \{H \in \mathbb{P}V^* \mid \exists x \in X \text{ such that } \mathbb{P}\hat{T}_x X \subset H \text{ and } \operatorname{rank}(II_{X,x}(\overline{H})) < n\};$$

that is, $X^\vee_{cusp}$ are the points $H \in X^\vee$ where $X \cap H$ has a singular point with singularity worse than a simple quadratic singularity (called a singularity of type A_1). Here $\overline{H} \in N^*_x X$ denotes a vector corresponding to H.

Similarly, define

$$X^\vee_{node,2} := \overline{\{H \in \mathbb{P}V^* \mid \exists x, y \in X \text{ such that } \mathbb{P}\hat{T}_x X, \mathbb{P}\hat{T}_y X \subset H\}}.$$

More generally, define $X^\vee_{cusp,k}$, $X^\vee_{node,k}$ respectively as the points in $X^\vee$ where there is a singularity of type at least A_{k+1} and as the Zariski closure of the points of $X^\vee$ tangent to at least k points.

Proposition 8.5.1.1. $X^\vee_{node,k} = \sigma_k(X)^\vee$.

Proof. This is an immediate consequence of Terracini's lemma in §5.3. $\square$

These varieties are generally of small codimension in $X^\vee$. One can continue the stratification by defining

$$X^\vee_k := \{H \in X^\vee \mid \dim\{x \in X \mid \mathbb{P}\hat{T}_x X \subset H\} \geq k\}.$$

Note that for any variety $Y \subset \mathbb{P}W$, one can define $Y_{sing,k}$ iteratively by $Y_{sing,0} = Y$ and $Y_{sing,k} := (Y_{sing,k-1})_{sing}$. The precise relationship between this stratification and the above for dual varieties is not clear.

By definition, $X^\vee_{sing} = X^\vee_{cusp} \cup X^\vee_{node,2}$. It is possible that one of the components is contained in the other or that they coincide (see Theorem 8.5.2.1 below).

Now let $X = G/P \subset \mathbb{P}V$ be homogeneous. Then the orbit of a highest weight vector in $\mathbb{P}V^*$ is a variety $X_* \subset \mathbb{P}V^*$ that is projectively isomorphic to X. Note that $X_* \subset X^\vee$ and in fact it is the most singular stratum.

Example 8.5.1.2. Let $X = G(k, W) \subset \mathbb{P}\Lambda^k W$. Then $X_* = X^\vee_{k(n-2k)}$.

Exercise 8.5.1.3: Let $X = v_d(\mathbb{P}W) \subset \mathbb{P}S^d W$. Determine the k such that $X_* = X^\vee_k$.

Example 8.5.1.4. If $X = Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n) \subset \mathbb{P}(A_1 \otimes \cdots \otimes A_n)$, with $\dim X = N$, then $X_* = X^\vee_{N-2}$, where if $H = e^1 \otimes \cdots \otimes e^n$, then the set of points to which H is tangent is

$$\Big\{ \bigcup_{i<j} Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_{i-1} \times \mathbb{P}(e^i)^\perp \times \mathbb{P}A_{i+1} \times \cdots \times \mathbb{P}A_{j-1} \\ \times \mathbb{P}(e^j)^\perp \times \mathbb{P}A_{j+1} \times \cdots \times \mathbb{P}A_n) \Big\}.$$

Note that, unlike the previous examples, here the set of points of X to which H is tangent has numerous components.

8.5.2. Singularities of discriminant hypersurfaces. In [**332**], J. Weyman and A. Zelevinski determined the codimension one components of the singular set of $X^\vee$ when $X = Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n)$, and $X^\vee$ is a hypersurface. By §8.2.7, if $\mathbf{a}_1 \leq \cdots \leq \mathbf{a}_n$, then $\mathbf{a}_n \leq \sum_{j<n} \mathbf{a}_j$. They call the case of equality *boundary format* and the case of strict inequality, *interior format*. Here are their main results:

Theorem 8.5.2.1 ([**332**, Thms. 0.1 and 0.5])**.** *Write* $(\mathbf{a}_1, \dots, \mathbf{a}_n)$ *for the dimensions of the* A_i*.*

(1) *If the format is interior and* $(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) \neq (2,2,2)$*, then* $X^\vee_{cusp}$ *is an irreducible hypersurface in* $X^\vee$*.*
(2) *In the case of* $(2,2,2)$ *there are three (isomorphic) irreducible components of* $X^\vee_{cusp}$*, each of codimension two.*
(3) *If the format is boundary, then* $X^\vee_{cusp}$ *is irreducible of codimension two.*
(4) *If the format is boundary,* $(X^\vee)_{sing}$ *is an irreducible hypersurface in* $X^\vee$ *coresponding to* $X^\vee_{node,2}$*.*
(5) *If the format is interior,* $(X^\vee)_{sing}$ *has two irreducible components of codimension one, corresponding to* $X^\vee_{cusp}$ *and* $X^\vee_{node,2}$*, with the following exceptional formats:* $(2,2,2)$, $(2,3,3)$, $(3,m,m)$, $(2,2,m,m)$ *where there are more components (described precisely in* [**332**]*).*

8.6. The Chow variety of zero cycles and its equations

The *Chow variety of zero cycles in* $\mathbb{P}V$ is the set of polynomials of degree d on V^* that decompose into a product of linear forms, i.e., the set of hypersurfaces of degree d that are a union of hyperplanes (possibly counted with multiplicity). It may also be described as the projection of $Seg(\mathbb{P}V \times \cdots \times \mathbb{P}V) \subset \mathbb{P}V^{\otimes d}$ to the quotient of $V^{\otimes d}$ by the $GL(V)$-complement to S^dV. Denote it by $Ch_d(V) \subset \mathbb{P}S^dV$. Set-theoretic defining equations for $Ch_d(V)$ are given in §8.6.2. The generators of the ideal are not known, although there is a description of the ideal as the kernel of an explicitly given map described in §8.6.1.

The Chow variety plays a role in complexity theory: if $d \leq \dim V$, it is the variety of all polynomials that can be expressed as monomials in some basis. The secant varieties of Chow varieties provide a measure of complexity of a polynomial: one can define the *Chow border rank* of $f \in S^dV^*$ to be the smallest r such that $f \in \sigma_r(Ch_d(V^*))$.

8.6.1. The ideal of the Chow variety. Consider the following map (FH stands for Foulkes-Howe; see Remark 8.6.1.3 below). First include $S^\delta(S^dV) \subset V^{\otimes d\delta}$. Next, regroup the copies of V and symmetrize the blocks to $(S^\delta V)^{\otimes d}$. Finally, thinking of $S^\delta V$ as a single vector space, symmetrize again to land in $S^d(S^\delta V)$.

For example, putting subscripts on V to indicate position:

$$\begin{aligned} S^2(S^3V) \subset V^{\otimes 6} &= V_1 \otimes V_2 \otimes V_3 \otimes V_4 \otimes V_5 \otimes V_6 \\ &= (V_1 \otimes V_4) \otimes (V_2 \otimes V_5) \otimes (V_3 \otimes V_6) \\ &\to S^2V \otimes S^2V \otimes S^2V \\ &\to S^3(S^2V). \end{aligned}$$

Note that $FH_{\delta,d}$ is a linear map, in fact a $GL(V)$-module map.

Exercise 8.6.1.1: Show that on products of d-th powers one has

$$FH_{\delta,d}(x_1^d \cdots x_\delta^d) = (x_1 \cdots x_\delta)^d.$$

Proposition 8.6.1.2 ([**39**])**.** $\ker FH_{\delta,d} = I_\delta(Ch_d(V^*))$.

Proof. Say $P = \sum_j x_{1j}^d \cdots x_{\delta j}^d$. Then Let $\ell^1, \dots, \ell^d \in V^*$.

$$\begin{aligned} P(\ell^1 \cdots \ell^d) &= \langle \overline{P}, (\ell^1 \cdots \ell^d)^\delta \rangle \\ &= \sum_j \langle x_{1j}^d \cdots x_{\delta j}^d, (\ell^1 \cdots \ell^d)^\delta \rangle \\ &= \sum_j \langle x_{1j}^d, (\ell^1 \cdots \ell^d) \rangle \cdots \langle x_{\delta j}^d, (\ell^1 \cdots \ell^d) \rangle \\ &= \sum_j \prod_{s=1}^d \prod_{i=1}^\delta x_{ij}(\ell_s) \\ &= \sum_j \langle x_{1j} \cdots x_{\delta j}, (\ell^1)^\delta \rangle \cdots \langle x_{1j} \cdots x_{\delta j}, (\ell^d)^\delta \rangle \\ &= \langle FH_{\delta,d}(P), \ell_1^\delta \cdots \ell_d^\delta \rangle. \end{aligned}$$

If $FH_{\delta,d}(P)$ is nonzero, there will be some monomial it will pair with to be nonzero. On the other hand, if $FH_{\delta,d}(P) = 0$, then P annihilates all points of $Ch_d(V^*)$. $\square$

Remark 8.6.1.3. The module $\ker(FH_{\delta,d})$ is not understood. Howe [**169**, p. 93] wrote "it is reasonable to expect" that $FH_{\delta,d}$ is injective for $\delta \leq d$ and surjective for $\delta \geq d$. This expectation/conjecture was disproved by Müller and Neunhöffer [**237**], who showed that FH_5 is not injective, so $Ch_5(V)$ has equations of degree 5 as long as $\dim V \geq 5$. Brion [**38**] proved an asymptotic version of the conjecture, namely that $S_m(S_{np}V) \subset S_n(S_{mp}V)$ for $n >> m$. (Howe's conjecture is a generalization of a conjecture of Foulkes [**127**], which is still open.)

8.6.2. Brill's equations. Consider the map $\pi_{d,d} : S^dV \otimes S^dV \to S_{d,d}V$ obtained by projection. (By the Pieri rule (Theorem 6.7.2.1), $S_{d,d}V \subset S^dV \otimes S^dV$ with multiplicity one.)

Lemma 8.6.2.1. *Let $\ell \in V$, $f \in S^dV$. Then $f = \ell h$ for some $h \in S^{d-1}V$ if and only if $\pi_{d,d}(f \otimes \ell^d) = 0$.*

Proof. Since $\pi_{d,d,}$ is linear, it suffices to prove the lemma when $f = \ell_1 \cdots \ell_d$. In that case $\pi_{d,d}(f \otimes \ell^d)$, up to a constant, is $(\ell_1 \wedge \ell) \cdots (\ell_d \wedge \ell)$. Since the expression of a polynomial as a monomial is essentially unique, the result follows. $\square$

Give $S^\bullet V \otimes S^\bullet V$ a multiplication defined on elements of $\hat{S}eg(\mathbb{P}S^\bullet V \times \mathbb{P}S^\bullet V)$ by

$$(a \otimes b) \cdot (c \otimes d) = ac \otimes bd$$

and extend to the whole space linearly. Define maps

$$\begin{aligned} E_j : S^\delta V &\to S^jV \otimes S^{j(\delta-1)}V, \\ f &\mapsto f_{j,\delta-j} \cdot (1 \otimes f^{j-1}). \end{aligned} \tag{8.6.1}$$

If $j > \delta$ define $E_j(f) = 0$.

Recall from §6.11 the elementary symmetric functions and power sums:

$$\begin{aligned} e_j &= \sum_{1 \le i_1 < i_2 < \cdots < i_j \le N} x^{i_1} \cdots x^{i_j}, \\ p_j &= (x^1)^j + \cdots + (x^N)^j. \end{aligned}$$

The power sums may be written in terms of elementary symmetric functions using the Girard formula (6.11.6) $p_j = \mathcal{P}_j(e_1, \ldots, e_j)$.

Following [**37**], define

$$\begin{aligned} Q_{d,\delta} : S^\delta V &\to S^dV \otimes S^{d(\delta-1)}V, \\ f &\mapsto \mathcal{P}_d(E_1(f), \ldots, E_d(f)). \end{aligned} \tag{8.6.2}$$

For example, since $\mathcal{P}_2(e_1, e_2) = e_1^2 - 2e_2$, one has $\mathcal{P}_2(a_1 \otimes b_1 + a_2 \otimes b_2, \alpha \otimes \beta) = a_1^2 \otimes b_1^2 + 2a_1a_2 \otimes b_1b_2 + a_2^2 \otimes b_2^2 - 2\alpha \otimes \beta$. When $d = \delta$, write $Q_d = Q_{d,d}$.

Proposition 8.6.2.2. *The map $Q_{d,\delta}$ has the following properties:*

(1) $Q_{d,\delta}(\ell_1 \cdots \ell_\delta) = \sum_j \ell_j^d \otimes (\ell_1^d \cdots \ell_{j-1}^d \ell_{j+1}^d \cdots \ell_\delta^d)$.

(2) *If $f_1 \in S^\delta V$ and $f_2 \in S^{d-\delta}V$, then*

$$Q_d(f_1 f_2)(w, z) = f_1^d(z) Q_{d,d-\delta}(f_2)(w, z) + f_2^d(z) Q_{d,\delta}(f_1)(w, z).$$

(3) $Q_d(f)(\cdot, z) = [f_{1,d-1}(z^{d-1})]^d + \sum \mu_j \psi_j(z)$, *where $\psi_j \in S^{d^2-d}V$, $\mu_j \in S^dV$, and f divides ψ_j for each j.*

Proof. First note that for $\ell \in V$, $E_j(\ell) = \ell^j \otimes \ell^{j-1}$ and $Q_{d,1}(\ell) = \ell^d \otimes 1$. Next, compute $E_1(\ell_1\ell_2) = \ell_1 \otimes \ell_2 + \ell_2 \otimes \ell_1$ and $E_2(\ell_1\ell_2) = \ell_1\ell_2 \otimes \ell_1\ell_2$, so $Q_{2,2}(\ell_1\ell_2) = \ell_1^2 \otimes \ell_2^2 + \ell_2^2 \otimes \ell_1^2$. Part (1) now follows from part (2) using induction.

To prove (2), define

$$\begin{aligned}\Delta_u : S^\bullet V &\to S^\bullet V \otimes S^\bullet V,\\ f &\mapsto \sum_j u^j f_{j,\deg(f)-j}.\end{aligned}$$

(This is a deformation of the coproduct induced from the addition map $(V^* \oplus V^*) \to V^*$.) Note that $\Delta_u(fg) = (\Delta_u f) \cdot (\Delta_u g)$. The generating series for the $E_j(f)$ may be written as

$$\mathcal{E}_f(t) = \frac{1}{1 \otimes f} \cdot \Delta_{t(1\otimes f)} f.$$

Note that $(1 \otimes f)^{\cdot s} = 1 \otimes f^s$ and $(1 \otimes fg) = (1 \otimes f) \cdot (1 \otimes g)$. Thus

$$\mathcal{E}_{fg}(t) = \left[\frac{1}{1\otimes f} \cdot \Delta_{[t(1\otimes g)](1\otimes f)}(f)\right] \cdot \left[\frac{1}{1\otimes g} \cdot \Delta_{[t(1\otimes f)](1\otimes g)}(g)\right],$$

and taking the logarithmic derivative (recalling (6.11.5)) we conclude.

To prove (3), note that $df_z^d = E_1(f)^d(\cdot, z)$, which, as observed in §6.11, occurs as a monomial in the Girard formula. Moreover, all the other terms in the Girard formula contain an $E_j(f)$ with $j > 1$, and so they are divisible by f. □

Theorem 8.6.2.3 (Brill, Gordan [**144**], Gelfand-Kapranov-Zelevinsky [**141**], Briand [**37**]). *Consider the map*

$$\mathcal{B} : S^d V \to S_{d,d} V \otimes S^{d^2-d} V, \tag{8.6.3}$$

$$f \mapsto \pi_{d,d} \otimes \mathrm{Id}_{S^{d^2-d}V}[f \otimes Q_d(f)]. \tag{8.6.4}$$

Then $[f] \in Ch_d(V)$ *if and only if* $\mathcal{B}(f) = 0$.

Proof. Say $f = \ell_1 \cdots \ell_d$. Then $Q_d(f) = \sum_j \ell_j^d \otimes (\ell_1^d \cdots \ell_{j-1}^d \ell_{j+1}^d \cdots \ell_d^d)$ and $\pi_{d,d}(\ell_1 \cdots \ell_d, \ell_j^d) = 0$ for each j by Lemma 8.6.2.1.

For the other direction, first assume that f is reduced, i.e., has no repeated factors. Let $z \in \mathrm{Zeros}(f)_{smooth}$; then by Proposition 8.6.2.2(3), $\mathcal{B}(f)(\cdot, z) = \pi_{d,d}(f \otimes (df_z)^d)$, so by Lemma 8.6.2.1, df_z divides f for all $z \in \mathrm{Zeros}(f)$. But this implies that the tangent space to f is constant in a neighborhood of z, i.e., that the component containing z is a linear space, and thus $\mathrm{Zeros}(f)$ is a union of hyperplanes, which is what we set out to prove.

Finally, say $f = g^k h$, where g is irreducible of degree q and h is of degree $d - qk$ and is relatively prime to g. Apply Proposition 8.6.2.2(2) $k-1$ times (the first with $f_1 = g^{k-1}h$) to obtain:

$$Q_d(g^k h) = g^{d(k-1)}[kh^d Q_{d,q}(g) + g^d Q_{d,d-qk}(h)],$$

and $g^{d(k-1)}$ will also factor out of $\mathcal{B}(f)$. Since $\mathcal{B}(f)$ is identically zero but $g^{d(k-1)}$ is not, we conclude

$$0 = \pi_{d,d} \otimes \mathrm{Id}_{S^{d^2-d}V^*} f \otimes [kh^d Q_{d,q}(g) + g^d Q_{d,d-qk}(h)].$$

Let $w \in \mathrm{Zeros}(g)$ be a general point, so in particular $h(w) \neq 0$. Evaluating at (z, w) with z arbitrary gives zero on the second term and the first term implies that $\pi_{d,d} \otimes Id_{S^{d^2-d}V^*}(f \otimes Q_{d,q}(g)) = 0$, which, in turn, implies that dg_w divides g. $\square$

Remark 8.6.2.4. There was a gap in the argument in [**144**], repeated in [**141**], as when proving the "only if" part of the argument, the authors assumed that the zero set of f contains a smooth point, i.e., that the differential of f is not identically zero. This was fixed in [**37**]. In [**141**] the authors use $G_0(d, \dim V)$ to denote $Ch_d(V)$.

Remark 8.6.2.5. If $d < \mathbf{v} = \dim V$, then $Ch_d(V) \subset Sub_d(S^d V)$, and so $I(Ch_d(V)) \supset \Lambda^{d+1}V^* \otimes \Lambda^{d+1}(S^{d-1}V^*)$. J. Weyman (in unpublished notes from 1994) observed that these equations are not in the ideal generated by Brill's equations. More precisely, the ideal generated by Brill's equations does not include all modules $S_\pi V^*$ with $\ell(\pi) > d$. As a consequence, it does not cut out $Ch_d(V)$ scheme-theoretically when $d < \mathbf{v}$.

8.6.3. Brill's equations as modules. Brill's equations are of degree $d+1$ on $S^d V^*$. (The total degree on V^* of $S_{d,d}V \otimes S^{d^2-d}V$ is $d(d+1)$, which is the total degree on V^* of $S^{d+1}(S^d V)$.) Consider the $GL(V)$-module map

$$S_{dd}V \otimes S^{d^2-d}V \to S^{d+1}(S^d V)$$

given by Brill's equations. The decomposition of $S^{d+1}(S^d V)$ is not known in general, and the set of modules present grows extremely fast. One can use the Pieri formula of §6.7.2 to get the components of $S_{d,d} \otimes S^{d^2-d}V$. Using the Pieri formula, we conclude:

Proposition 8.6.3.1. *Brill's equations are multiplicity-free.*

By inheritance (see §7.4), the component of $\mathcal{B}$ in any partition with at most two parts must be zero because any polynomial in two variables can be written as a product of linear factors, i.e., $Ch_d(\mathbb{C}^2) = S^d(\mathbb{C}^2)$.

8.7. The Fano variety of linear spaces on a variety

8.7.1. Definition. Given a variety $X \subset \mathbb{P}V$, let

$$\mathbb{F}_k(X) := \{E \in G(k, V) \mid \mathbb{P}E \subset X\},$$

the *Fano variety* of $\mathbb{P}^{k-1}$'s on X. (These varieties are not to be confused with the varieties with positive first Chern class, which are also called Fano varieties.)

Exercise 8.7.1.1: Show that if $X \subset \mathbb{P}W$ is a general hypersurface of degree d, then $\dim \mathbb{F}_k(X) = \dim G(k+1, W) - \operatorname{rank}(S^d\mathcal{S})$, where $\mathcal{S} \to G(k+1, W)$ is the tautological subspace bundle.

In particular, if $P \in S^n\mathbb{C}^{n^2}$, there is at least a $4n+2$-dimensional family of $\mathbb{P}^2$'s through a general point of $\mathrm{Zeros}(P)$, and if P is general, there are no $\mathbb{P}^3$'s on $\mathrm{Zeros}(P)$.

Exercise 8.7.1.2: Show that $\mathbb{F}_k(v_d(\mathbb{P}V)) = \emptyset$ when $d \geq 2$.

8.7.2. Homogeneous varieties. For those familiar with Dynkin diagrams: if $X = G/P \subset \mathbb{P}V$ is a homogeneous variety in its minimal homogeneous embedding, and the Dynkin diagram of G is simply laced, the linear spaces on X can be determined pictorially. (The result holds in a more general context, but for simplicity of exposition I restrict to simply laced diagrams, which account for all spaces of tensors we have been discussing.) To further simplify the exposition, assume that P is maximal, so X may be represented by the Dynkin diagram of G with one marked node.

The Fano variety of $\mathbb{P}^k$'s on X will have a component for each subdiagram of the marked diagram that corresponds to the marked diagram of a $\mathbb{P}^k$. The component is a homogeneous variety corresponding to marking the diagram of G at the nodes on the boundary of the components of the diagram that are removed.

For example, the diagrams of $\mathbb{P}^2$ and $G(5, 7)$ are shown in Figure 8.7.1.

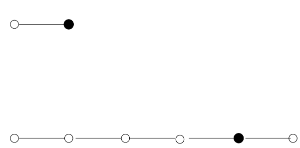


Figure 8.7.1. Diagram for $\mathbb{P}^2$ and for $G(5, 7)$.

One can remove nodes from the second diagram to obtain the first in two different ways, so the variety of $\mathbb{P}^2$'s on $G(5, 7)$ is the union of $G(4, 7)$ and $Flag_{3,6}(\mathbb{C}^7)$ (Figure 8.7.2). For more details see [**204**].

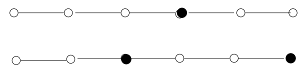


Figure 8.7.2. Variety of $\mathbb{P}^2$'s on $G(5,7)$ is the union of $G(4,7)$ and $Flag_{3,6}(\mathbb{C}^7)$.

8.7.3. Spaces of matrices of bounded and constant rank. When $X = \sigma_r(Seg(\mathbb{P}A\times\mathbb{P}B))$, $\mathbb{F}_k(X)$ is the set of k-dimensional subspaces of the space of $\mathbf{a}\times\mathbf{b}$-matrices such that every vector in the subspace has rank at most r. An obvious such space can be obtained by taking an r-dimensional subspace $A'\subset A$ and the $r\mathbf{b}$-dimensional space $A'{\otimes}B$. A more interesting space in $\sigma_{n-1}(Seg(\mathbb{P}^{n-1}\times\mathbb{P}^{n-1}))$ when n is odd is the space of skew-symmetric matrices. Atkinson [**10**] (also see [**120**]) proved that for $n=3$ all spaces of bounded rank are of one of these two forms. It is an open problem to determine the linear spaces on $\sigma_{n-1}(Seg(\mathbb{P}^{n-1}\times\mathbb{P}^{n-1}))$ in general. This case is important for the GCT program; see §13.6.3.

For easy reference, here is a summary of facts about linear spaces of bounded and constant rank. For proofs, see [**175**].

Theorem 8.7.3.1. *Let*

$$l(r,m,n) = \max\{\dim(A) \mid A\subset \mathbb{C}^n{\otimes}\mathbb{C}^m \textit{ is of constant rank } r\},$$

$$\underline{l}(r,m,n) = \max\{\dim(A) \mid A\subset \mathbb{C}^n{\otimes}\mathbb{C}^m \textit{ is of bounded rank } r\},$$

$$\overline{l}(r,m,n) = \max\{\dim(A) \mid A\subset \mathbb{C}^n{\otimes}\mathbb{C}^m \textit{ is of rank bounded below by } r\},$$

and similarly let $c(r,m)$ (resp. $\lambda(r,m)$), etc., denote the corresponding numbers for symmetric (resp. skew-symmetric) matrices. Then

(1) *(*[**331, 330**]*) For $2\leq r\leq m\leq n$,*
  (a) *$l(r,m,n)\leq m+n-2r+1$;*
  (b) *$l(r,m,n)=n-r+1$ if $n-r+1$ does not divide $(m-1)!/(r-1)!$;*
  (c) *$l(r,r+1,2r-1)=r+1$.*

(2) *(Classical; see* [**175**, Prop. 2.8]*)*
  (a) *$\overline{l}(r,m,n)=(m-r)(n-r)$;*
  (b) *$\overline{c}(r,m)=\binom{m-r+1}{2}$;*
  (c) *$\overline{\lambda}(r,m)=\binom{m-r}{2}$ (r even).*

(3) *(Classical; see* [**175**, Prop. 2.10]*)*
  (a) *If $0<r\leq m\leq n$, then $l(r,m,n)\geq n-r+1$;*
  (b) *If r is even, then $c(r,m)\geq m-r+1$;*
  (c) *If r is even, then $\lambda(r,m)\geq m-r+1$.*

(4) *(Classical; see* [**175**, Prop. 2.15]*) If r is odd, then $c(r,m)=1$.*

(5) *(*[**175**, Prop. 2.16]*) If r is even, then $c(r,m)=m-r+1$.*

Chapter 9

Rank

This chapter discusses X-rank, with a focus on symmetric tensor rank. Little is known about rank for tensors, other than what is known about border rank and the cases of "small" tensors discussed in Chapter 10. The chapter begins in §9.1 with some general facts about X-rank, including an upper bound on the maximum possible X-rank for a variety X (Theorem 9.1.3.1). Two results on symmetric rank are presented in §9.2. In §9.2.1, a lower bound for the symmetric tensor rank of a polynomial $\phi \in S^dV$ is given in terms of the singularities of the zero set of ϕ, $\mathrm{Zeros}(\phi) \subset \mathbb{P}V^*$. In §9.2.2, the theorem of Comas and Seguir, completely determining the symmetric rank of binary forms, is stated and proved. Finally, the ranks of several classes of polynomials, especially monomials, are discussed in §9.3.

The only cases of spaces of polynomials where the possible symmetric ranks and border ranks are known are as follows: (i) $S^2\mathbb{C}^n$ for all n. Here rank and border rank coincide with the rank of the corresponding symmetric matrix, and there is a normal form for elements of rank r, namely $x_1^2 + \cdots + x_r^2$. (ii) $S^d\mathbb{C}^2$, where the possible ranks and border ranks are known; see Theorem 9.2.2.1. (iii) $S^3\mathbb{C}^3$, where the possible ranks and border ranks were determined in [**99**].

Further information on rank is in Chapter 10, where for spaces of tensors and symmetric tensors admitting normal forms, the rank of the normal form is determined in many cases.

9.1. Remarks on rank for arbitrary varieties

9.1.1. Rank is nonincreasing under projection. Let $X \subset \mathbb{P}V$ be a variety. Let $U \subset V$ be a linear subspace. Consider the image of the cone

$\hat{X} \subset V$ under a projection $\pi_U : V \to (V/U)$. (This gives rise to a rational map $\pi_U : \mathbb{P}V \dashrightarrow \mathbb{P}(V/U)$; see §4.9.1.) Then for $p \in \mathbb{P}V$,

$$\mathbf{R}_{\pi_U(X)}(\pi_U(p)) \leq \mathbf{R}_X(p),$$

and similarly for border rank. To see this, note that if $p = q_1 + \cdots + q_r$, then $\pi_U(p) = \pi_U(q_1) + \cdots + \pi_U(q_r)$ because π_U is a linear map.

In [**13**] the authors observe that rank may decrease considerably under a linear projection. For example, if one projects from a general point, the maximum possible $\pi(X)$-rank of the projection is the generic rank of a point in $\mathbb{P}V$. This is essentially because the X-rank of the projected point is the minimum of the X-ranks of the points in its fiber. For example, when $X = v_d(\mathbb{P}^1)$, the rank of a point on a tangent line drops from d to $\lfloor \frac{d+1}{2} \rfloor$ under a general projection from a point.

9.1.2. Rank is nondecreasing under linear sections.

Exercise 9.1.2.1: Prove that if $X \subset \mathbb{P}V$ is a variety, $L \subset \mathbb{P}V$ a linear space, and $p \in L$, then $\mathbf{R}_X(p) \leq \mathbf{R}_{X \cap L}(p)$. Show that strict inequality can hold, by considering $v_3(\mathbb{P}^1) \cap L$, where $L \subset \mathbb{P}^3$ is a general 2-plane.

9.1.3. Maximum possible rank. For any variety $X \subset \mathbb{P}V = \mathbb{P}^N$ that is not contained in a hyperplane, *a priori* the maximum X-rank of any point is $N+1$, as one may take a basis of V consisting of elements of $\hat{X}$. If X is a collection of points, equality holds.

Theorem 9.1.3.1 ([**208**])**.** *Let $X \subset \mathbb{P}^N = \mathbb{P}V$ be an irreducible variety of dimension n not contained in a hyperplane. Then for all $p \in \mathbb{P}V$, $\mathbf{R}_X(p) \leq N + 1 - n$.*

Corollary 9.1.3.2. *Given $\phi \in S^d\mathbb{C}^{\mathbf{v}}$, $\mathbf{R}_S(\phi) \leq \binom{\mathbf{v}+d-1}{d} + 1 - \mathbf{v}$.*

Corollary 9.1.3.2 is sharp for $\mathbf{v} = 2$ and $d > 2$, but fails to be sharp for $S^3\mathbb{C}^3$, where the actual maximum rank is 5, whereas Corollary 9.1.3.2 only gives the bound of 8.

Corollary 9.1.3.3. *Let $C \subset \mathbb{P}^N = \mathbb{P}V$ be a curve not contained in a hyperplane. Then the maximum C-rank of any $p \in \mathbb{P}V$ is at most N.*

For $C = v_N(\mathbb{P}^1)$ the maximum C-rank of a point is indeed N; see Theorem 9.2.2.1 below. E. Ballico [**11**] showed that for "most" varieties Theorem 9.1.3.1 is optimal. He also showed, in [**12**], that the bound only gets weaker by one over $\mathbb{R}$.

The proof of Theorem 9.1.3.1 and other results in this chapter require the following classical theorems in algebraic geometry:

Theorem 9.1.3.4 (Bertini's theorem; see, e.g., [**157**, pp. 216–217])**.** *Let $\mathcal{H} \subset \mathbb{P}V^*$ be a linear subspace, which may be thought of as parametrizing a* *family of hyperplanes in $\mathbb{P}V$. Let Base$(\mathcal{H}) = \bigcap_{H\in\mathcal{H}} H \subset \mathbb{P}V$. Let $X \subset \mathbb{P}V$ be a smooth variety; then for general hyperplanes $H \in \mathcal{H}$, $X \cap H$ is smooth off $X \cap$ Base$(\mathcal{H})$.*

Theorem 9.1.3.5 (Consequence of Lefshetz hyperplane theorem; see, e.g., [**146**, p. 156]). *Let $X^n \subset \mathbb{P}V$ be a smooth variety of dimension at least two, and let $H \subset \mathbb{P}V$ be a hyperplane. Then $X \cap H$ is an irreducible variety.*

Proof of Theorem 9.1.3.1. If $p \in X$, then $\mathbf{R}_X(p) = 1 \leq N+1-n$. Henceforth only consider $p \notin X$. Let $\mathcal{H}_p$ be the set of hyperplanes containing p. Proceed by induction on the dimension of X. If $\dim X = 1$, for a general $H \in \mathcal{H}_p$, H intersects X transversely by Bertini's theorem. I claim that H is spanned by $H \cap X$. Otherwise, if H' is any other hyperplane containing $H \cap X$, say H and H' are defined by linear forms L and L', respectively. Then L'/L defines a meromorphic function on X with no poles, since each zero of L is simple and is also a zero of L'. So L'/L is actually a holomorphic function on X, and since X is projective, L'/L must be constant. This shows that $H = H'$ and thus $H \cap X$ indeed spans H.

As noted above, taking a basis of H from points of $H \cap X$ gives

$$\mathbf{R}_X(p) \leq \mathbf{R}_{H\cap X}(p) \leq \dim H + 1.$$

For the inductive step, define $\mathcal{H}_p$ as above. For general $H \in \mathcal{H}_p$, $H \cap X$ spans H by the same argument, and is also irreducible if $\dim X = n > 1$ by the Lefschetz theorem. We are reduced to the case of $(X \cap H) \subset \mathbb{P}H$, which is covered by the induction hypothesis. $\square$

9.1.4. Additivity. Let $X \subset \mathbb{P}A$, $Y \subset \mathbb{P}B$ be varieties, and let $p \in (A \oplus B)$, so $p = p_A + p_B$ uniquely. Then

$$\mathbf{R}_{X\sqcup Y}(p) = \mathbf{R}_X(p_A) + \mathbf{R}_Y(p_B), \tag{9.1.1}$$

$$\underline{\mathbf{R}}_{X\sqcup Y}(p) = \underline{\mathbf{R}}_X(p_A) + \underline{\mathbf{R}}_Y(p_B). \tag{9.1.2}$$

To see this, note that if $p = p_1 + \cdots + p_r$, with each $p_j \in X \sqcup Y$, then each p_j lies in either X or Y giving the additivity for rank. The additivity for border rank follows, as the elements of rank r form a Zariski open subset of the elements of border rank r.

9.2. Bounds on symmetric rank

9.2.1. A lower bound on the rank of a symmetric tensor. For $\phi \in S^dW$, recall the partial polarization $\phi_{s,d-s} \in S^sW \otimes S^{d-s}W$. Adopt the notation $\ker(\phi_{s,d-s})$ for the kernel of the linear map $\phi_{s,d-s} : S^sW^* \to S^{d-s}W$ and Zeros$(\phi) \subset \mathbb{P}W^*$ for the zero set of ϕ, a hypersurface of degree d in $\mathbb{P}W^*$. Recall the dual variety $v_d(\mathbb{P}W)^\vee$ from §8.2.

If $\ker(\phi_{s,d-s})$ is spanned by elements of $v_s(\mathbb{P}W^*)$, then the bounds on border rank and rank, respectively obtained from the equations for $\sigma_r(v_d(\mathbb{P}W))$ given by requiring that $\operatorname{rank}(\phi_{s,d-s}) \leq r$ and by Proposition 3.2.1.1 using $\phi_{s,d-s}$, are the same.

Definition 9.2.1.1. For $\phi \in S^d W$, let

$$\begin{aligned}\Sigma_\delta = \Sigma_\delta(\phi) &:= \mathbb{P}\{v \in W^* \mid \phi_{\delta,d-\delta}(v^\delta, \cdot) \in S^{d-\delta}W \text{ is zero}\},\\ &=: \{x \in \operatorname{Zeros}(\phi) \subset \mathbb{P}W^* \mid \operatorname{mult}_x(\operatorname{Zeros}(\phi)) \geq \delta + 1\}.\end{aligned}$$

(For readers unfamiliar with multiplicity, the second line may be taken as defining $\operatorname{mult}_x(\operatorname{Zeros}(\phi))$.) One says that $x \in \Sigma_\delta \backslash \Sigma_{\delta+1}$ is a point of $\operatorname{Zeros}(\phi)$ with *multiplicity* $\delta+1$. With this notation, $\Sigma_0(\phi) = Z(\phi)$, $\Sigma_1(\phi) = Z(\phi)_{sing}$, $\Sigma_d = \emptyset$, and $\Sigma_{d-1} = \mathbb{P}\langle\phi\rangle^\perp$. In particular, Σ_{d-1} is empty if and only if $\langle\phi\rangle = W$, i.e., $\operatorname{Zeros}(\phi)$ is not a cone.

Observe that:

Proposition 9.2.1.2. *Let $\phi \in S^d W$; then $\mathbb{P}(\ker(\phi_{d-s,s})) \cap v_{d-s}(\mathbb{P}W^*) = \emptyset$ if and only if $\Sigma_s(\phi) = \emptyset$.*

Proposition 9.2.1.3. *Let $\phi \in S^d W$ and assume that $\langle\phi\rangle = W$ (so $\underline{\mathbf{R}}_S(\phi) \geq \mathbf{w}$). If $\mathbf{R}_S(\phi) = \mathbf{w}$, then $\operatorname{Zeros}(\phi)$ is smooth (i.e., $[\phi] \notin v_d(\mathbb{P}W^*)^\vee$).*

Proof. If $\mathbf{R}_S(\phi) = \mathbf{w}$, then $\phi = \eta_1^d + \cdots + \eta_{\mathbf{w}}^d$ for some $\eta_j \in W$. The η_i furnish a basis of W because $\langle\phi\rangle = W$, so $\operatorname{Zeros}(\phi)$ is smooth (see Exercise 4.6.1.6). $\square$

Proposition 9.2.1.2 may be rephrased as

$$v_{d-s}(\Sigma_s) = \mathbb{P}\ker \phi_{d-s,s} \cap v_{d-s}(\mathbb{P}W^*).$$

Adopt the convention that $\dim(\emptyset) = -1$.

Theorem 9.2.1.4 ([**208**]). *Let $\phi \in S^d W$ with $\langle\phi\rangle = W$. For $\delta \leq \frac{d}{2}$,*

$$\mathbf{R}_S(\phi) > \operatorname{rank}(\phi_{\delta,d-\delta}) + \dim \Sigma_\delta.$$

Corollary 9.2.1.5. *Let $\phi \in S^d W$ with $\langle\phi\rangle = W$. If ϕ is reducible, then $\mathbf{R}_S(\phi) \geq 2\mathbf{w} - 2$. If ϕ has a repeated factor, then $\mathbf{R}_S(\phi) \geq 2\mathbf{w} - 1$.*

Proof of the corollary. $\operatorname{rank}(\phi_{1,d-1}) = \mathbf{w}$. If $\phi = \chi\psi$ factors, then $\Sigma_1(\phi)$ includes the intersection $\{\chi = \psi = 0\}$, which has codimension 2 in $\mathbb{P}W \cong \mathbb{P}^{\mathbf{w}-1}$. Therefore $\mathbf{R}_S(\phi) \geq \mathbf{w} + \mathbf{w} - 3 + 1 = 2\mathbf{w} - 2$.

If ϕ has a repeated factor, say ϕ is divisible by ψ^2, then Σ_1 includes the hypersurface $\{\psi = 0\}$, which has codimension 1. So $\mathbf{R}_S(\phi) \geq \mathbf{w}+\mathbf{w}-2+1 = 2\mathbf{w} - 1$. $\square$

Lemma 9.2.1.6. *Let $\phi \in S^d W$. Suppose there is an expression $\phi = \eta_1^d + \cdots + \eta_r^d$. Let $L := \mathbb{P}\{p \in S^{d-s}W^* \mid p(\eta_i) = 0, 1 \leq i \leq r\}$. Then*

(1) $L \subset \mathbb{P}\ker \phi_{d-s,s}$.
(2) $\mathrm{codim} L \leq r$.
(3) *If* $\langle \phi \rangle = W$, *then* $L \cap v_{d-s}(\mathbb{P}W^*) = \emptyset$.

Proof. The first statement is clear as $\phi_{s,d-s} = \sum \eta_j^s \otimes \eta_j^{d-s}$.

For the second, since each point $[\eta_i]$ imposes a single linear condition on the coefficients of p, L is the common zero locus of a system of r independent linear equations, and so $\mathrm{codim}\, L \leq r$.

If $\langle \phi \rangle = W$, then the η_i span W, i.e., the points $[\eta_i]$ in $\mathbb{P}W$ do not lie on a hyperplane. Say there exists nonzero $\alpha \in W^*$ with $[\alpha^{d-s}] \in L$; then the linear form α vanishes at each $[\eta_i]$, so the $[\eta_i]$ lie on the hyperplane defined by α, a contradiction. $\square$

Proof of Theorem 9.2.1.4. Suppose $\phi = \eta_1^d + \cdots + \eta_r^d$. Consider $L = \mathbb{P}\{p \in S^{d-s}W^* \mid p(\eta_i) = 0, 1 \leq i \leq r\}$ as in Lemma 9.2.1.6. Then $L \subseteq \mathbb{P}\ker \phi_{d-s,s}$ so $r \geq \mathrm{codim}_{\mathbb{P}S^{d-s}W^*} L$.

On the other hand, $L \cap v_{d-s}(\Sigma_s) = \emptyset$ because $L \cap v_{d-s}(\mathbb{P}W^*) = \emptyset$ by Lemma 9.2.1.6. Therefore

$$\mathrm{codim}_{\mathbb{P}\ker \phi_{d-s,s}} L + \mathrm{codim}_{\mathbb{P}\ker \phi_{d-s,s}} \Sigma_s \geq \dim \mathbb{P}\ker \phi_{d-s,s}.$$

Putting these together gives

$$\begin{aligned} r &\geq \mathrm{codim}_{\mathbb{P}S^{d-s}W^*} L \\ &= \mathrm{codim}_{\mathbb{P}S^{d-s}W^*} \mathbb{P}\ker \phi_{d-s,s} + \mathrm{codim}_{\mathbb{P}\ker \phi_{d-s,s}} L \\ &\geq \mathrm{rank}(\phi_{d,s,s}) + \dim \Sigma_s. \end{aligned}$$

$\square$

9.2.2. Symmetric ranks of binary forms. Recall that $\sigma_{\lfloor \frac{d+1}{2} \rfloor}(v_d(\mathbb{P}^1)) = \mathbb{P}^d$.

Theorem 9.2.2.1 (Comas-Seiguer [**95**])**.** *Let* $r \leq \lfloor \frac{d+1}{2} \rfloor$. *Then*

$$\sigma_r(v_d(\mathbb{P}^1)) = \{[\phi] : \mathbf{R}_S(\phi) \leq r\} \cup \{[\phi] : \mathbf{R}_S(\phi) \geq d - r + 2\}.$$

Equivalently,

$$\sigma_r(v_d(\mathbb{P}^1)) \setminus \sigma_{r-1}(v_d(\mathbb{P}^1)) = \{[\phi] : \mathbf{R}_S(\phi) = r\} \cup \{[\phi] : \mathbf{R}_S(\phi) = d - r + 2\}.$$

The case $r = 2$ dates back to Sylvester: say there were a secant $\mathbb{P}^{d-2}$ containing a point on a tangent line to $v_d(\mathbb{P}^1)$. Then the span of the $\mathbb{P}^{d-2}$ with the tangent line would be a $\mathbb{P}^{d-1}$ that contained $d+1$ points of $v_d(\mathbb{P}^1)$ (counting multiplicities), but $\deg(v_d(\mathbb{P}^1)) = d < d+1$, a contradiction.

Here is the proof of the general case (in what follows $W = \mathbb{C}^2$).

Lemma 9.2.2.2. *Let* $\phi \in Sym^d(W)$. *Let* $1 \leq r \leq d-1$ *and* $\underline{\mathbf{R}}_S(\phi) \leq r$, *that is,* $\ker(\phi_{r,d-r}) \neq 0$. *Then* $\mathbf{R}_S(\phi) > r$ *if and only if* $\mathbb{P}\ker \phi_{r,d-r} \subset v_r(\mathbb{P}W)^\vee$.

Proof. First say $\mathbf{R}_S(\phi) \leq r$ and write $\phi = w_1^d + \cdots + w_r^d$. Then $\ker \phi_{r,d-r}$ contains the polynomial with distinct roots $w_1, \ldots, w_r$. Conversely, say $0 \neq P \in \ker \phi_{r,d-r}$ has distinct roots $w_1, \ldots, w_r$. It will be sufficient to show that $\phi \wedge w_1^d \wedge \cdots \wedge w_r^d = 0$, i.e., that $\phi \wedge w_1^d \wedge \cdots \wedge w_r^d(p_1, \ldots, p_{r+1}) = 0$ for all $p_1, \ldots, p_{r+1} \in Sym^d W^*$. Rewrite this as

$$\phi(p_1)m_1 - \phi(p_2)m_2 + \cdots + (-1)^r \phi(p_{r+1})m_{r+1} = \phi(m_1 p_1 + \cdots + (-1)^r m_{r+1} p_{r+1}),$$

where $m_j = w_1^d \wedge \cdots \wedge w_r^d(p_1, \ldots, \hat{p}_j, \ldots, p_{r+1}) \in \mathbb{C}$. Now for each j,

$$\begin{aligned} w_j^d(m_1 p_1 + \cdots + (-1)^r m_{r+1} p_{r+1}) &= \sum_{i=1}^{r+1} w_j^d((-1)^{i-1} m_i p_i) \\ &= \sum_{i=1}^{r+1} (-1)^{2(i-1)} w_j^d \wedge w_1^d \wedge \cdots \wedge w_r^d(p_1, \ldots, p_{r+1}) \\ &= 0. \end{aligned}$$

Now consider the p_j as polynomials of degree d on W,

$$(m_1 p_1 + \cdots + (-1)^r m_{r+1} p_{r+1})(w_i) = 0$$

for each i. But then $(m_1 p_1 + \cdots + (-1)^r m_{r+1} p_{r+1}) = PQ$ for some $Q \in Sym^{d-r} W^*$ and $\phi(PQ) = 0$ because $P \in \ker \phi_{r,d-r}$. □

Recall from §6.10.4 that the generators of the ideal of $\sigma_r(v_d(\mathbb{P}^1))$ can be obtained from the $(r+1) \times (r+1)$ minors of $\phi_{s,d-s}$.

Lemma 9.2.2.3. *Let $r \leq \lfloor \frac{d+1}{2} \rfloor$. If $\phi = \eta_1^d + \cdots + \eta_k^d$, $k \leq d - r + 1$, and $P \in \ker \phi_{r,d-r}$, then $P(\eta_i) = 0$ for each $1 \leq i \leq k$.*

Proof. For $1 \leq i \leq k$ let $M_i \in W^*$ annihilate η_i. In particular, $M_i(\eta_j) \neq 0$ if $j \neq i$, because the $[\eta_j]$ are distinct. Suppose $L \in W^*$ does not vanish at any η_i. For each i, let

$$g_i = P M_1 \cdots \widehat{M_i} \cdots M_k L^{d-r+1-k},$$

so $\deg g_i = d$. Since $P \in \ker \phi_{r,d-r}$, one has $\phi(g_i) = 0$. On the other hand, $\eta_j^d(g_i) = 0$ for $j \neq i$, so

$$\eta_i^d(g_i) = 0 = P(\eta_i) M_1(\eta_i) \cdots \widehat{M_i(\eta_i)} \cdots M_k(\eta_i) L(\eta_i)^{d-r+1-k}.$$

All the factors on the right are nonzero except possibly $P(\eta_i)$. Thus $P(\eta_i) = 0$. □

Proof of Theorem 9.2.2.1. Suppose $[\phi] \in \sigma_r(v_d(\mathbb{P}^1))$ and $\mathbf{R}_S(\phi) \leq d - r + 1$. Write $\phi = \eta_1^d + \cdots + \eta_k^d$ for some $k \leq d - r + 1$ and the $[\eta_i]$ distinct. Then $[\phi] \in \sigma_r(v_d(\mathbb{P}^1))$ implies that $\operatorname{rank} \phi_{r,d-r} \leq r$, so $\dim(\ker \phi_{r,d-r}) \geq 1$. Therefore, there is some nonzero $P \in \ker \phi_{r,d-r}$. Every $[\eta_i]$ is a zero of P, but $\deg P = r$, so P has at most r roots. Thus, $k \leq r$. This shows the inclusion $\subseteq$ in the statement of the theorem.

To show that $\{[\phi] : \mathbf{R}_S(\phi) \geq d-r+2\} \subseteq \sigma_r(v_d(\mathbb{P}^1))$, suppose $\mathbf{R}_S(\phi) \geq d-r+2$ and $[\phi] \notin \sigma_{r-1}(v_d(\mathbb{P}^1))$. Then $\operatorname{codim}(\ker \phi_{d-r+1,r-1}) = r$ and $\mathbb{P}\ker \phi_{d-r+1,r-1} \subset v_r(\mathbb{P}W)^\vee$ by Lemma 9.2.2.2 (applied to $\ker \phi_{d-r+1,r-1} = \ker \phi_{d-r+1,r-1}$). This means that every polynomial $P \in \ker \phi_{d-r+1,r-1}$ has a singularity (multiple root in $\mathbb{P}^1$). By Bertini's Theorem 9.1.3.4, there is a basepoint of the linear system (a common divisor of all the polynomials in $\ker \phi_{d-r+1,r-1}$). Let F be the greatest common divisor. Say $\deg F = f$. Let $M = \{P/F \mid P \in \ker \phi_{d-r+1,r-1}\}$. Every $P/F \in M$ has degree $d-r+1-f$. So $\mathbb{P}M \subset \mathbb{P}Sym^{d-r+1-f}W^*$, which has dimension $d-r+1-f$. Also $\dim \mathbb{P}M = \dim \mathbb{P}\ker \phi_{d-r+1,r-1} = d-2r+1$. Therefore $d-2r+1 \leq d-r+1-f$, so $f \leq r$.

Since the polynomials in M have no common roots, $(Sym^{r-f}W^*).M = Sym^{d-2f+1}W^*$ (see, e.g., [**157**, Lemma 9.8]). Thus

$$\begin{aligned} Sym^{r-1}W^*.\ker \phi_{d-r+1,r-1} &= Sym^{f-1}W^*.Sym^{r-f}W^*.M.F \\ &= Sym^{d-f}W^*.F. \end{aligned}$$

Therefore, if $Q \in S^{d-f}W^*$, then $FQ = GP$ for some $G \in Sym^{r-1}W^*$ and $P \in \ker \phi_{d-r+1,r-1}$, and so $\phi(FQ) = \phi(GP) = 0$. Thus $0 \neq F \in \ker \phi_{f,d-f}$ and $[\phi] \in \sigma_f(v_d(\mathbb{P}^1))$. Finally $\sigma_f(v_d(\mathbb{P}^1)) \subset \sigma_r(v_d(\mathbb{P}^1))$, since $f \leq r$. □

Corollary 9.2.2.4. *If $a, b > 0$, then $\mathbf{R}_S(x^a y^b) = \max(a+1, b+1)$.*

Proof. Assume that $a \leq b$. The symmetric flattening $(x^a y^b)_{a,b}$ has rank $a+1$ (the image is spanned by $x^a y^0, x^{a-1}y^1, \ldots, x^0 y^a$); it follows that $\underline{\mathbf{R}}_S(x^a y^b) \geq a+1$. Similarly, $(x^a y^b)_{a+1,b-1}$ has rank $a+1$ as well. Therefore $\underline{\mathbf{R}}_S(x^a y^b) = a+1$, so $\mathbf{R}_S(x^a y^b)$ is either $b+1$ or $a+1$.

Let $\{\alpha, \beta\}$ be a dual basis to $\{x, y\}$. If $a < b$, then $\mathbb{P}\ker(x^a y^b)_{a+1,b-1} = \{[\alpha^{a+1}]\} \subset v_{a+1}(\mathbb{P}W)^\vee$. Therefore $\mathbf{R}_S(x^a y^b) > a+1$. If $a = b$, then $\mathbf{R}_S(x^a y^b) = a+1 = b+1$. □

9.3. Examples of classes of polynomials and their ranks

9.3.1. Ranks and border ranks of some cubic polynomials.

Proposition 9.3.1.1 ([**208**]). *Consider $\phi = x_1 y_1 z_1 + \cdots + x_m y_m z_m \in S^3 W$, where $W = \mathbb{C}^{3m}$. Then $\mathbf{R}_S(\phi) = 4m = \frac{4}{3}\dim W$ and $\underline{\mathbf{R}}_S(\phi) = 3m = \dim W$.*

Proof. Since $\langle \phi \rangle = W$, $\operatorname{rank} \phi_{1,2} = \dim W = 3m$, and Σ_1 contains the set $\{x_1 = y_1 = x_2 = y_2 = \cdots = x_m = y_m = 0\}$. Thus Σ_1 has dimension at least $m-1$. So $\mathbf{R}_S(\phi) \geq 4m$ by Proposition 9.2.1.4. On the other hand, each $x_i y_i z_i$ has rank 4 by Theorem 10.4.0.5, so $\mathbf{R}_S(\phi) \leq 4m$.

Since $\underline{\mathbf{R}}_S(xyz) = 3$, one has $\underline{\mathbf{R}}_S(\phi) \leq 3m$ and the other direction follows because $\operatorname{rank}(\phi_{1,2}) = 3m$. □

9.3.2. Determinants and permanents.

Theorem 9.3.2.1 ([**151, 208**])**.**

$$2^{n-1}n! \geq \mathbf{R}_S(\det_n) \geq \binom{n}{\lfloor n/2 \rfloor}^2 + n^2 - (\lfloor n/2 \rfloor + 1)^2,$$

$$4^{n-1} \geq \mathbf{R}_S(\operatorname{perm}_n) \geq \binom{n}{\lfloor n/2 \rfloor}^2 + n(n - \lfloor n/2 \rfloor - 1).$$

Proof. The first bound is obtained by writing $\det_n$ as a sum of $n!$ terms, each of the form $x_1 \cdots x_n$, and applying Theorem 9.3.3.7: $\mathbf{R}_S(x_1 \cdots x_n) = 2^{n-1}$. For the second bound, a variant of the Ryser formula for the permanent (see [**279**]) allows one to write perm_n as a sum of 2^{n-1} terms, each of the form $x_1 \cdots x_n$:

$$\operatorname{perm}_n = 2^{-n+1} \sum_{\substack{\epsilon \in \{-1,1\}^n \\ \epsilon_1 = 1}} \prod_{1 \leq i \leq n} \sum_{1 \leq j \leq n} \epsilon_i \epsilon_j x_{i,j}, \tag{9.3.1}$$

where the outer sum is taken over n-tuples $(\epsilon_1 = 1, \epsilon_2, \ldots, \epsilon_n)$. Note that each term in the outer sum is a product of n independent linear forms and there are 2^{n-1} terms. Applying Theorem 9.3.3.7 again gives the upper bound for $\mathbf{R}_S(\operatorname{perm}_n)$.

The determinant $\det_n$ vanishes to order $a + 1$ on a matrix A if and only if every minor of A of size $n - a$ vanishes. Thus $\Sigma_a(\det_n)$ is the locus of matrices of rank at most $n-a-1$. This locus has dimension $n^2-1-(a+1)^2$. Therefore, for each a,

$$\mathbf{R}_S(\det_n) \geq \binom{n}{a}^2 + n^2 - (a + 1)^2.$$

The right hand side is maximized at $a = \lfloor n/2 \rfloor$.

A lower bound for $\dim \Sigma_a(\operatorname{perm}_n)$ is obtained as follows. If a matrix A has $a + 1$ columns identically zero, then each term in perm_n vanishes to order $a+1$, so perm_n vanishes to order at least $a+1$. Therefore, $\Sigma_a(\operatorname{perm}_n)$ contains the set of matrices with $a+1$ zero columns, which is a finite union of projective linear spaces of dimension $n(n-a-1)-1$. Therefore, for each a,

$$\mathbf{R}_S(\operatorname{perm}_n) \geq \binom{n}{a}^2 + n(n - a - 1).$$

Again, the right hand side is maximized at $a = \lfloor n/2 \rfloor$. □

9.3.3. Ranks of monomials.

Normal forms for points of $\tau_k(v_d(\mathbb{P}W))$. Let $x(t) \subset W$ be a curve and write $x_0 = x(0)$, $x_1 = x'(0)$, and $x_j = x^{(j)}(0)$. Consider the corresponding curve $y(t) = x(t)^d$ in $\hat{v}_d(\mathbb{P}W)$ and note that

$$\begin{aligned}
y(0) &= x_0^d, \\
y'(0) &= dx_0^{d-1}x_1, \\
y''(0) &= d(d-1)x_0^{d-2}x_1^2 + dx_0^{d-1}x_2, \\
y^{(3)}(0) &= d(d-1)(d-2)x_0^{d-3}x_1^3 + 3d(d-1)x_0^{d-2}x_1x_2 + dx_0^{d-1}x_3, \\
y^{(4)}(0) &= d(d-1)(d-2)(d-3)x_0^{d-4}x_1^4 + 6d(d-1)(d-2)x_0^{d-3}x_1^2x_2 \\
&\quad + 3d(d-1)x_0^{d-2}x_2^2, + 4d(d-1)x_0^{d-2}x_1x_3 + dx_0^{d-1}x_4, \\
y^{(5)}(0) &= d(d-1)(d-2)(d-3)(d-4)x_0^{d-5}x_1^5 \\
&\quad + 9d(d-1)(d-2)(d-3)x_0^{d-4}x_1^3x_2 + 10d(d-1)(d-2)x_0^{d-3}x_1^2x_3 \\
&\quad + 15d(d-1)(d-2)x_0^{d-3}x_1x_2^2 + 4d(d-1)x_0^{d-2}x_2x_3 \\
&\quad + 5d(d-1)x_0^{d-2}x_1x_4 + dx_0^{d-1}x_5, \\
&\ \vdots
\end{aligned}$$

At r derivatives, one obtains a sum of terms

$$x_0^{d-s}x_1^{a_1}\cdots x_p^{a_p}, \quad a_1 + 2a_2 + \cdots + pa_p = r, \quad s = a_1 + \cdots + a_p.$$

In particular, $x_0x_1\cdots x_{d-1}$ appears for the first time at order $1 + 2 + \cdots + (d-1) = \binom{d}{2}$.

Write $\mathbf{b} = (b_1, \dots, b_m)$. Let $S_{\mathbf{b},\delta}$ denote the number of distinct m-tuples $(a_1, \dots, a_m)$ satisfying $a_1+\cdots+a_m = \delta$ and $0 \le a_j \le b_j$. Adopt the notation that $\binom{a}{b} = 0$ if $b > a$ and is the usual binomial coefficient otherwise.

Proposition 9.3.3.1 (L. Matusevich; see [**208**])**.** *Write $I = (i_1, i_2, \dots, i_k)$ with $i_1 \le i_2 \le \cdots \le i_k$. Then*

$$S_{\mathbf{b},\delta} = \sum_{k=0}^{m}(-1)^k \left[\sum_{|I|=k} \binom{\delta + m - k - (b_{i_1} + \cdots + b_{i_k})}{m}\right].$$

Proof. The proof is safely left to the reader. (It is a straightforward inclusion-exclusion counting argument, in which the k-th term of the sum counts the m-tuples with $a_j \ge b_j + 1$ for at least k values of the index j.) $\square$

Remark 9.3.3.2. $S_{\mathbf{b},\delta}$ is the Hilbert function of the variety defined by the monomials $x_1^{b_1+1}, \dots, x_m^{b_m+1}$.

For $\mathbf{b} = (b_1, \dots, b_m)$, consider the quantity

$$T_{\mathbf{b}} := \prod_{i=1}^{m} (1 + b_i).$$

$T_{\mathbf{b}}$ counts the number of tuples $(a_1, \dots, a_m)$ satisfying $0 \le a_j \le b_j$ (with no restriction on $a_1 + \cdots + a_m$).

Theorem 9.3.3.3 ([**208**]). *Let $b_0 \ge b_1 \ge \cdots \ge b_n$ and write $d = b_0 + \cdots + b_n$. Then*

$$S_{(b_0, b_1, \dots, b_n), \lfloor \frac{d}{2} \rfloor} \le \underline{\mathbf{R}}_S(x_0^{b_0} x_1^{b_1} \cdots x_n^{b_n}) \le T_{(b_1, \dots, b_n)}.$$

Proof. Let $\phi = x_0^{b_0} \cdots x_n^{b_n}$. The lower bound follows from considering the image of $\phi_{\lfloor \frac{d}{2} \rfloor, \lceil \frac{d}{2} \rceil}$, which is

$$\phi_{\lfloor \frac{d}{2} \rfloor, \lceil \frac{d}{2} \rceil}(S^{\lceil \frac{d}{2} \rceil}\mathbb{C}^{n+1}) = \left\langle x_0^{a_0} x_1^{a_1} \cdots x_n^{a_n} \;\middle|\; 0 \le a_j \le b_j,\ a_0 + a_1 + \cdots + a_n = \left\lfloor \frac{d}{2} \right\rfloor \right\rangle,$$

whose dimension is $S_{(b_0, b_1, \dots, b_n), \lfloor \frac{d}{2} \rfloor}$.

To prove the upper bound, let

$$F_{\mathbf{b}}(t) = \bigwedge_{s_1=0}^{b_1} \cdots \bigwedge_{s_n=0}^{b_n} (x_0 + t^1 \lambda_{1,s_1} x_1 + t^2 \lambda_{2,s_2} x_2 + \cdots + t^n \lambda_{n,s_n} x_n)^d, \tag{9.3.2}$$

where the $\lambda_{i,s}$ are chosen sufficiently generally. Here $\bigwedge_{j=1}^{p} y_j$ denotes $y_1 \wedge y_2 \wedge \cdots \wedge y_p$. Take each $\lambda_{i,0} = 0$ and each $\lambda_{i,1} = 1$. For $t \ne 0$, $[F_{\mathbf{b}}(t)]$ is a plane spanned by $T_{\mathbf{b}}$ points in $v_d(\mathbb{P}W)$. I will show that $x_0^{b_0} \cdots x_n^{b_n}$ lies in the plane $\lim_{t \to 0}[F_{\mathbf{b}}(t)]$, to prove that $\underline{\mathbf{R}}_S(x_0^{b_0} \cdots x_n^{b_n}) \le T_{\mathbf{b}}$. More precisely, I will show that

$$\lim_{t \to 0}[F_{\mathbf{b}}(t)] = \left[\bigwedge_{a_1=0}^{b_1} \cdots \bigwedge_{a_n=0}^{b_n} x_0^{d-(a_1+\cdots+a_n)} x_1^{a_1} \cdots x_n^{a_n} \right], \tag{9.3.3}$$

so $x_0^{b_0} \cdots x_n^{b_n}$ occurs precisely as the last member of the spanning set for the limit plane.

For an n-tuple $I = (a_1, \dots, a_n)$ and an n-tuple $(p_1, \dots, p_n)$ satisfying $0 \le p_i \le b_i$, let

$$c^{(a_1, \dots, a_n)}_{(p_1, \dots, p_n)} = \lambda_{1,p_1}^{a_1} \cdots \lambda_{n,p_n}^{a_n},$$

the coefficient of $x_1^{a_1} \cdots x_n^{a_n} x_0^{d-(a_1+\cdots+a_n)}$ in $(x_0 + t\lambda_{1,p_1} x_1 + \cdots + t^n \lambda_{n,p_n} x_n)^d$, omitting binomial coefficients. Choose an enumeration of the n-tuples $(p_1, \dots, p_n)$ satisfying $0 \le p_i \le b_i$; say, in lexicographic order. Then given n-tuples $I_1, \dots, I_T$, the coefficient of the term

$$x^{I_1} \wedge \cdots \wedge x^{I_T}$$

in $F_{\mathbf{b}}(t)$ is the product $\prod_{j=1}^{T} c_j^{I_j}$, omitting binomial coefficients. Interchange the x^{I_j} so that $I_1 \le \cdots \le I_T$ in some order. Then the total coefficient of $x^{I_1} \wedge \cdots \wedge x^{I_T}$ is the alternating sum of the permuted products,

$$\sum_{\pi} (-1)^{|\pi|} c_{\pi(j)}^{I_j}$$

(summing over all permutations π of $\{1, \ldots, T\}$) times a product of binomial coefficients (which I henceforth ignore). This sum is the determinant of the $T \times T$ matrix $C := (c_i^{I_j})_{i,j}$.

Assume that the monomials $x^{I_1}, \ldots, x^{I_T}$ are all distinct (otherwise the term $x^{I_1} \wedge \cdots \wedge x^{I_T}$ vanishes identically). Suppose some monomial in $x_2, \ldots, x_n$ appears with more than $b_1 + 1$ different powers of x_1; without loss of generality,

$$x^{I_1} = x_0^{p-a_1} x_1^{a_1} r, \quad \ldots \quad , \quad x^{I_{b_1+2}} = x_0^{p-a_{b_1+2}} x_1^{a_{b_1+2}} r,$$

where r is a monomial in $x_2, \ldots, x_n$, and $p = d - \deg(r)$. The matrix $(\lambda_{1,i}^{a_j})_{i,j}$ has size $(b_1+1) \times (b_1+2)$. The dependence relation among the columns of this matrix holds for the first $b_1 + 2$ columns of C, since (in these columns) each row is multiplied by the same factor r.

More generally, if r is any monomial in $(n-1)$ of the variables, say $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$, then $x_i^a r$ can occur for at most $b_i + 1$ distinct values of the exponent a. The lowest power of t occurs when the values of a are $a = 0, 1, \ldots$. In particular, $x_i^a r$ only occurs for $a \le b_i$.

Therefore, if a term $x^{I_1} \wedge \cdots \wedge x^{I_T}$ has a nonzero coefficient in $F_{\mathbf{b}}(t)$ and occurs with the lowest possible power of t, then in every single x^{I_j}, each x_i occurs to a power $\le b_i$. The only way the x^{I_j} can be distinct is for it to be the term in the right hand side of (9.3.3). This shows that no other term with the same or lower power of t survives in $F_{\mathbf{b}}(t)$; it remains to show that the desired term has a nonzero coefficient. For this term C is a tensor product,

$$C = (\lambda_{1,i}^{j})_{(i,j)=(0,0)}^{(b_1,b_1)} \otimes \cdots \otimes (\lambda_{n,i}^{j})_{(i,j)=(0,0)}^{(b_n,b_n)},$$

and each matrix on the right hand side is nonsingular since they are Vandermonde matrices and the $\lambda_{p,i}$ are distinct. Therefore the coefficient of the term in (9.3.3) is $\det C \neq 0$. $\square$

For example

$$
\begin{aligned}
F_{(b)}(t) &= x_0^d \wedge \bigwedge_{s=1}^{b} (x_0 + t\lambda_s x_1)^d \\
&= t^{\binom{b+1}{2}} \left[x_0^d \wedge \left(\sum_s (-1)^s \lambda_s \right) x_0^{d-1} x_1 \right. \\
&\qquad \left. \wedge \sum_u (-1)^{u+1} \lambda_u^2 x_0^{d-2} x_1^2 \wedge \cdots \wedge \sum_v (-1)^{v+b} \lambda_v^b x_0^{d-b} x_1^b \right] \\
&\quad + O\big(t^{\binom{b+1}{2}+1}\big)
\end{aligned}
$$

and (with each $\lambda_{i,s} = 1$)

$$
\begin{aligned}
F_{(1,1)}(t) &= x_0^d \wedge (x_0 + tx_1)^d \wedge (x_0 + t^2 x_2)^d \wedge (x_0 + tx_1 + t^2 x_2)^d \\
&= x_0^d \wedge \left(x_0^d + dt x_0^{d-1} x_1 + \binom{d}{2} t^2 x_0^{d-2} x_1^2 + \cdots \right) \\
&\qquad \wedge \left(x_0^d + dt^2 x_0^{d-1} x_2 + \cdots \right) \\
&\qquad \wedge \left(x_0^d + dt x_0^{d-1} x_1 + t^2 \left(\binom{d}{2} x_0^{d-2} x_1^2 + d x_0^{d-1} x_2 \right) \right. \\
&\qquad\qquad \left. + t^3 \left(\binom{d}{3} x_0^{d-3} x_1^3 + d(d-1) x_0^{d-2} x_1 x_2 \right) + \cdots \right) \\
&= t^6 \left(x_0^d \wedge d x_0^{d-1} x_1 \wedge d x_0^{d-1} x_2 \wedge d(d-1) x_0^{d-2} x_1 x_2 \right) + O(t^7).
\end{aligned}
$$

Theorem 9.3.3.4 ([**208**]). *Let $b_0 \geq b_1 + \cdots + b_n$. Then $\underline{\mathbf{R}}_S(x_0^{b_0} x_1^{b_1} \cdots x_n^{b_n}) = T_{(b_1,\ldots,b_n)}$.*

Theorem 9.3.3.4 is an immediate consequence of Theorem 9.3.3.3 and the following lemma:

Lemma 9.3.3.5 ([**208**]). *Let $\mathbf{a} = (a_1, \ldots, a_n)$. Write $\mathbf{b} = (a_0, \mathbf{a})$ with $a_0 \geq a_1 + \cdots + a_n$. Then for $a_1 + \cdots + a_n \leq \delta \leq a_0$, $S_{\mathbf{b},\delta}$ is independent of δ and equals $T_{\mathbf{a}}$.*

Proof. The right hand side $T_{\mathbf{a}}$ counts n-tuples $(e_1, \ldots, e_n)$ such that $0 \leq e_j \leq a_j$. To each such tuple associate the $(n+1)$-tuple $(\delta - (e_1 + \cdots + e_n), e_1, \ldots, e_n)$. Since

$$0 \leq \delta - (a_1 + \cdots + a_n) \leq \delta - (e_1 + \cdots + e_n) \leq \delta \leq a_0,$$

this is one of the tuples counted by the left hand side $S_{\mathbf{b},\delta}$, establishing a bijection between the sets counted by $S_{\mathbf{b},\delta}$ and $T_{\mathbf{a}}$. □

In particular, the following corollary holds.

Corollary 9.3.3.6 ([**208**])**.** *Write $d = a + n$ and consider the monomial $\phi = x_0^a x_1 \cdots x_n$. If $a \geq n$, then $\underline{\mathbf{R}}_S(x_0^a x_1 \cdots x_n) = 2^n$. Otherwise,*

$$\binom{n}{\lfloor \frac{d}{2} \rfloor - a} + \binom{n}{\lfloor \frac{d}{2} \rfloor - a + 1} + \cdots + \binom{n}{\lfloor \frac{d}{2} \rfloor} \leq \underline{\mathbf{R}}_S(x_0^a x_1 \cdots x_n) \leq 2^n.$$

Proof. The right hand inequality follows as $T_{(1,\dots,1)} = 2^n$. To see the left hand inequality, for $0 \leq k \leq a$, let $e = \lfloor \frac{d}{2} \rfloor - a + k$. Then $\binom{n}{e}$ is the number of monomials of the form $x_0^{\lfloor d/2 \rfloor - e} x_{i_1} \cdots x_{i_e}$, $1 \leq i_1 < \cdots < i_e \leq n$, and $S_{(a,1,\dots,1),\lfloor \frac{d}{2} \rfloor}$ is precisely the total number of all such monomials for all values of e. $\square$

The following theorem appeared just as this book was being finished:

Theorem 9.3.3.7 ([**272**])**.** $\mathbf{R}_S(x_1 \cdots x_n) = 2^{n-1}$.

Compare this with the lower bound for the border rank of $\underline{\mathbf{R}}_S(\phi) \geq \binom{n}{\lfloor n/2 \rfloor}$ obtained by considering the flattening $\phi_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$.

That 2^{n-1} gives an upper bound may be seen as follows:

$$x_1 \cdots x_n = \frac{1}{2^{n-1} n!} \sum_{\epsilon \in \{-1,1\}^{n-1}} (x_1 + \epsilon_1 x_2 + \cdots + \epsilon_{n-1} x_n)^n \epsilon_1 \cdots \epsilon_{n-1},$$

a sum of 2^{n-1} terms.

Chapter 10

Normal forms for small tensors

This chapter contains theory as well as practical information for applications regarding spaces and varieties whose points can be explicitly parametrized. I have made an effort to give statements that are easily read by researchers working in applications.

The best possible situation for normal forms is a space with a group action that decomposes it into a finite number of orbits. A typical example is the space of matrices $\mathbb{C}^p \otimes \mathbb{C}^q$ under the action of $GL_p \times GL_q$, where the orbits correspond to the ranks. The next best case I will refer to as being "tame", which essentially means there are normal forms with parameters. A typical tame case is that of linear maps from a vector space to itself, $V \otimes V^*$ under the action of $GL(V)$. Here there is Jordan normal form.

In §10.1, I present Kac's classification of pairs (G, V) with a finite number of orbits and explicitly describe the spaces of tensors, symmetric tensors, and partially symmetric tensors with a finite number of orbits. Kac's classification of visible representations (visible representations are tame) is discussed in §10.2, including an explicit list of visible spaces of tensors. One such case, of $\mathbb{C}^2 \otimes \mathbb{C}^{\mathbf{b}} \otimes \mathbb{C}^{\mathbf{c}}$, was classified in the nineteenth century by Kronecker. His normal forms and the corresponding ranks are presented in §10.3. Another tame case, $S^3\mathbb{C}^3$, has normal forms dating back to Yerushalmy [**337**] in the 1930s. The forms and their ranks for this case are discussed in §10.4. The case of $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ was studied more recently by K. Ng [**250**]. Here the results are less developed; what is known is presented in §10.5. I also describe normal forms for $\mathbb{C}^2 \otimes S^2 V$ in §10.6.

In addition to vector spaces, small secant varieties can admit normal forms. In principle, one could hope to find normal forms for points on $\sigma_r(v_d(\mathbb{P}^{n-1}))$ when $r \lesssim n$, and for points on $Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n)$ when $r \lesssim \frac{\sum \mathbf{a}_i^2 - n}{\sum \mathbf{a}_i - n + 1}$, e.g., if all $\mathbf{a}_i = \mathbf{a}$, when $r \lesssim \mathbf{a}$. A few exercises on general points in such spaces are given in §10.7.

Differential-geometric preliminaries and general information about limiting secant planes are presented in §10.8. Then, in §10.9 and §10.10, the theory is applied to Veronese and Segre varieties, i.e., symmetric tensors and general tensors. The symmetric case is studied up to σ_5 and the case of general tensors up to σ_3. The Segre case is considerably more difficult than the Veronese case. The reason the Veronese case is easier can be traced to Proposition 4.3.7.6, which severely restricts the nature of limiting r-planes when $r < d$, and to the fact that all tangent directions to a point on the Veronese are equivalent.

10.1. Vector spaces with a finite number of orbits

For any G-variety Z (including the case of a G-module V) an obvious necessary condition for Z to have a finite number of G-orbits is that $\dim G \geq \dim Z$. Moreover, G must have an open orbit in Z. These conditions are not sufficient however.

10.1.1. Spaces of tensors with a finite number of orbits.

Theorem 10.1.1.1 ([**182**]). (1) *The spaces of tensors $A_1 \otimes \cdots \otimes A_n$ with a finite number of $GL(A_1) \times \cdots \times GL(A_n)$-orbits are*
(a) $\mathbb{C}^{\mathbf{a}} \otimes \mathbb{C}^{\mathbf{b}}$;
(b) $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^{\mathbf{a}}$;
(c) $\mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^{\mathbf{a}}$.

(2) *The spaces of symmetric tensors S^dV with a finite number of $GL(V)$-orbits are $S^2\mathbb{C}^{\mathbf{v}}$ and $S^3\mathbb{C}^2$.*

(3) *The spaces of partially symmetric tensors $S^dV \otimes W$ with a finite number of $GL(V) \times GL(W)$-orbits are $S^2\mathbb{C}^2 \otimes \mathbb{C}^{\mathbf{w}}$ and $S^2\mathbb{C}^3 \otimes \mathbb{C}^2$.*

10.1.2. Vector spaces with a finite number of orbits in general.
Here is a more general result for those familiar with Dynkin diagrams:

Theorem 10.1.2.1 ([**182**]). *The list of connected linear algebraic groups G acting irreducibly on a vector space V with a finite number of orbits is as follows:*

- *Let $\tilde{\mathfrak{g}}$ be a simple Lie algebra. Let $\mathfrak{g}$ be a reductive Lie algebra with a one-dimensional center whose semisimple part has a Dynkin diagram obtained from the diagram of $\tilde{\mathfrak{g}}$ by removing one node. Mark *the nodes adjacent to the deleted node, with multiplicity equal to one if there was a single bond or a multiple bond with the arrow pointing towards the node, and otherwise with the multiplicity of the bond. The resulting G-module V will have finitely many orbits, where G is the associated Lie (algebraic) group. In these cases $G = F \times \mathbb{C}^*$, with F semisimple. The F-action on V will also have finitely many orbits in the cases $SL_{\mathbf{v}} \times SL_{\mathbf{w}}$, $\mathbf{v} \neq \mathbf{w}$, $SL_{\mathbf{v}} \times SO_{\mathbf{w}}$, $\mathbf{v} > \mathbf{w}$, $SL_{\mathbf{v}} \times Sp_{\mathbf{w}}$, $\mathbf{v} \neq \mathbf{w}$, or $\mathbf{v}$ odd, $(SL(V), \Lambda^2 V)$, $\mathbf{v}$ odd, $SL_2 \times SL_3 \times SL_{\mathbf{w}}$ for $\mathbf{w} = 5$ or $\mathbf{w} > 6$, $Spin_{10}$, $SL_2 \times SL_{\mathbf{w}}$ on $\mathbb{C}^2 \otimes \Lambda^2 W$ when $\mathbf{w} = 5, 7$.*

- $\mathbb{C}^2 \otimes \mathcal{S}_{Spin_7} = \mathbb{C}^2 \otimes V_{\omega_7}^{B_3}$ *as an $SL_2 \times Spin_7$-module.*
- $\mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^{\mathbf{a}}$ *(new cases are $\mathbf{a} \geq 6$).*

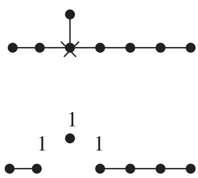


Figure 10.1.1. $\mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^5$ has a finite number of orbits thanks to $\mathfrak{e}_8$.

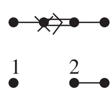


Figure 10.1.2. $\mathbb{C}^2 \otimes S^2 \mathbb{C}^3$ has a finite number of orbits thanks to $\mathfrak{f}_4$.

Remark 10.1.2.2. These spaces are exactly the spaces $T_1 \subset T_{[\mathrm{Id}]} \tilde{G}/P$, where P is a maximal parabolic subgroup of $\tilde{G}$ which induces a filtration of $T_{[\mathrm{Id}]} \tilde{G}/P$, and T_1 denotes the first filtrand.

A method to determine the orbits in these cases is given by Vinberg: the orbits are classified by the sets of weight vectors with prescribed scalar products (via the Killing form of $\tilde{\mathfrak{g}}$); see [**324, 325**]. W. Kraśkiewicz and J. Weyman determined the complete list of orbits in these spaces in [**193**], including many properties of the corresponding orbit closures.

Exercise 10.1.2.3: Prove that if G acts on V with finitely many orbits, then there is at most one invariant hypersurface. ⊚

10.2. Vector spaces where the orbits can be explicitly parametrized

10.2.1. Visible pairs. Let $V = \mathrm{End}(\mathbb{C}^n)$, the space of $n \times n$ matrices, with the action of $G = GL_n$ by conjugation. Thanks to Jordan normal form, the orbits can be parametrized. That is, there are a finite number of discrete invariants (the types of the Jordan blocks) and, fixing the discrete invariants, the orbits with those invariants can be parametrized (by the eigenvalues).

Consider the ring of polynomials on V invariant under the action of G, denoted $\mathbb{C}[V]^G$. This ring is generated by the elementary symmetric functions of the eigenvalues. In particular the generators are algebraically independent.

From this perspective, the space of nilpotent matrices can be described as

$$N(V) := \{v \in V \mid \forall f \in \mathbb{C}[V]^G_{>0},\ f(v) = 0\}. \tag{10.2.1}$$

Observe the following properties:

(1) $\mathbb{C}[V]^G$ is a polynomial ring.

(2) The G-orbits in V can be explicitly described.

(3) G has finitely many orbits in $N(V)$.

Now let (G, V) be a pair, where G is reductive and V is an irreducible G-module. Define the *nullcone* $N(V)$ by (10.2.1). If there are finitely many orbits in $N(V)$, one calls the pair (G, V) *visible*.

If (G, V) is visible, then, as was shown in [**285**], $\mathbb{C}[V]^G$ is a polynomial ring (and the level varieties of the invariant polynomials all have the same dimension). Moreover, if one considers the map $\pi : V \to V//G$, defined by $v \mapsto (p_1(v), \ldots, p_k(v))$, where $p_1, \ldots, p_k$ generate $\mathbb{C}[V]^G$, the pre-image of each point consists of finitely many G-orbits, which allows for an explicit description of all the orbits. The space $V//G$ is called the *GIT quotient.*

The pairs $(G, \mathfrak{g})$, where $\mathfrak{g}$ is the adjoint representation of G, are visible. Several spaces of tensors are as well:

Theorem 10.2.1.1 ([**182**]). *The visible spaces of tensors (under the action of the product of the general linear groups) are*

(1) $S^3\mathbb{C}^2$, *where* $\mathbb{C}[S^3\mathbb{C}^2]^{SL_2} = \mathbb{C}[disc]$, *and* $disc \in S^4(S^3\mathbb{C}^2)^*$ *is the classical discriminant;*

(2) $S^4\mathbb{C}^2$, *where* $\mathbb{C}[S^4\mathbb{C}^2]^{SL_2}$ *has two generators, one in degree two and one in degree three;*

(3) $\mathbb{C}^2 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4$;

(4) $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$;

(5) $\mathbb{C}^2\otimes S^2\mathbb{C}^4$;

(6) $S^3\mathbb{C}^3$;

(7) $\mathbb{C}^3\otimes S^2\mathbb{C}^3$;

(8) $\mathbb{C}^2\otimes\mathbb{C}^3\otimes\mathbb{C}^6$.

In [**182**], Kac describes a complete list of visible pairs (G, V), where G is a reductive group and V an irreducible G-module, and determines which of these spaces have finitely many orbits. In his original list there were several errors, which were corrected in [**106**].

The visible pairs with an infinite number of orbits are all obtained by deleting a node from an affine Dynkin diagram and marking the nodes adjacent to the deleted node according to the same recipe as above for the finite case. Sometimes this case is called *tame*. I will use the word tame in the slightly larger context of either visible or having an open orbit as in the next subsection. The cases in Theorem 10.2.1.1 are respectively obtained from:

$$\mathfrak{g}_2{}^{(1)}, \mathfrak{a}_2{}^{(2)}, \mathfrak{e}_7{}^{(1)}, \mathfrak{e}_6{}^{(1)}, \mathfrak{e}_6{}^{(2)}, \mathfrak{d}_4{}^{(3)}, \mathfrak{f}_4{}^{(1)}, \mathfrak{e}_8{}^{(1)}.$$

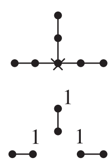


Figure 10.2.1. Orbits in $\mathbb{C}^3\otimes\mathbb{C}^3\otimes\mathbb{C}^3$ are parametrizable thanks to $\mathfrak{e}_6^{(1)}$.

10.2.2. Cases where (G, V) contains an open orbit. Another class of pairs (G, V) that overlaps with the visible case but is distinct, is when there is an open G-orbit in V.

Consider $G = GL_p \times GL_q \times GL_r$-orbits in $V = \mathbb{C}^p\otimes\mathbb{C}^q\otimes\mathbb{C}^r$. Note that $\dim V = pqr$ and $\dim G = p^2 + q^2 + r^2$, but in fact the image of G in $GL(V)$ has dimension $p^2+q^2+r^2-2$ because $(\lambda\,\mathrm{Id}, \mu\,\mathrm{Id}, \nu\,\mathrm{Id})$ with $\lambda\mu\nu = 1$ acts as the identity on V. Thus there is no hope for an open orbit unless $p^2 + q^2 + r^2 - 2 - pqr \geq 0$.

Theorem 10.2.2.1 ([**281**])**.** *If $p^2 + q^2 + r^2 - 2 - pqr \geq 0$, then $G = GL_p \times GL_q \times GL_r$ has an open orbit in $V = \mathbb{C}^p\otimes\mathbb{C}^q\otimes\mathbb{C}^r$.*

The space $\mathbb{C}^2\otimes\mathbb{C}^{\mathbf{v}}\otimes\mathbb{C}^{\mathbf{v}}$ has an open orbit but is not visible when $\mathbf{v} > 4$. The space $\mathbb{C}^3\otimes\mathbb{C}^3\otimes\mathbb{C}^3$ is visible but does not have an open orbit.

The main tool for proving Theorem 10.2.2.1 is the following result, which involves the so-called *Castling transform*.

Theorem 10.2.2.2 ([**281**]). *Let G be reductive and let V be an irreducible G-module. There is a one-to-one correspondence between $G \times GL_r$-orbits on $V \otimes \mathbb{C}^r$ and $G \times GL_{\mathbf{v}-r}$-orbits on $V^* \otimes \mathbb{C}^{\mathbf{v}-r}$.*

In particular, $\mathbb{C}^p \times \mathbb{C}^q \times \mathbb{C}^r$ has an open orbit if and only if $\mathbb{C}^p \times \mathbb{C}^q \times \mathbb{C}^{pq-r}$ does.

Proof. Let $T \in V \otimes \mathbb{C}^r$ and consider $T(\mathbb{C}^{r*}) \subset V$. If $[T] \notin \sigma_{r-1}(Seg(\mathbb{P}V \times \mathbb{P}^{r-1}))$, the image is r-dimensional. Let $(V \otimes \mathbb{C}^r)^0 = (V \otimes \mathbb{C}^r) \backslash \hat{\sigma}_{r-1}(Seg(\mathbb{P}V \times \mathbb{P}^{r-1}))$. We obtain a map $\phi : (V \otimes \mathbb{C}^r)^0 \to G(r, V)$. The map ϕ is G-equivariant, and for $E \in G(r, V)$, $\phi^{-1}(E)$ is a single GL_r-orbit, so ϕ gives a correspondence of orbits. Now observe that $G(r, V) \simeq G(\mathbf{v} - r, V^*)$ to obtain the desired correspondence of nondegenerate orbits. The remaining cases are left to the reader. (Hint: Work next on $\sigma_{r-1} \backslash \sigma_{r-2}$.) □

10.2.3. Case where all orbit closures are secant varieties of the closed orbit. Among pairs (G, V) with a finite number of orbits, there is a preferred class (called *subcominuscule*), where the only orbit closures are the successive secant varieties of $X = G/P \subset \mathbb{P}V$. The examples where all orbit closures are secant varieties of the closed orbit are:

X	V	geometric interpretation of r-th orbit closure $\sigma_r(X)$
$Seg(\mathbb{P}A \times \mathbb{P}B)$	$A \otimes B$	$\mathbf{a} \times \mathbf{b}$-matrices of rank at most r
$v_2(\mathbb{P}W)$	S^2W	symmetric matrices of rank at most r
$G(2, W)$	$\Lambda^2 W$	skew-symmetric matrices of rank at most $2r$
$\mathbb{S}_5$	$\mathcal{S}_+$	
$\mathbb{OP}^2$	$\mathcal{J}_3(\mathbb{O})$	octonionic-Hermitian matrices of "rank" at most r

The first three are the classical cases of matrices, symmetric matrices, and skew-symmetric matrices. Any element in the r-th orbit in the first two cases is equivalent to

$$\begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}.$$

10.3. Points in $\mathbb{C}^2 \otimes \mathbb{C}^{\mathbf{b}} \otimes \mathbb{C}^{\mathbf{c}}$

I follow the presentation in [**49**] in this section. As mentioned in Chapter 3, Kronecker determined a normal form for pencils of matrices, i.e., two-dimensional linear subspaces of $B \otimes C$ up to the action of $GL(B) \times GL(C)$.

After reviewing the normal form, I use it to derive the Grigoriev-Ja'Ja'-Teichert classification of ranks of elements of these spaces.

10.3.1. Kronecker's normal form. Kronecker determined a normal form for pencils of matrices, i.e., two-dimensional linear subspaces of $B\otimes C$ up to the action of $GL(B)\times GL(C)$. It is convenient to use matrix notation so choose bases of B, C and write the family as $sX+tY$, where $X, Y\in B\otimes C$ and $s,t\in\mathbb{C}$. (Kronecker's classification works over arbitrary closed fields, as does J. Ja'Ja's partial classification of rank, but we only present the results over $\mathbb{C}$.) The result is as follows (see, e.g., [**136**, Chap. XII]):

Define the $\epsilon\times(\epsilon+1)$ matrix

$$L_\epsilon = L_\epsilon(s,t) = \begin{pmatrix} s & t & & \\ & \ddots & \ddots & \\ & & s & t\end{pmatrix}.$$

The normal form is

$$sX+tY = \begin{pmatrix} L_{\epsilon_1} & & & & & & \\ & \ddots & & & & & \\ & & L_{\epsilon_q} & & & & \\ & & & L_{\eta_1}^T & & & \\ & & & & \ddots & & \\ & & & & & L_{\eta_p}^T & \\ & & & & & & s\,\mathrm{Id}_f + tF \end{pmatrix}, \tag{10.3.1}$$

where F is an $f\times f$ matrix in Jordan normal form (one can also use rational canonical form) and T denotes the transpose.

10.3.2. Complete normalization is possible when $f\leq 3$. Say F has Jordan blocks, $F_{i,j}$, where λ_i is the i-th eigenvalue. If there is no block of the form L_{ϵ_i} or $L_{\eta_j}^T$, then we may assume that at least one of the λ_i is zero by changing basis in $\mathbb{C}^2$. If the blocks $F_{i,j}$ are such that there are no 1's above the diagonal, then we can normalize one of the $\lambda_i=1$ by rescaling t.

For example, say $f=3$, then the possible normal forms are

$$\begin{pmatrix}\lambda & & \\ & \mu & \\ & & \nu\end{pmatrix}, \begin{pmatrix}\lambda & & \\ & \lambda & \\ & & \mu\end{pmatrix}, \begin{pmatrix}\lambda & 1 & \\ & \lambda & \\ & & \mu\end{pmatrix}, \begin{pmatrix}\lambda & & \\ & \lambda & \\ & & \lambda\end{pmatrix}, \begin{pmatrix}\lambda & 1 & \\ & \lambda & \\ & & \lambda\end{pmatrix}, \begin{pmatrix}\lambda & 1 & \\ & \lambda & 1\\ & & \lambda\end{pmatrix}$$

which (again provided there is no block of the form L_{ϵ_i} or $L_{\eta_j}^T$) can respectively be normalized to

$$\begin{pmatrix}0 & & \\ & 1 & \\ & & -1\end{pmatrix}, \begin{pmatrix}0 & & \\ & 0 & \\ & & 1\end{pmatrix}, \begin{pmatrix}0 & 1 & \\ & 0 & \\ & & 1\end{pmatrix}, \begin{pmatrix}0 & & \\ & 0 & \\ & & 0\end{pmatrix}, \begin{pmatrix}0 & 1 & \\ & 0 & \\ & & 0\end{pmatrix}, \begin{pmatrix}0 & 1 & \\ & 0 & 1\\ & & 0\end{pmatrix}. \tag{10.3.2}$$

Note that when X is $f \times f$, the fourth case is not a pencil. The first case requires explanation—we claim that all pencils of the form

$$s\begin{pmatrix}1&&\\&1&\\&&1\end{pmatrix}+t\begin{pmatrix}\lambda&&\\&\mu&\\&&\nu\end{pmatrix},$$

where λ, μ, ν are distinct, are equivalent. In particular, any such is equivalent to one where $\lambda = 0, \mu = 1, \nu = -1$. To prove the claim, first get rid of λ by replacing s with $s_1 := s + \lambda t$:

$$s_1\begin{pmatrix}1&&\\&1&\\&&1\end{pmatrix}+t\begin{pmatrix}0&&\\&\mu_1&\\&&\nu_1\end{pmatrix}.$$

Note that 0, μ_1, and ν_1 are still distinct. Next we replace t with $t_2 := s_1 + t\mu_1$:

$$s_1\begin{pmatrix}1&&\\&0&\\&&\mu_2\end{pmatrix}+t_2\begin{pmatrix}0&&\\&1&\\&&\nu_2\end{pmatrix}.$$

where $\mu_2 = 1 - \frac{\nu_1}{\mu_1}$ and $\nu_2 = -\frac{\nu_1}{\mu_1}$ and 0, μ_2, and ν_2 are distinct. Then we transport the constants to the first two entries by setting $s_3 := \frac{1}{\mu_2}s_1$ and $t_3 := \frac{1}{\nu_2}t_2$:

$$s_3\begin{pmatrix}\frac{1}{\mu_2}&&\\&0&\\&&1\end{pmatrix}+t_3\begin{pmatrix}0&&\\&\frac{1}{\nu_2}&\\&&1\end{pmatrix}.$$

It only remains to change basis in B by sending b_1 to $\frac{1}{\mu_2}b_1$ and b_2 to $\frac{1}{\nu_2}b_2$ to show that the pencil is equivalent to:

$$s_3\begin{pmatrix}1&&\\&0&\\&&1\end{pmatrix}+t_3\begin{pmatrix}0&&\\&1&\\&&1\end{pmatrix}.$$

Thus every two such pencils are equivalent.

If F is 4×4, it is no longer possible to normalize away all constants in the case of

$$\begin{pmatrix}\lambda&1&&\\&\lambda&&\\&&\mu&1\\&&&\mu\end{pmatrix}.$$

This case gives the idea of the proof of Theorem 10.1.1.1 that the only spaces of tensors $\mathbb{C}^{\mathbf{a}}\otimes\mathbb{C}^{\mathbf{b}}\otimes\mathbb{C}^{\mathbf{c}}$, $2 \leq \mathbf{a} \leq \mathbf{b} \leq \mathbf{c}$, that have a finite number of $GL_{\mathbf{a}} \times GL_{\mathbf{b}} \times GL_{\mathbf{c}}$-orbits, are $\mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^{\mathbf{c}}$ and $\mathbb{C}^2\otimes\mathbb{C}^3\otimes\mathbb{C}^{\mathbf{c}}$. Moreover, in the first case any tensor lies in a $\mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^4$ and in the second, any tensor lies in a $\mathbb{C}^2\otimes\mathbb{C}^3\otimes\mathbb{C}^6$.

10.3.3. Ranks of tensors in $\mathbb{C}^2 \otimes \mathbb{C}^{\mathbf{b}} \otimes \mathbb{C}^{\mathbf{c}}$. For a fixed linear map $F : \mathbb{C}^f \to \mathbb{C}^f$, let $d(\lambda)$ denote the number of Jordan blocks of size at least two associated to the eigenvalue λ, and let $M(F)$ denote the maximum of the $d(\lambda)$.

Theorem 10.3.3.1 (Grigoriev, Ja'Ja', Teichert; see [**149, 181, 304**])**.** *A pencil of the form* (10.3.1) *has rank*

$$\sum_{i=1}^{q}(\epsilon_i + 1) + \sum_{j=1}^{p}(\eta_j + 1) + f + M(F).$$

In particular, the maximum possible rank of a tensor in $\mathbb{C}^2 \otimes \mathbb{C}^{\mathbf{b}} \otimes \mathbb{C}^{\mathbf{b}}$ *is* $\lfloor \frac{3\mathbf{b}}{2} \rfloor$.

Remark 10.3.3.2. In [**181**], Theorem 10.3.3.1 is stated as an inequality (Cor. 2.4.3 and Thm. 3.3), but the results are valid over arbitrary closed fields. In [**149**] the results are stated, but not proved, and the reader is referred to [**148**] for indications towards the proofs. In [**54**] a complete proof is given of an equivalent statement in terms of the elementary divisors of the pair, and the text states the proof is taken from the unpublished PhD thesis [**304**].

The proof of Theorem 10.3.3.1 presented in [**54**] uses four lemmas.

Proposition 10.3.3.3. *Let* $A = A_1 \oplus A_2$, $B = B_1 \oplus B_2$, $T_1 \in A_1 \otimes B_1 \otimes C$, $T_2 \in A_2 \otimes B_2 \otimes C$, *and* $T_3 \in A_2 \otimes B_1 \otimes C$. *Suppose* $\mathbf{a}_i$ *and* $\mathbf{b}_i$ *are the dimensions of, respectively,* A_i, B_i *for* $i = 1, 2$. *Then*

(1) *If* $T_2 : A_2^* \to B_2 \otimes C$ *is injective, then* $\mathbf{R}(T_1 + T_2 + T_3) \geq \mathbf{R}(T_1) + \mathbf{a}_2$.
(2) *If both maps* $T_2 : A_2^* \to B_2 \otimes C$ *and* $T_2 : B_2^* \to A_2 \otimes C$ *are injective and* $\mathbf{R}(T_2) = \max\{\mathbf{a}_2, \mathbf{b}_2\}$ *(the minimum possible for such* T_2*), then* $\mathbf{R}(T_1 + T_2) = \mathbf{R}(T_1) + \mathbf{R}(T_2)$.

Proof. To prove (1) let $T := T_1 + T_2 + T_3$, $r := \mathbf{R}(T)$ and write $T = \sum_{i=1}^{r} a_i \otimes b_i \otimes c_i$ in some minimal presentation. Consider the projection $p_{A_2} : A \to A_2$. Since $T|_{A_2^*}$ is injective, we may assume that $p_{A_2}(a_1), \dots, p_{A_2}(a_{\mathbf{a}_2})$ form a basis of A_2. Let $A_2' \subset A$ be the span of $a_1, \dots, a_{\mathbf{a}_2}$. Note that the composition $A_1 \hookrightarrow A \to A/A_2'$ is an isomorphism. Consider the following composed projection π:

$$A \otimes B \otimes C \to (A/A_2') \otimes B \otimes C \to (A/A_2') \otimes B_1 \otimes C.$$

The kernel of π contains $A \otimes B_2 \otimes C$, and $\pi|_{A_1 \otimes B_1 \otimes C}$ is an isomorphism. Thus $\pi(T)$ is T_1 (up to the isomorphism $A/A_2' \simeq A_1$) and also $\pi(T) = \sum_{i=\mathbf{a}_2+1}^{r} \pi(a_i \otimes b_i \otimes c_i)$. Hence $\mathbf{R}(T_1) \leq r - \mathbf{a}_2$ as claimed in (1).

Statement (2) follows from (1) with $T_3 = 0$ used twice, once with the roles of A and B exchanged to note that $\mathbf{R}(T_1 + T_2) \geq \mathbf{R}(T_1) + \mathbf{a}_2$ and $\mathbf{R}(T_1 + T_2) \geq \mathbf{R}(T_1) + \mathbf{b}_2$, and that $\mathbf{R}(T_1 + T_2) \leq \mathbf{R}(T_1) + \mathbf{R}(T_2)$. $\square$

Proposition 10.3.3.3 was stated and proven for the special case $\dim C = 2$ in [**54**, Lemma 19.6]. The lemma is worth generalizing because it provides an example of a situation where the *additivity* conjectured by Strassen (see §5.7) holds.

The next two lemmas follow by perturbing the relevant pencil by a rank one pencil to obtain a diagonalizable pencil. (Recall that a generic $n \times n$ pencil is diagonalizable and thus of rank n.)

Lemma 10.3.3.4. $\mathbf{R}(L_\epsilon) = \epsilon + 1$.

Lemma 10.3.3.5. *Let a pencil T be given by $(\mathrm{Id}_{\mathbf{a}}, F)$ with F a size $\mathbf{a}$ matrix in Jordan normal form with no eigenvalue having more than one associated Jordan block. Then $\mathbf{R}(T) = \mathbf{a} + 1$ if the Jordan form is not diagonal, and $\mathbf{R}(T) = \mathbf{a}$ if the Jordan form is diagonal.*

Here is a proof of Lemma 10.3.3.5 from [**49**] that generalizes to the symmetric case. Consider the classical identity, where $\mathbf{A}$ is an $\mathbf{a} \times \mathbf{a}$ matrix, and $\mathbf{u}, \mathbf{v} \in \mathbb{C}^{\mathbf{a}}$:

$$\det(\mathbf{A} + \mathbf{u}\mathbf{v}^{\mathrm{T}}) = (1 + \mathbf{v}^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{u}) \det(\mathbf{A}). \tag{10.3.3}$$

Apply this with $\mathbf{A} = F - t\,\mathrm{Id}_{\mathbf{a}}$. Replace $\mathbf{A}^{-1}$ by the cofactor matrix $\Lambda^{\mathbf{a}-1}\mathbf{A}$ divided by the determinant. If a given eigenvalue λ has more than one Jordan block, then $(t - \lambda)$ will appear in the determinant to a higher power than in the subdeterminant of the Jordan block of the inverse, so it will still divide the whole polynomial. Otherwise, $(t - \lambda_1) \cdots (t - \lambda_{\mathbf{a}}) + \mathbf{v}^{\mathrm{T}}\Lambda^{\mathbf{a}-1}\mathbf{A}$ will have distinct roots for general $\mathbf{u}, \mathbf{v}$. For later use, I remark that this remains true even if $\mathbf{u} = \mathbf{v}$.

Finally, they show

Lemma 10.3.3.6 ([**54**, Prop. 19.10])**.** *Let a pencil T be given by $(\mathrm{Id}_{2\mathbf{a}}, F)$ with F a matrix consisting of $\mathbf{a}$ Jordan blocks of size two, all with the same eigenvalue. Then $\mathbf{R}(T) = 3\mathbf{a}$.*

The proof of Lemma 10.3.3.6 is similar to that of Proposition 10.3.3.3; namely, one splits the computation and defines an appropriate projection operator.

10.3.4. Orbits in $A \otimes B \otimes C$. In this section, for spaces of tensors $A \otimes B \otimes C$ with a finite number of $GL(A) \times GL(B) \times GL(C)$-orbits, I present the list of orbits with their Kronecker normal form (which appeared in [**262**]), geometric descriptions of the orbit closures along with their dimensions, and the ranks and border ranks of the points in the orbits, following [**49**].

The first case is $\dim A = \dim B = 2$ and $\dim C = \mathbf{c}$. Table 10.3.1 lists a representative of each orbit of the $GL(A) \times GL(B) \times GL(C)$-action on $\mathbb{P}(A\otimes B\otimes C)$, where $\dim A = \dim B = 2$ and $\dim C = \mathbf{c}$.

Here and in what follows

$$X = Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C).$$

Table 10.3.1. Orbits in $\mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^{\mathbf{c}}$**.** Each orbit is uniquely determined by its closure, which is an algebraic variety listed in the second column. The orbit itself is an open dense subset of this variety. The dimension of the algebraic variety is in the third column. The fourth column is the normal form of the underlying tensor; the distinct variables are assumed to be linearly independent. The normal form is also given as a pencil, except cases 1 and 3, which are not pencils of matrices. The border rank and rank are given in the next columns. If $\mathbf{c} = 3$, then $\sigma_3(X) = \mathbb{P}(A\otimes B\otimes C)$, and case 9 does not occur.

#	orbit closure	dim	Kronecker normal form	pencil	$\underline{\mathbf{R}}$	$\mathbf{R}$
1	X	$\mathbf{c}+1$	$a_1\otimes b_1\otimes c_1$	$\begin{pmatrix} s \end{pmatrix}$	1	1
2	Sub_{221}	$\mathbf{c}+2$	$a_1\otimes b_1\otimes c_1 + a_2\otimes b_2\otimes c_1$	$\begin{pmatrix} s & t \end{pmatrix}$	2	2
3	Sub_{122}	$2\mathbf{c}$	$a_1\otimes b_1\otimes c_1 + a_1\otimes b_2\otimes c_2$	$\begin{pmatrix} s & \\ & s \end{pmatrix}$	2	2
4	Sub_{212}	$2\mathbf{c}$	$a_1\otimes b_1\otimes c_1 + a_2\otimes b_1\otimes c_2$	$\begin{pmatrix} s \\ t \end{pmatrix}$	2	2
5	$\tau(X)$	$2\mathbf{c}+2$	$a_1\otimes(b_1\otimes c_1 + b_2\otimes c_2)$ $+a_2\otimes b_1\otimes c_2$	$\begin{pmatrix} s & t \\ & s \end{pmatrix}$	2	3
6	$\sigma_2(X) = Sub_{222}$	$2\mathbf{c}+3$	$a_1\otimes b_1\otimes c_1 + a_2\otimes b_2\otimes c_2$	$\begin{pmatrix} s & \\ & t \end{pmatrix}$	2	2
7	$X_*^\vee$	$3\mathbf{c}+1$	$a_1\otimes(b_1\otimes c_1 + b_2\otimes c_3)$ $+a_2\otimes b_1\otimes c_2$	$\begin{pmatrix} s & t & \\ & & s \end{pmatrix}$	3	3
8	$\sigma_3(X)$	$3\mathbf{c}+2$	$a_1\otimes(b_1\otimes c_1 + b_2\otimes c_2)$ $+a_2\otimes(b_1\otimes c_2 + b_2\otimes c_3)$	$\begin{pmatrix} s & t & \\ & s & t \end{pmatrix}$	3	3
9	$\mathbb{P}(A\otimes B\otimes C)$	$4\mathbf{c}-1$	$a_1\otimes(b_1\otimes c_1 + b_2\otimes c_3)$ $+a_2\otimes(b_1\otimes c_2 + b_2\otimes c_4)$	$\begin{pmatrix} s & t & & \\ & & s & t \end{pmatrix}$	4	4

Recall from §7.1 that the *subspace variety* $Sub_{ijk} \subset \mathbb{P}(A\otimes B\otimes C)$ is the set of tensors $[p] \in \mathbb{P}(A\otimes B\otimes C)$ such that there exist linear subspaces $A' \subset A$, $B' \subset B$, $C' \subset C$ respectively of dimensions i, j, k such that $p \in A'\otimes B'\otimes C'$, and that $\dim Sub_{ijk} = i(\mathbf{a}-i)+j(\mathbf{b}-j)+k(\mathbf{c}-k)+ijk-1$. The other interpretations are as follows: $\tau(X)$ is the *tangential variety* to the Segre variety, $X_* \subset \mathbb{P}(A^*\otimes B^*\otimes C^*)$ is the Segre variety in the dual projective space, and $X_*^\vee \subset \mathbb{P}(A\otimes B\otimes C)$ is its dual variety.

The point of $\tau(X)$ is tangent to the point $[a_1\otimes b_2\otimes c_1]$, the point of $X_*^\vee$ contains the tangent plane to the $(\mathbf{c}-3)$-parameter family of points $[a_2^*\otimes b_1^*\otimes(s_2c_2^* + s_2c_4^* + s_2c_5^* + \cdots + s_{\mathbf{c}}c_{\mathbf{c}}^*)]$, where (a_j^*) is the dual basis to

(a_j) of A, etc. The dual variety $X_*^\vee$ is degenerate (i.e., not a hypersurface) except when $\mathbf{c} \leq 3$; see, e.g., [**141**, p. 46, Cor. 5.10].

To see the geometric explanations of the orbit closures: Cases 1, 6, 8 are clearly on the respective secant varieties. Cases 2, 3, 4 are all clearly on the respective subspace varieties, and it is straightforward to check that cases 5 and 7 are tangent to the points asserted. Finally, to see that the orbit closures of these points are the asserted ones and that there are no others, one can compute the dimensions of the Lie algebras of their stabilizers to determine the dimensions of their orbit closures.

Note that $\sigma_3(X) = \sigma_3(Seg(\mathbb{P}(A \otimes B) \times \mathbb{P}C))$, which causes it to be degenerate with defect three.

The orbits 1–8 are inherited from the $\mathbf{c} = 3$ case, in the sense that they are contained in Sub_{223}. Orbit 9 is inherited from the $\mathbf{c} = 4$ case.

Proposition 10.3.4.1. *If $[p] \in \mathbb{P}(A \otimes B \otimes C)$, with $A \simeq \mathbb{C}^2$, $B \simeq \mathbb{C}^2$, $C \simeq \mathbb{C}^{\mathbf{c}}$, then p is in precisely one of the orbits* 1–9 *from Table* 10.3.1. *The rank and border rank of $[p]$ are as indicated in the table.*

Proof. Consider $p : A^* \to B \otimes C$. If $\dim p(A^*) = 1$, then let $e \in p(A^*)$ be a nonzero element. Since $\dim B = 2$, the rank of e is one or two, giving cases 1 and 3, respectively.

Otherwise, $\dim p(A^*) = 2$ and the Kronecker normal form (3.11.1) gives the following cases:

2. There is only one block of the form L_1.
4. There is only one block of the form L_1^T.
8. There is only one block of the form L_2.
9. There are two blocks, both of the form L_1.
7. There is one block L_1, and F is a 1×1 matrix. The pencil is then $\begin{pmatrix} s & t & \\ & & s+\lambda t \end{pmatrix}$, and we can normalize λ to zero by changing coordinates: $s' := s + \lambda t$ and $c_1' := c_1 + \lambda c_2$.

5–6. Otherwise, there is no block of the form L_ϵ or L_η^T and F is a 2×2 matrix. We can normalize one of the eigenvalues to 0. We continue, depending on the Jordan normal form of F:

5. $F = \begin{pmatrix} 0 & 1 \\ & 0 \end{pmatrix}$.
6. $F = \begin{pmatrix} 0 & \\ & \lambda \end{pmatrix}$. Note that $\lambda \neq 0$ because $\dim p(A^*) = 2$. Changing the coordinates $t' := \lambda t + s$, we obtain the pencil $\begin{pmatrix} s & \\ & t' \end{pmatrix}$.

The ranks are calculated using Theorem 10.3.3.1. It remains to calculate $\underline{\mathbf{R}}(p)$. The border ranks in cases 1–4 and 6 follow because $\underline{\mathbf{R}}(p) \leq \mathbf{R}(p)$ and $\underline{\mathbf{R}}(p) = 1$ if and only if $[p] \in X$. Case 5 is clear too, as the tangential variety is contained in $\sigma_2(X)$. Further, $X_*^\vee$ cannot be contained in $\sigma_2(X)$, as its dimension is larger, so for $p \in X_*^\vee$, we have $2 < \underline{\mathbf{R}}(p) \leq \mathbf{R}(p) = 3$ proving case 7. Case 8 is clear, and case 9 follows from the dimension count. $\square$

Table 10.3.2. The orbits listed in Table 10.3.1, viewed as orbits in $\mathbb{C}^2\otimes\mathbb{C}^3\otimes\mathbb{C}^{\mathbf{c}}$. Case 9 does not occur for $\mathbf{c} = 3$.

#	orbit closure	dim
1	$X = Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)$	$\mathbf{c}+2$
2	Sub_{221}	$\mathbf{c}+4$
3	Sub_{212}	$2\mathbf{c}+1$
4	Sub_{122}	$2\mathbf{c}+2$
5	$\tau(X)$	$2\mathbf{c}+4$
6	$Sub_{222} = \sigma_2(X)$	$2\mathbf{c}+5$
7	$\mathcal{S}eg_*^\vee \subset Sub_{223}$	$3\mathbf{c}+3$
8	Sub_{223}	$3\mathbf{c}+4$
9	Sub_{224}	$4\mathbf{c}+1$

Now suppose $\dim A = 2$, $\dim B = 3$, and $\dim C = \mathbf{c}$. The list of orbits for $\mathbf{c} = 3$ with their Kronecker normal forms appears in [**262**, Thm. 6]. First, we inherit all the orbits from the $\mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^{\mathbf{c}}$ case, i.e., all the orbits from Table 10.3.1. They become subvarieties of $Sub_{22\mathbf{c}}$ with the same normal forms, pencils, ranks, and border ranks—the new dimensions are presented in Table 10.3.2.

In Table 10.3.2 and below, $\mathcal{S}eg_*^\vee \subset Sub_{ijk}$ denotes the subvariety of Sub_{ijk}, obtained from the subfiber bundle of $\mathcal{S}_{G(i,A)}\otimes\mathcal{S}_{G(j,B)}\otimes\mathcal{S}_{G(k,C)}$, whose fiber in $A'\otimes B'\otimes C'$ (where $\dim A' = i$, $\dim B' = j$, and $\dim C' = k$) is

$$\hat{S}eg(\mathbb{P}A'^* \times \mathbb{P}B'^* \times \mathbb{P}C'^*)^\vee \subset A'\otimes B'\otimes C'.$$

In the special case $(i,j,k) = (\mathbf{a},\mathbf{b},\mathbf{c})$, the variety $\mathcal{S}eg_*^\vee \subset Sub_{ijk}$ becomes $X_*^\vee$.

Table 10.3.2 lists the orbits in $\mathbb{C}^2\otimes\mathbb{C}^3\otimes\mathbb{C}^{\mathbf{c}}$ that are contained in Sub_{233}, that is, tensors in some $\mathbb{C}^2\otimes\mathbb{C}^3\otimes\mathbb{C}^3$.

Proposition 10.3.4.2. *Table* 10.3.3 *lists the orbits in* $\mathbb{P}(\mathbb{C}^2\otimes\mathbb{C}^3\otimes\mathbb{C}^3)$ *that are not contained in* Sub_{223}.

Proof. Let $[p] \in \mathbb{P}(\mathbb{C}^2\otimes\mathbb{C}^3\otimes\mathbb{C}^3)$, and let $p(A^*) = p((\mathbb{C}^2)^*)$. If $\dim p(A^*) = 1$, then p must be as in case 10. Otherwise $\dim p(A^*) = 2$ and the Kronecker normal form gives cases 11–13, if there is at least one block of the form L_ϵ or L_ϵ^T. Note that in the case $\begin{pmatrix} s & \\ t & s+\lambda t \end{pmatrix}$ the eigenvalue may be set to zero (as in case 7) to obtain case 11.

Table 10.3.3. Orbits in $\mathbb{C}^2\otimes\mathbb{C}^3\otimes\mathbb{C}^{\mathbf{c}}$ contained in Sub_{233}. Note that the cases 11 and 12 are are analogous to orbits in $\mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^3$. The unnamed orbits 13–16 are various components of the singular locus of $\mathcal{S}eg_*^\vee \subset Sub_{233}$ (case 17); see [**193**] for descriptions.

#	orbit cl.	dim	Kronecker normal form	pencil	$\underline{\mathbf{R}}$	$\mathbf{R}$
10	Sub_{133}	$3\mathbf{c}$	$a_1\otimes(b_1\otimes c_1+b_2\otimes c_2+b_3\otimes c_3)$	$\begin{pmatrix} s & & \\ & s & \\ & & s\end{pmatrix}$	3	3
11	$\mathcal{S}eg_*^\vee \subset Sub_{232}$	$2\mathbf{c}+6$	$a_1\otimes(b_1\otimes c_1+b_3\otimes c_2)$ $+a_2\otimes b_2\otimes c_1$	$\begin{pmatrix} s & \\ t & \\ & s\end{pmatrix}$	3	3
12	Sub_{232}	$2\mathbf{c}+7$	$a_1\otimes(b_1\otimes c_1+b_2\otimes c_2)$ $+a_2\otimes(b_2\otimes c_1+b_3\otimes c_2)$	$\begin{pmatrix} s & \\ t & s \\ & t\end{pmatrix}$	3	3
13			$a_1\otimes(b_1\otimes c_1+b_2\otimes c_3)$ $+a_2\otimes(b_1\otimes c_2+b_3\otimes c_3)$	$\begin{pmatrix} s & t & \\ & & s \\ & & t\end{pmatrix}$	3	4
14			$a_1\otimes(b_1\otimes c_1+b_2\otimes c_2)$ $+a_2\otimes b_3\otimes c_3$	$\begin{pmatrix} s & & \\ & s & \\ & & t\end{pmatrix}$	3	3
15			$a_1\otimes(b_1\otimes c_1+b_2\otimes c_2+b_3\otimes c_3)$ $+a_2\otimes b_1\otimes c_2$	$\begin{pmatrix} s & t & \\ & s & \\ & & s\end{pmatrix}$	3	4
16			$a_1\otimes(b_1\otimes c_1+b_2\otimes c_2+b_3\otimes c_3)$ $+a_2\otimes(b_1\otimes c_2+b_2\otimes c_3)$	$\begin{pmatrix} s & t & \\ & s & t \\ & & s\end{pmatrix}$	3	4
17	$\mathcal{S}eg_*^\vee \subset Sub_{233}$	$3\mathbf{c}+7$	$a_1\otimes(b_1\otimes c_1+b_2\otimes c_2)$ $+a_2\otimes(b_1\otimes c_2+b_3\otimes c_3)$	$\begin{pmatrix} s & t & \\ & s & \\ & & t\end{pmatrix}$	3	4
18	Sub_{233}	$3\mathbf{c}+8$	$a_1\otimes(b_1\otimes c_1+b_2\otimes c_2)$ $+a_2\otimes(b_2\otimes c_2+b_3\otimes c_3)$	$\begin{pmatrix} s & & \\ & s+t & \\ & & t\end{pmatrix}$	3	3

Now suppose there is just the block $s\operatorname{Id}_3+tF$, for F a 3×3 matrix in its Jordan normal form. Then F can be normalized to one of the six matrices in (10.3.2). One of these matrices gives case 10, while the remaining give cases 14–18.

Since $\sigma_3(Seg(\mathbb{P}^1\times\mathbb{P}^2\times\mathbb{P}^2))$ fills out the ambient space, all the tensors listed in the table have border rank 3. The ranks follow from Theorem 10.3.3.1. □

Next consider tensors contained in Sub_{234} that are not contained in Sub_{233}. These orbits are listed in Table 10.3.4.

Proposition 10.3.4.3. *Table* 10.3.4 *lists the orbits in* $\mathbb{P}(\mathbb{C}^2\otimes\mathbb{C}^3\otimes\mathbb{C}^4)$ *that are not contained in* Sub_{233} *or* Sub_{224}.

Proof. Let $[p]\in\mathbb{P}(\mathbb{C}^2\otimes\mathbb{C}^3\otimes\mathbb{C}^4)$, and let $p(A^*)=p((\mathbb{C}^2)^*)$. If $\dim p(A^*)=1$, then p must be in Sub_{233}. Otherwise $\dim p(A^*)=2$ and by the Kronecker normal form, there must be at least one block of the form L_ϵ (otherwise $p\in Sub_{233}$). Various configurations of the blocks give cases 19–23. In all the cases the eigenvalues can be absorbed by a coordinate change.

Table 10.3.4. Orbits in $\mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^{\mathbf{c}}$ contained in Sub_{234} but not contained in Sub_{233} or Sub_{224}. The unlabeled orbit closures 19–21 are various components of the singular locus of $\mathcal{S}eg_*^\vee \subset Sub_{234}$, case 22.

#	orbit cl.	dim	Kronecker normal form	pencil	$\underline{\mathbf{R}}$	$\mathbf{R}$
19			$a_1 \otimes (b_1 \otimes c_1 + b_2 \otimes c_2 + b_3 \otimes c_4)$ $+a_2 \otimes (b_1 \otimes c_2 + b_2 \otimes c_3)$	$\begin{pmatrix} s & t & & \\ & s & t & \\ & & & s \end{pmatrix}$	4	4
20			$a_1 \otimes (b_1 \otimes c_1 + b_2 \otimes c_3 + b_3 \otimes c_4)$ $+a_2 \otimes b_1 \otimes c_2$	$\begin{pmatrix} s & t & & \\ & & s & \\ & & & s \end{pmatrix}$	4	4
21			$a_1 \otimes (b_1 \otimes c_1 + b_2 \otimes c_3 + b_3 \otimes c_4)$ $+a_2 \otimes (b_1 \otimes c_2 + b_2 \otimes c_4)$	$\begin{pmatrix} s & t & & \\ & & s & t \\ & & & s \end{pmatrix}$	4	5
22	$\mathcal{S}eg_*^\vee \subset Sub_{234}$	$4\mathbf{c}+6$	$a_1 \otimes (b_1 \otimes c_1 + b_2 \otimes c_3)$ $+a_2 \otimes (b_1 \otimes c_2 + b_3 \otimes c_4)$	$\begin{pmatrix} s & t & & \\ & & s & \\ & & & t \end{pmatrix}$	4	4
23	Sub_{234}	$4\mathbf{c}+7$	$a_1 \otimes (b_1 \otimes c_1 + b_2 \otimes c_2 + b_3 \otimes c_3)$ $+a_2 \otimes (b_1 \otimes c_2 + b_2 \otimes c_3 + b_3 \otimes c_4)$	$\begin{pmatrix} s & t & & \\ & s & t & \\ & & s & t \end{pmatrix}$	4	4

Since $\sigma_4(Seg(\mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^3))$ fills out the ambient space, all the tensors listed in the table have border rank 4. The ranks follow from Theorem 10.3.3.1. $\square$

Table 10.3.5. Orbits in $\mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^{\mathbf{c}}$ that are not contained in Sub_{234}. When $\mathbf{c} = 5$, $Sub_{235} = \mathbb{P}(A \otimes B \otimes C)$ and case 26 does not occur.

#	orbit cl.	dim	Kronecker normal form	pencil	$\underline{\mathbf{R}}$	$\mathbf{R}$
24	$X_*^\vee$	$5\mathbf{c}$	$a_1 \otimes (b_1 \otimes c_1 + b_2 \otimes c_3 + b_3 \otimes c_5)$ $+2+a_2 \otimes (b_1 \otimes c_2 + b_2 \otimes c_4)$	$\begin{pmatrix} s & t & & & \\ & & s & t & \\ & & & & s \end{pmatrix}$	5	5
25	Sub_{235} $= \sigma_5(X)$	$5\mathbf{c}+4$	$a_1 \otimes (b_1 \otimes c_1 + b_2 \otimes c_2 + b_3 \otimes c_4)$ $+a_2 \otimes (b_1 \otimes c_2 + b_2 \otimes c_3 + b_3 \otimes c_5)$	$\begin{pmatrix} s & t & & & \\ & s & t & & \\ & & & s & t \end{pmatrix}$	5	5
26	$\mathbb{P}(A \otimes B \otimes C)$	$6\mathbf{c}-1$	$a_1 \otimes (b_1 \otimes c_1 + b_2 \otimes c_3 + b_3 \otimes c_5)$ $+a_2 \otimes (b_1 \otimes c_2 + b_2 \otimes c_4 + b_3 \otimes c_6)$	$\begin{pmatrix} s & t & & & & \\ & & s & t & & \\ & & & & s & t \end{pmatrix}$	6	6

The following proposition completes the list of orbits in $\mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^{\mathbf{c}}$.

Proposition 10.3.4.4. *Table* 10.3.5 *lists the orbits in* $\mathbb{P}(\mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^{\mathbf{c}})$ *that are not contained in* Sub_{234}.

Proof. Since we need to fill a $3 \times \mathbf{c}$ matrix with $\mathbf{c} \geq 5$, we need at least two blocks of the form L_{ϵ_i}. Thus we either have two blocks L_1 and F is a 1×1 matrix (case 24), or blocks L_1 and L_2 (case 25), or three blocks L_1 (case 26). The ranks follow from Theorem 10.3.3.1. The border rank is bounded

Table 10.4.1. Ranks and border ranks of plane cubic curves.

Description	normal form	$\mathbf{R}$	$\underline{\mathbf{R}}$	$\mathbb{P}\ker\phi_{2,1}\cap\sigma_2(v_2(\mathbb{P}W^*))$
triple line	x^3	1	1	
three concurrent lines	$xy(x+y)$	2	2	
double line + line	x^2y	3	2	
irreducible	$y^2z-x^3-z^3$	3	3	triangle
irreducible	$y^2z-x^3-xz^2$	4	4	smooth
cusp	y^2z-x^3	4	3	double line + line
triangle	xyz	4	4	triangle
conic + transversal line	$x(x^2+yz)$	4	4	conic + transversal line
irreducible	y^2z-x^3 $-axz^2-bz^3$	4	4	irred. cubic, smooth for general a,b
conic + tangent line	$y(x^2+yz)$	5	3	triple line

from below by 5 (cases 24–25) and by 6 in case 26. It is also bounded from above by rank. This gives the result. □

10.4. Ranks and border ranks of elements of $S^3\mathbb{C}^3$

Normal forms for plane cubic curves were determined in [**337**] in the 1930s. In [**99**] an explicit algorithm was given for determining the rank of a cubic curve (building on unpublished work of B. Reznick), and the possible ranks for polynomials in each $\sigma_r(v_3(\mathbb{P}^2))\backslash\sigma_{r-1}(v_3(\mathbb{P}^2))$ were determined. What follows is the explicit list of normal forms and their ranks and border ranks, illustrating how one can use singularities of auxiliary geometric objects to determine the rank of a polynomial.

Theorem 10.4.0.5 ([**208**])**.** *The possible ranks and border ranks of plane cubic curves are described in Table* 10.4.1.

Proof. Table 10.4.2 shows that the ranks in Table 10.4.1 are upper bounds. It remains to show that the ranks are also lower bounds.

The first three cases are covered by Theorem 9.2.2.1. For all the remaining polynomials ϕ in Table 10.4.2 , $\phi\notin Sub_2(S^3W)$, so $\mathbf{R}(\phi)\geq 3$. Since the Fermat hypersurface is smooth, it follows that if ϕ is singular, then $\mathbf{R}(\phi)\geq 4$, and this is the case for the cusp, the triangle, and the union of a

Table 10.4.2. Upper bounds on ranks of plane cubic forms.

$$
\begin{aligned}
xy(x+y) &= \frac{1}{3\sqrt{3}i}\Big((\omega x - y)^3 - (\omega^2 x - y)^3\Big) \qquad (\omega = e^{2\pi i/3})\\
x^2y &= \frac{1}{6}\Big((x+y)^3 - (x-y)^3 - 2y^3\Big)\\
y^2z - x^3 &= \frac{1}{6}\Big((y+z)^3 + (y-z)^3 - 2z^3\Big) - x^3\\
xyz &= \frac{1}{24}((x+y+z)^3 - (-x+y+z)^3 - (x-y+z)^3 - (x+y-z)^3)\\
x(x^2+yz) &= \frac{1}{288}\Big((6x+2y+z)^3 + (6x-2y-z)^3\\
&\quad - \sqrt{3}(2\sqrt{3}x - 2y + z)^3 - \sqrt{3}(2\sqrt{3}x + 2y - z)^3\Big)\\
y^2z - x^3 - xz^2 &= \frac{-1}{12\sqrt{3}}\Big((3^{1/2}x + 3^{1/4}iy + z)^3 + (3^{1/2}x - 3^{1/4}iy + z)^3\\
&\quad + (3^{1/2}x + 3^{1/4}y - z)^3 + (3^{1/2}x - 3^{1/4}y - z)^3\Big)\\
y^2z - x^3 - z^3 &= \frac{1}{6\sqrt{3}i}\Big((2\omega z - (y-z))^3 - (2\omega^2 z - (y-z))^3\Big) - x^3\\
y^2z - x^3 - axz^2 - bz^3 &= z(y - b^{1/2}z)(y + b^{1/2}z) - x(x - a^{1/2}iz)(x + a^{1/2}iz)\\
&= \frac{1}{6b^{1/2}\sqrt{3}i}\Big((2\omega b^{1/2}z - (y - b^{1/2}z))^3 - (2\omega^2 b^{1/2}z - (y - b^{1/2}z))^3\Big)\\
&\quad - \frac{1}{6\sqrt{3}i}\Big((\omega(x - a^{1/2}iz) - (x + a^{1/2}iz))^3 - (\omega^2(x - a^{1/2}iz) - (x + a^{1/2}iz))^3\Big)\\
y(x^2+yz) &= (x-y)(x+y)y + y^2(y+z)\\
&= \frac{1}{6\sqrt{3}i}\Big((2\omega y - (x-y))^3 - (2\omega^2 y - (x-y))^3\Big) + \frac{1}{6}\Big((2y+z)^3 + z^3 - 2(y+z)^3\Big)
\end{aligned}
$$

conic and a line. The following three cases remain:

$$y^2z - x^3 - xz^2, \quad y^2z - x^3 - axz^2 - bz^3, \quad y(x^2+yz).$$

The *Hessian* of a polynomial ϕ is the variety whose equation is the determinant of the Hessian matrix of the equation of ϕ. When the curve $\mathrm{Zeros}(\phi) \subset \mathbb{P}W^*$ is not a cone, the Hessian cubic of $\phi \in S^3\mathbb{C}^3$ is $\mathbb{P}\ker\phi_{2,1} \cap \sigma_2(v_2(\mathbb{P}W^*))$.

If $\phi = \eta_1^3 + \eta_2^3 + \eta_3^3$ with $[\eta_i]$ linearly independent, then the Hessian of ϕ is defined by $\eta_1\eta_2\eta_3 = 0$, so it is a union of three nonconcurrent lines. In particular, it has three distinct singular points. But a short calculation verifies that the Hessian of $y^2z - x^3 - xz^2$ is smooth and the Hessian of $y^2z - x^3 - axz^2 - bz^3$ has at most one singularity. Therefore these two curves have rank at least 4.

Let $\phi = y(x^2+yz)$. The Hessian of ϕ is defined by the equation $y^3 = 0$. Therefore the Hessian $\mathbb{P}\ker\phi_{2,1} \cap \sigma_2(v_2(\mathbb{P}W))$ is a (triple) line. Since it is not a triangle, $R(y(x^2+yz)) \geq 4$, as argued in the last two cases. But in this case one can say more.

Suppose $\phi = y(x^2+yz) = \eta_1^3 + \eta_2^3 + \eta_3^3 + \eta_4^3$, with the $[\eta_i]$ distinct points in $\mathbb{P}W$. Since $\langle\phi\rangle = W$, the $[\eta_i]$ are not all collinear. Therefore there is a unique 2-dimensional linear space of quadratic forms vanishing at the η_i. These quadratic forms thus lie in $\ker \phi_{2,1}$. In the plane $\mathbb{P}\ker\phi_{2,1} \cong \mathbb{P}^2$, $H := \mathbb{P}\ker\phi_{2,1} \cap \sigma_2(v_2(\mathbb{P}W))$ is a triple line, and the pencil of quadratic forms vanishing at each η_i is also a line L.

Now either $H = L$ or $H \neq L$. If $H = L$, then L contains the point $\mathbb{P}\ker\phi_{2,1} \cap v_2(\mathbb{P}W) \cong \Sigma_1$. But $\langle\phi\rangle = W$, so L is disjoint from $v_2(\mathbb{P}W)$. Therefore $H \neq L$. But then L contains exactly one reducible conic, corresponding to the point $H \cap L$. But this is impossible: a pencil of conics through four points in $\mathbb{P}^2$ contains at least three reducible conics (namely the pairs of lines through pairs of points).

Thus $\phi = y(x^2+yz) = \eta_1^3+\eta_2^3+\eta_3^3+\eta_4^3$ is impossible, so $R(y(x^2+yz)) \geq 5$.

This completes the proof of the calculation of ranks. For the border ranks, just use the equations in §3.9.2. □

10.5. Tensors in $\mathbb{C}^3\otimes\mathbb{C}^3\otimes\mathbb{C}^3$

Fix $\mathbf{a} = \mathbf{b} = \mathbf{c} = 3$. Recall the variety $Rank_A^2(A\otimes B\otimes C) \subset \mathbb{P}(A\otimes B\otimes C)$ from §7.2.2. It is the zero locus of $S^3A^*\otimes\Lambda^3B^*\otimes\Lambda^3C^*$.

In [**250**], K. Ng shows that $\mathbb{C}^3\otimes\mathbb{C}^3\otimes\mathbb{C}^3\backslash Rank_A^2$ can be parametrized by triplets (E, L_1, L_2), where E is a smooth elliptic curve (i.e., a Riemann surface of genus one) and L_1, L_2 are nonisomorphic line bundles of degree 3 on E modulo the equivalence relation $(E, L_1, L_2) \sim (F, M_1, M_2)$ if there exists an isomorphism $\lambda : E \to F$ such that $\lambda^*M_i = L_i$.

The elliptic curve E is associated to $T \in A\otimes B\otimes C$ as follows: consider $T_{23,1} : B^*\otimes C^* \to A$; then

$$E = \mathbb{P}(\ker T_{23,1}) \cap Seg(\mathbb{P}B^* \times \mathbb{P}C^*) \subset \mathbb{P}(B^*\otimes C^*).$$

The curve E generically admits isomorphic projections to $\mathbb{P}B^*$ and $\mathbb{P}C^*$ where it becomes a plane cubic curve, and thus E has genus one by the Riemann-Hurwitz formula; see, e.g., [**146**, p. 216]. The line bundles in question are the pullbacks of $\mathcal{O}_{\mathbb{P}B^*}(1)$ and $\mathcal{O}_{\mathbb{P}C^*}(1)$. After fixing a point of E and a degree three line bundle on E, such bundles are parametrized by points on E, so, after choices, one can parametrize such tensors by the moduli space of smooth genus one curves with two marked points, $\mathcal{M}_{1,2}$. In particular, elements of $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3 \setminus Rank_A^2$ up to equivalence depend on three parameters.

To see the correspondence the other way, one just takes $B = H^0(E, L_1)$, $C = H^0(E, L_2)$, $\ker T_{23,1} = H^0(E, L_1\otimes L_2)$, and A^* the kernel of the multiplication map $H^0(E, L_1)\otimes H^0(E, L_2) \to H^0(E, L_1\otimes L_2)$. Then T can be recovered from the inclusion $A \subset B\otimes C$.

It would be interesting to determine how the ranks and border ranks vary as one moves through the moduli space—for example, what subvariety of the moduli space corresponds to the tensors satisfying Strassen's degree nine equation?

After having established this correspondence, the paper goes on to exhibit explicit normal forms for elements of $Rank^2_A$. There are 24 such (without taking the $\mathfrak{S}_3$-symmetry into account). Some of the normal forms depend on one parameter, while others are discrete.

10.6. Normal forms for $\mathbb{C}^2 \otimes S^2 W$

This case is closely related to the determination of ranks of tensors with symmetric matrix slices, an often-studied case in applications; see, e.g., [**306**] and the references therein.

P. Comon conjectured that the symmetric rank of a symmetric tensor is the same as its rank; see §5.7. In [**47**, §4] Comon's conjecture was generalized in several forms; in particular, the authors asked if the analogous property holds for border rank and for partially symmetric tensors. It does hold in the case at hand.

Let $T \in \mathbb{C}^{\mathbf{a}} \otimes S^2 \mathbb{C}^{\mathbf{b}}$. Write $\mathbf{R}_{ps}(T)$ for the smallest r such that T is a sum of r elements of the form $a \otimes b^2$, and $\underline{\mathbf{R}}_{ps}$ for the smallest r such that it is the limit of such.

Theorem 10.6.0.6 ([**49**])**.** *Let $T \in \mathbb{C}^2 \otimes S^2 \mathbb{C}^{\mathbf{b}}$. Then $\mathbf{R}_{ps}(T) = \mathbf{R}(T)$ and $\underline{\mathbf{R}}_{ps}(T) = \underline{\mathbf{R}}(T)$. In particular, the maximum partially symmetric rank of an element of $\mathbb{C}^2 \otimes S^2 \mathbb{C}^{\mathbf{b}}$ is $\lfloor \frac{3\mathbf{b}}{2} \rfloor$ (and the maximum partially symmetric border rank had been known to be $\mathbf{b}$).*

Proof. The classification of pencils of quadrics is known; see [**136**, vol. 2, XII.6]. Each pencil of quadrics $q \in A \otimes S^2 W$ is isomorphic to one built from blocks, each block of the form either

$$L_\epsilon^{sym} := \begin{pmatrix} 0 & L_\epsilon \\ L_\epsilon^T & 0 \end{pmatrix},$$

where L_ϵ is as in §10.3, or

$$G_{\lambda,\eta} = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & t & s+\lambda t \\ 0 & 0 & 0 & \dots & t & s+\lambda t & 0 \\ \vdots & & & & & & \\ 0 & t & s+\lambda t & \dots & 0 & 0 & 0 \\ t & s+\lambda t & 0 & \dots & 0 & 0 & 0 \\ s+\lambda t & 0 & 0 & \dots & 0 & 0 & 0 \end{pmatrix}.$$

The $\eta \times \eta$ blocks of the form $G_{\lambda,\eta}$ are analogous to the Jordan blocks, but written in the other direction to maintain symmetry. (The key point is that two pencils of complex symmetric matrices are equivalent as symmetric pencils iff they are equivalent as pencils of matrices; see [**136**, vol. 2, XII.6, Thm. 6].)

Now apply (10.3.3)—the same argument as in the Jordan case holds to obtain upper bounds, and the lower bounds follow from the nonsymmetric case. $\square$

10.7. Exercises on normal forms for general points on small secant varieties

What follows are some varieties in spaces of tensors where the general points either have normal forms or may be parametrized.

(1) Show that a general point of the variety $\sigma_r(Seg(\mathbb{P}^{r-1}\times\cdots\times\mathbb{P}^{r-1})) = \sigma_r(Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n))$ may be written as $a_1^1 \otimes \cdots \otimes a_1^n + \cdots + a_r^1 \otimes \cdots \otimes a_r^n$, where $a_1^j, \dots, a_r^j$ is a basis of A_j.

(2) Show that a general point of $\sigma_{r+1}(Seg(\mathbb{P}^{r-1} \times \cdots \times \mathbb{P}^{r-1} \times \mathbb{P}^r)) = \sigma_r(Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n))$ may be written as $a_1^1 \otimes \cdots \otimes a_1^n + \cdots + a_r^1 \otimes \cdots \otimes a_r^n + (a_1^1 + \cdots + a_r^1) \otimes \cdots \otimes (a_1^{n-1} + \cdots + a_r^{n-1}) \otimes a_{r+1}^n$, where $a_1^j, \dots, a_r^j$ is a basis of A_j for $j < n$ and $a_1^n, \dots, a_{r+1}^n$ is a basis of A_n.

(3) Determine a normal form for a general point of $\sigma_r(v_d(\mathbb{P}^{r-1}))$. Is a normal form without parameters possible for a general point of $\sigma_{r+1}(v_d(\mathbb{P}^{r-1}))$?

10.8. Limits of secant planes

There are several reasons for studying points on $\sigma_r(Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n))$ that are not on secant $\mathbb{P}^{r-1}$'s. First, in order to prove that a set of equations E is a set of defining equations for $\sigma_r(Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n))$, one must prove that any point in the zero set of E is either a point on a secant $\mathbb{P}^{r-1}$ or on a limit $\mathbb{P}^{r-1}$. For example, the proof of the set-theoretic GSS conjecture in §3.9.1 proceeded in this fashion. Second, to prove lower bounds for the border rank of a given tensor, e.g., matrix multiplication, one could try to first prove that it cannot lie on any secant $\mathbb{P}^{r-1}$ and then that it cannot lie on any limiting $\mathbb{P}^{r-1}$ either. This was the technique used in [**201**]. Third, a central ingredient for writing explicit approximating algorithms is to exploit certain limiting $\mathbb{P}^{r-1}$'s discussed below. Finally, tensors with rank greater than their border rank arise from such limiting $\mathbb{P}^{r-1}$'s.

10.8.1. Limits for arbitrary projective varieties. Let $X \subset \mathbb{P}V$ be a projective variety. Let $\sigma_r^0(X)$ denote the set of points on $\sigma_r(X)$ that lie on a secant $\mathbb{P}^{r-1}$. Let us assume that we know the nature of points on $\sigma_{r-1}(X)$ and study points on $\sigma_r(X)\backslash(\sigma_r^0(X) \cup \sigma_{r-1}(X))$.

It is convenient to study the limiting r-planes as points on the Grassmannian in its Plücker embedding, $G(r,V) \subset \mathbb{P}(\Lambda^r V)$. That is, consider the curve of r-planes as being represented by $[x_1(t) \wedge \cdots \wedge x_r(t)]$ and examine the limiting plane as $t \to 0$. (There must be a unique such plane as the Grassmannian is compact.)

Let $[p] \in \sigma_r(X)$. Then there exist curves $x_1(t), \ldots, x_r(t) \subset \hat{X}$ with $p \in \lim_{t\to 0}\langle x_1(t), \ldots, x_r(t)\rangle$. We are interested in the case where

$$\dim\langle x_1(0), \ldots, x_r(0)\rangle < r \quad \text{but} \quad \dim\langle x_1(t), \ldots, x_r(t)\rangle = r \quad \text{for} \quad t > 0.$$

Use the notation $x_j = x_j(0)$. Assume for the moment that $x_1, \ldots, x_{r-1}$ are linearly independent. Then $x_r = c_1x_1 + \cdots + c_{r-1}x_{r-1}$ for some constants $c_1, \ldots, c_{r-1}$. Write each curve $x_j(t) = x_j + tx_j' + t^2x_j'' + \cdots$, where the derivatives are taken at $t = 0$.

Consider the Taylor series

$$\begin{aligned}
&x_1(t) \wedge \cdots \wedge x_r(t)\\
&\quad = (x_1 + tx_1' + t^2x_1'' + \cdots) \wedge \cdots \wedge (x_{r-1} + tx_{r-1}' + t^2x_{r-1}'' + \cdots)\\
&\qquad \wedge (x_r + tx_r' + t^2x_r'' + \cdots)\\
&\quad = t((-1)^r(c_1x_1' + \cdots + c_{r-1}x_{r-1}' - x_r') \wedge x_1 \wedge \cdots \wedge x_{r-1} + t^2(\cdots) + \cdots .
\end{aligned}$$

If the t coefficient is nonzero, then p lies in the the r-plane

$$\langle x_1, \ldots, x_{r-1}, (c_1x_1' + \cdots + c_{r-1}x_{r-1}' - x_r')\rangle.$$

If the t coefficient is zero, then $c_1x_1' + \cdots + c_{r-1}x_{r-1}' - x_r' = e_1x_1 + \cdots + e_{r-1}x_{r-1}$ for some constants $e_1, \ldots, e_{r-1}$. In this case we must examine the t^2 coefficient of the expansion. It is

$$\Big(\sum_{k=1}^{r-1} e_kx_k' + \sum_{j=1}^{r-1} c_jx_j'' - x_r''\Big) \wedge x_1 \wedge \cdots \wedge x_{r-1}.$$

One continues to higher order terms if this is zero.

Such limits with $X = Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)$ are used to construct explicit algorithms for matrix multiplication in Chapter 11. The algorithm of Example 11.2.1 uses the t coefficient, the algorithm of Example 11.2.2 uses the t^2 coefficient, and Algorithm 8.2 in [**282**] uses the coefficient of t^{20}.

Exercise 10.8.1.1: When $r = 3$, the t^2 term for arbitrary curves is

$$x_1' \wedge x_2' \wedge x_3 + x_1' \wedge x_2 \wedge x_3' + x_1 \wedge x_2' \wedge x_3' + x_1'' \wedge x_2 \wedge x_3 + x_1 \wedge x_2'' \wedge x_3 + x_1 \wedge x_2 \wedge x_3''. \tag{10.8.1}$$

Show that if the coefficients of the t^0 and t^1 terms vanish, then (10.8.1) is indeed a decomposable vector.

Following [**48**], here is a proof of the standard fact that a point on a secant variety to a smooth variety X is either on X, on an honest secant line, or on a tangent line to X via limits.

Proposition 10.8.1.2. *Let $x(t), y(t) \subset X$ be curves with $x(0) = y(0) = x$; then any point of $\sigma_2(X)$ that may be obtained as a point of the limiting $\mathbb{P}^1 = \mathbb{P}(\lim_{t\to 0}\langle x(t), y(t)\rangle)$ may be obtained from first order information.*

Proof. The 2-plane obtained by first order information (as a point in the Grassmannian) is $[x\wedge(y_1 - x_1)]$. Assume that $y_1 = x_1 + cx$ for some constant c. Then the second order term is $x \wedge (\tilde{y}_2 - \tilde{x}_2)$, but under our hypotheses $\tilde{y}_2 - \tilde{x}_2 \in \hat{T}_x X$ as the terms $II(x_1, x_1)$ cancel.

If this term is zero, then $y_2 = x_2 + c_2 x$ and the third order terms again cancel up to elements of $\hat{T}_x X$, and similarly for higher order terms. $\square$

With considerably more work, one obtains the following result:

Theorem 10.8.1.3 ([**48**])**.** *Let $X \subset \mathbb{P}V$ be a variety whose only nonzero differential invariants are fundamental forms. (Cominuscule, and in particular Segre and Veronese varieties satisfy this property.) Let $x \in \hat{X}$ be a general point. (If X is homogeneous, all points are general.) Let $x(t), y(t), z(t) \subset \hat{X}$ be curves with $x(0) = y(0) = z(0) = x$ and suppose $p \in \sigma_3(X)$ may be obtained as a point of the limiting $\mathbb{P}^2 = \mathbb{P}(\lim_{t\to 0}\langle x(t), y(t), z(t)\rangle)$. Then p may be obtained from at most second order information, and in fact $p = [w''(0)]$ or $p = [w'(0)]$ for some curve $w(t)$ with $w(0) = x$, or $p = [II(v, w) + u]$ for some $u, v, w \in \hat{T}_x X$, such that $II(v, v) = 0$ or $p = [x'(0) + y'(0)]$, where $[x(0)]$ and $[y(0)]$ are colinear. In the case that X is a Segre variety, the last two types of points coincide.*

See [**48**] for the proof.

10.9. Limits for Veronese varieties

10.9.1. Points on $\sigma_2(v_d(\mathbb{P}W))$. For any smooth variety $X \subset \mathbb{P}V$, a point on $\sigma_2(X)$ is either a point of X, a point on an honest secant line (i.e., a point of X-rank two), or a point on a tangent line of X. For a Veronese variety all nonzero tangent vectors are equivalent. They are all of the form $[x^d + x^{d-1}y]$ (or equivalently $[x^{d-1}z]$); in particular they lie on a subspace variety Sub_2 and thus have rank d by Theorem 9.2.2.1. In summary:

Proposition 10.9.1.1. *If $p \in \sigma_2(v_d(\mathbb{P}W))$, then $R(p) = 1$, 2, or d. In these cases p respectively has the normal forms $x^d, x^d + y^d, x^{d-1}y$. (The last two are equivalent when $d = 2$.)*

10.9.2. Points on $\sigma_3(v_d(\mathbb{P}W))$. Assume that $d > 2$ as the $d = 2$ case is well understood; see §10.2.3. Consider the case of points on $\sigma_3(v_d(\mathbb{P}W))\backslash \sigma_2(v_d(\mathbb{P}W))$. There cannot be three distinct limiting points x_1, x_2, x_3 with $\dim\langle x_1, x_2, x_3\rangle < 3$ unless at least two of them coincide, because there are no trisecant lines to $v_d(\mathbb{P}W)$. (For any variety $X \subset \mathbb{P}V$ with ideal generated in degree two, any trisecant line of X is contained in X, and Veronese varieties $v_d(\mathbb{P}W) \subset \mathbb{P}S^dW$ are cut out by quadrics but contain no lines when $d > 2$.)

Write the curves as

$$\begin{aligned} x(t) &= (x_0 + tx_1 + t^2x_2 + t^3x_3 + \cdots)^d \\ &= x_0^d + t(dx_0^{d-1}x_1) + t^2\left(\binom{d}{2}x_0^{d-2}x_1^2 + dx_0^{d-1}x_2\right) \\ &\quad + t^3\left(\binom{d}{3}x_0^{d-3}x_1^3 + d(d-1)x_0^{d-2}x_1x_2 + dx_0^{d-1}x_3\right) + \cdots \end{aligned}$$

and similarly for $y(t)$, $z(t)$.

Case 1: There are two distinct limit points x_0^d, z_0^d, and $y_0 = x_0$. (One can always rescale to have equality of points rather than just collinearity since we are working in projective space.) When we expand the Taylor series, assuming $d > 2$, the coefficient of t (ignoring constants which disappear when projectivizing) is

$$x_0^{d-1}(x_1 - y_1) \wedge x_0^d \wedge z_0^d,$$

which can be zero only if the first term is zero, i.e., $x_1 \equiv y_1 \mod x_0$. If this happens, by (10.8.1) the second order term is of the form

$$x_0^{d-1}(x_2 - y_2 + \lambda x_1) \wedge x_0^d \wedge z_0^d.$$

Similarly, if this term vanishes, the t^3 term will still be of the same nature. Inductively, if the lowest nonzero term is t^k, then for each $j < k$, $y_j = x_j \mod (x_0, \dots, x_{j-1})$, and the coefficient of the t^k term is (up to a constant factor)

$$x_0^{d-1}(x_k - y_k + \ell) \wedge x_0^d \wedge z_0^d,$$

where ℓ is a linear combination of $x_0, \dots, x_{k-1}$. Rewrite this as $x^{d-1}y \wedge x^d \wedge z^d$. If $\dim\langle z, x, y\rangle < 3$, we are reduced to a point of $\sigma_3(v_d(\mathbb{P}^1))$ and can appeal to Theorem 9.2.2.1. If the span is three-dimensional, then any point in the plane $[x^{d-1}y \wedge x^d \wedge z^d]$ can be put in the normal form $x^{d-1}w + z^d$.

Case 2: There is one limit point $x_0 = y_0 = z_0 = z$. By Theorem 10.8.1.3, it must be of the form $[x^d \wedge x^{d-1}y \wedge (x^{d-1}\ell + \mu x^{d-2}y^2)]$.

Theorem 10.9.2.1 ([**208, 23**]). *There are three types of points $\phi \in S^3W$ of border rank three with* $\dim\langle\phi\rangle = 3$. *They have the following normal forms:*

limiting curves	normal form	$\mathbf{R}_S$
x^d, y^d, z^d	$x^d + y^d + z^d$	3
$x^d, (x+ty)^d, z^d$	$x^{d-1}y + z^d$	$d+1$
$x^d, (x+ty)^d, (x+2ty+t^2z)^d$	$x^{d-2}y^2 + x^{d-1}z$	$2d-1$

The normal forms follow from the discussion above (and are taken from [**208**]). The determination of the ranks is from [**23**]. The key to their proof is a scheme-theoretic analog of Proposition 4.3.7.6. I record the precise lemma for the experts.

Lemma 10.9.2.2 ([**23**, Lemma 2]). *Let $Z \subset \mathbb{P}^n$ be a 0-dimensional scheme with* $\deg(Z) \leq 2d+1$; *then* $\dim\langle v_d(Z)\rangle = \deg(Z) - \mu$ *if and only if there exist lines $L_s \subset \mathbb{P}^n$ such that* $\deg(Z \cap L_s) \geq d + 2 + \mu_s$, *with* $\sum_s \mu_s = \mu$.

Corollary 10.9.2.3. *Let $\phi \in S^dW$ with* $\underline{\mathbf{R}}_S(\phi) = 3$. *Then $\mathbf{R}_S(\phi)$ is either* $3, d-1, d+1$ *or* $2d-1$.

Proof. The only additional case occurs if $\dim\langle\phi\rangle = 2$, which is handled by Theorem 9.2.2.1. □

Remark 10.9.2.4. Even for higher secant varieties, $x_1^d \wedge \cdots \wedge x_r^d$ cannot be zero if the x_j are distinct points, even if they lie on a $\mathbb{P}^1$, as long as $d \geq r$. This is because a hyperplane in S^dW corresponds to a (defined up to scale) homogeneous polynomial of degree d on W. Now take $W = \mathbb{C}^2$. No homogeneous polynomial of degree d vanishes at $d+1$ distinct points of $\mathbb{P}^1$; thus the image of any $d+1$ distinct points under the d-th Veronese embedding cannot lie on a hyperplane.

10.9.3. Points on $\sigma_4(v_d(\mathbb{P}W))$.

Theorem 10.9.3.1 ([**208**]). *There are five types of points of border rank four in S^dW, $d > 2$, whose span is 4-dimensional. They have the following normal forms:*

limiting curves	normal form	$\mathbf{R}$
x^d, y^d, z^d, w^d	$x^d + y^d + z^d + w^d$	4
$x^d, (x+ty)^d, z^d, w^d$	$x^{d-1}y + z^d + w^d$	$d \leq \mathbf{R} \leq d+2$
$x^d, (x+ty)^d, z^d, (z+tw)^d$	$x^{d-1}y + z^{d-1}w$	$d \leq \mathbf{R} \leq 2d$
$x^d, (x+ty)^d, (x+ty+t^2z)^d, w^d$	$x^{d-2}y^2 + x^{d-1}z + w^d$	$d \leq \mathbf{R} \leq 2d$
$x^d, (x+ty)^d, (x+ty+t^2z)^d,$ $(x+ty+t^2z+t^3w)^d$	$x^{d-3}y^3 + x^{d-2}yz + x^{d-1}w$	$d \leq \mathbf{R} \leq 3d-3$

Theorem 10.9.3.2 ([**23**]). *Any point in $S^4\mathbb{C}^3$ of border rank 4 has rank 4, 6, or 7. Any point of border rank 5 has rank 5, 6, or 7.*

For $\sigma_5(v_d(\mathbb{P}W))$, a new phenomenon arises when $d = 3$ because $\dim S^3\mathbb{C}^2 = 4 < 5$. One can have five curves a, b, c, d, e, with $a_0, \dots, e_0$ all lying in a $\mathbb{C}^2$, but otherwise general, so $\dim\langle a_0^3, \dots, e_0^3\rangle = 4$. Thus the t term will be of the form $a_0^3 \wedge b_0^3 \wedge c_0^3 \wedge d_0^3 \wedge (s_1 a_0^2 a_1 + \cdots + s_4 d_0^2 d_1 - e_0^2 e_1)$. Up to scaling we can give $\mathbb{C}^2$ linear coordinates x, y so that $a_0 = x$, $b_0 = y$, $c_0 = x + y$, $d_0 = x + \lambda y$ for some λ. Then, independently of e_0, the limiting plane will be contained in

$$\langle x^3, y^3, (x+y)^3, (x+\lambda y)^3, x^2\alpha, xy\beta, y^2\gamma\rangle$$

for some $\alpha, \beta, \gamma \in W$ (depending on $a_1, \dots, e_1$). Any point contained in this plane is of the form $x^2u + y^2v + xyz$ for some $u, v, z \in W$.

10.10. Ranks and normal forms in $\sigma_3(Seg(\mathbb{P}A_1 \otimes \cdots \otimes \mathbb{P}A_n))$

I assume throughout this section that $n \geq 3$.

10.10.1. Case of $\sigma_2(Seg(\mathbb{P}A_1 \otimes \cdots \otimes \mathbb{P}A_n))$. The $n = 3$ case of the following theorem is classical.

Theorem 10.10.1.1 ([48]). *Let $X = Seg(\mathbb{P}A_1 \otimes \cdots \otimes \mathbb{P}A_n) \subset \mathbb{P}(A_1 \otimes \cdots \otimes A_n)$ be a Segre variety. There is a normal form for points $x \in \hat{\sigma}_2(X)$: $x = a_0^1 \otimes \cdots \otimes a_0^n$ for a point of X, $x = a_0^1 \otimes \cdots \otimes a_0^n + a_1^1 \otimes \cdots \otimes a_1^n$ for a point on a secant line to X (where for this normal form, we do not require all the a_1^i to be independent of the a_0^i), which respectively have ranks* 1, 2, *and, for each $J \subseteq [n]$, $|J| > 2$, the normal form*

$$x = a_0^1 \otimes \cdots \otimes a_0^n + \sum_{j_i \in J} a_0^1 \otimes \cdots \otimes a_0^{j_i - 1} \otimes a_1^{j_i} \otimes a_0^{j_i+1} \otimes \cdots \otimes a_0^n, \tag{10.10.1}$$

which has rank $|J|$. In particular, all ranks from 1 *to n occur for elements of $\sigma_2(X)$.*

Proof. All the assertions except for the rank of x in (10.10.1) are immediate. The rank of x is at most $|J|$ because there are $|J|+1$ terms in the summation but the first can be absorbed into any of the others (e.g., using $a_0^{j_i} + a_1^{j_i}$ instead of $a_1^{j_i}$). Assume without loss of generality that $|J| = n$ and work by induction. First say $n = 3$; then Theorem 10.3.3.1 establishes this base case.

Now assume the result has been established up to $n - 1$, and consider $x(A_1^*)$. It is spanned by

$$a_0^2 \otimes \cdots \otimes a_0^n, \ \sum_j a_0^2 \otimes \cdots \otimes a_0^{j-1} \otimes a_1^j \otimes a_0^{j+1} \otimes \cdots \otimes a_0^n.$$

By induction, the second vector has rank $n - 1$. It remains to show that there is no expression of the second vector as a sum of $n-1$ rank one tensors where one of the terms is a multiple of $a_0^2 \otimes \cdots \otimes a_0^n$. Say there were, where $a_0^2 \otimes \cdots \otimes a_0^n$ appeared with coefficient λ. Then the tensor

$$\sum_j a_0^2 \otimes \cdots \otimes a_0^{j-1} \otimes a_1^j \otimes a_0^{j+1} \otimes \cdots \otimes a_0^n - \lambda a_0^2 \otimes \cdots \otimes a_0^n$$

would have rank $n-2$, but setting $\tilde{a}_1^2 = a_1^2 - \lambda_0^2$ and $\tilde{a}_1^j = a_1^j$ for $j \in \{3, \ldots, n\}$, this would imply that

$$\sum_j a_0^2 \otimes \cdots \otimes a_0^{j-1} \otimes \tilde{a}_1^j \otimes a_0^{j+1} \otimes \cdots \otimes a_0^n$$

had rank $n-2$, a contradiction. $\square$

10.10.2. Case of $\sigma_3(Seg(\mathbb{P}A_1 \otimes \cdots \otimes \mathbb{P}A_n))$. Let

$$x \in \sigma_3(Seg(\mathbb{P}A_1 \otimes \cdots \otimes \mathbb{P}A_n)) \backslash \sigma_2(Seg(\mathbb{P}A_1 \otimes \cdots \otimes \mathbb{P}A_n)).$$

The standard types of points are: a point on an honest secant $\mathbb{P}^2$ (which has rank 3), a point on the plane spanned by a point of the Segre and a tangent $\mathbb{P}^1$ to the Segre, which has rank at most $n+1$, and a point of the form $y + y' + y''$, where $y(t)$ is a curve on $\hat{S}eg(\mathbb{P}A_1 \otimes \cdots \otimes \mathbb{P}A_n)$.

The latter type of point has rank at most $\binom{n+1}{2}$ because a generic such point is of the form

$$\begin{aligned} a_0^1 \otimes \cdots \otimes a_0^n &+ \sum_j a_0^1 \otimes \cdots \otimes a_0^{j-1} \otimes a_1^j \otimes a_0^{j+1} \otimes \cdots \otimes a_0^n \\ &+ \sum_{j<k} a_0^1 \otimes \cdots \otimes a_0^{j-1} \otimes a_1^j \otimes a_0^{j+1} \otimes \cdots \otimes a_0^{k-1} \otimes a_1^k \otimes a_0^{k+1} \otimes \cdots \otimes a_0^n \\ &+ \sum_j a_0^1 \otimes \cdots \otimes a_0^{j-1} \otimes a_2^j \otimes a_0^{j+1} \otimes \cdots \otimes a_0^n. \end{aligned}$$

The first term and second set can be folded into the last term, giving the estimate.

Theorem 10.10.2.1 ([**48**]). *Let*

$$p = [v] \in \sigma_3(Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n)) \backslash \sigma_2(Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n)).$$

Then either

(1) $v = x + y + z$ *with* $[x], [y], [z] \in Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n)$,

(2) $v = x + y + y'$, *with* $[x], [y] \in Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n)$ *and* $y' \in \hat{T}_{[y]} Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n)$,

(3) $v = x + x' + x''$, *where* $[x(t)] \subset Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n)$ *is a curve and* $x' = x'(0)$, $x'' = x''(0)$, *or*

(4) *There are distinct points* $[x], [y] \in Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n)$ *that lie on a line contained in* $Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n)$ *and* $v = x' + y'$, *where* $x' \in \hat{T}_{[x]} Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n)$ *and* $y' \in \hat{T}_{[y]} Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n)$.

Normal forms for such points when $n = 3$ are as follows:

(1) $a_1 \otimes b_1 \otimes c_1 + a_2 \otimes b_2 \otimes c_2 + a_3 \otimes b_3 \otimes c_3$;

(2) $a_1 \otimes b_1 \otimes c_1 + a_2 \otimes b_2 \otimes c_3 + a_2 \otimes b_3 \otimes c_2 + a_3 \otimes b_2 \otimes c_2$;

(3) $a_1 \otimes b_1 \otimes c_2 + a_1 \otimes b_2 \otimes c_1 + a_2 \otimes b_1 \otimes c_1 + a_1 \otimes b_3 \otimes c_3 + a_3 \otimes b_1 \otimes c_3 + a_3 \otimes b_3 \otimes c_1$;

(4) $a_3 \otimes b_1 \otimes c_1 + a_1 \otimes b_2 \otimes c_2 + a_1 \otimes b_1 \otimes c_2 + a_2 \otimes b_3 \otimes c_1 + a_2 \otimes b_1 \otimes c_3$,

and these are depicted in terms of "slices" below.

The points of type (1) form a Zariski open subset of $\sigma_3(Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n))$, those of type (2) have codimension 1 in $\sigma_3(Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n))$, those of type (3) have codimension 2, and those of type (4) give rise to n distinct components of the boundary of $\sigma_3(Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n))$ and each of the components has codimension at least 3.

Proof. The first three possibilities clearly can occur and are different, so it remains to show that the fourth possibility is distinct and that there are no other possibilities. We already saw that (4) is indeed a point of σ_3 in the three-factor case, but the general case is the same. So suppose $x_1(t), x_2(t), x_3(t)$ are three curves on $\hat{S}eg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n)$, such that $p = \lim_{t \to 0}[x_1(t) + x_2(t) + x_3(t)]$ and suppose p is not on an honest secant $\mathbb{P}^2$. Let $x_i := x_i(0)$; then x_1, x_2, x_3 must be colinear. Since any line containing three points of a variety cut out by quadrics must be contained in the variety, up to permuting the factors, we may assume that $x_1 = e \otimes a_2 \otimes \cdots \otimes a_n$, $x_2 = f \otimes a_2 \otimes \cdots \otimes a_n$, $x_3 = (se + tf) \otimes a_2 \otimes \cdots \otimes a_n$, for some $e, f \in A_1$ and $s, t \in \mathbb{C}$. Then we may have any point in the limit in $\hat{T}_{x_1} Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n) + \hat{T}_{x_2} Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n) + \hat{T}_{x_3} Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n)$. The third tangent space is redundant because for a line on a Segre variety, the span of the tangent spaces to any two points on the line equals the span of the tangent spaces to all the points on the line.

To see that one can obtain a point not of types (1)–(3) this way, consider the case $n = 3$ and write $x_1 = a_1 \otimes b_1 \otimes c_1$, $x_2 = a_2 \otimes b_1 \otimes c_1$. Then the point

$$v = a_3 \otimes b_1 \otimes c_1 + a_1 \otimes b_2 \otimes c_2 + a_1 \otimes b_1 \otimes c_2 + a_2 \otimes b_3 \otimes c_1 + a_2 \otimes b_1 \otimes c_3$$

is a general point in the sum of the two tangent spaces. Considering $v(A^*) \subset B \otimes C$ in bases, one obtains the following subspace:

$$\left\{ \begin{pmatrix} \alpha_3 & \alpha_1 & \alpha_2 \\ \alpha_1 & 0 & 0 \\ \alpha_2 & 0 & 0 \end{pmatrix} \;\middle|\; \alpha_j \in \mathbb{C} \right\}.$$

The other types give

$$\left\{ \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{pmatrix} \middle| \; \alpha_j \in \mathbb{C} \right\},$$

$$\left\{ \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & \alpha_3 \\ 0 & \alpha_3 & 0 \end{pmatrix} \middle| \; \alpha_j \in \mathbb{C} \right\},$$

$$\left\{ \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_2 & \alpha_3 & 0 \\ \alpha_3 & 0 & 0 \end{pmatrix} \middle| \; \alpha_j \in \mathbb{C} \right\}.$$

These types are all different.

By Theorem 10.8.1.3, no higher order term of this form gives something new.

The codimensions of the orbit closures of each type may be obtained by computing the Lie algebras of the stabilizers. See [**48**] for details. □

Remark 10.10.2.2. In contrast to case (4) above, already with four points spanning a 3-dimensional vector space, one can obtain new limits by taking a second derivative. Consider the points $x_1 = a_1 \otimes b_1 \otimes c_1$, $x_2 = a_2 \otimes b_2 \otimes c_1$, $x_3 = \frac{1}{2}(a_1 + a_2) \otimes (b_1 - b_2) \otimes c_1$, and $x_4 = \frac{1}{2}(a_1 - a_2) \otimes (b_1 + b_2) \otimes c_1$. Note that $x_1 = x_2 + x_3 + x_4$. Here both first and second derivatives of curves give new points.

The bound $\mathbf{R}_{Seg(\mathbb{P}A_1 \otimes \cdots \otimes \mathbb{P}A_n)}(y + y' + y'') \leq \binom{n+1}{2}$ is not effective, as for $n = 3$ we have $\mathbf{R}_{Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)}(y + y' + y'') = 5$ (see Corollary 10.10.2.4).

Proposition 10.10.2.3 ([**48**])**.** *The ranks of the* 3*-planes given by*

$$\begin{pmatrix} c_2 & c_1 & c_0 \\ c_1 & c_0 & 0 \\ c_0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \alpha_3 & \alpha_1 & \alpha_2 \\ \alpha_1 & 0 & 0 \\ \alpha_2 & 0 & 0 \end{pmatrix}$$

are 5.

Proof. We first show that the rank is at most 5, by noting that the ranks of

$$\begin{pmatrix} 0 & c_1 & c_0 \\ c_1 & c_0 & 0 \\ c_0 & 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & \alpha_1 & \alpha_2 \\ \alpha_1 & 0 & 0 \\ \alpha_2 & 0 & 0 \end{pmatrix}$$

are at most 4 by Theorem 10.3.3.1 and the rank of

$$\begin{pmatrix} c_2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

is 1.

To see that the ranks are at least five, were it four, we would be able to find a 3×3 matrix

$$T = \begin{pmatrix} s_1t_1 & s_1t_2 & s_1t_3 \\ s_2t_1 & s_2t_2 & s_2t_3 \\ s_3t_1 & s_3t_2 & s_3t_3 \end{pmatrix}$$

of rank 1, such that in the first case, the 4-plane spanned by

$$S_1 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ S_2 := \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},\ S_3 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},\ T$$

is spanned by matrices of rank 1. In particular, S_1 would be in the span of S_2, S_3, T, and another matrix of rank 1. Thus we would be able to find constants $\alpha, \beta, s_1, s_2, s_3, t_1, t_2, t_3$, such that the rank of

$$\begin{pmatrix} 1 & 0 & 0 \\ \alpha & 1 & 0 \\ \beta & \alpha & 1 \end{pmatrix} + \begin{pmatrix} s_1t_1 & s_1t_2 & s_1t_3 \\ s_2t_1 & s_2t_2 & s_2t_3 \\ s_3t_1 & s_3t_2 & s_3t_3 \end{pmatrix}$$

is 1. There are two cases: if $s_1 \neq 0$, then we can subtract $\frac{s_2}{s_1}$ times the first row from the second, and $\frac{s_3}{s_1}$ times the first row from the third to obtain

$$\begin{pmatrix} 1 + s_1t_1 & s_1t_2 & s_1t_3 \\ * & 1 & 0 \\ * & * & 1 \end{pmatrix},$$

which has rank at least 2. If $s_1 = 0$, the matrix already visibly has rank at least 2. Thus it is impossible to find such constants α, β, s_i, t_i and the rank in question is necessarily at least 5. The second case is similar. $\square$

Corollary 10.10.2.4 ([**48**]). *The rank of a general point of the form $y + y' + y''$ of $\hat{\sigma}_3(Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C))$ as well as the rank of a general point of the form $x' + y'$, where $[x], [y]$ lie on a line in $Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)$, is* 5. *Moreover, the maximum rank of any point of $\sigma_3(Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C))$ is* 5.

Remark 10.10.2.5. Corollary 10.10.2.4 seems to have been a "folklore" theorem in the tensor literature. For example, in [**188**, Table 3.2], the result is stated and the reader is referred to [**194**], but in that paper the result is stated and is referred to a paper that had never appeared. Also, there were privately circulating proofs, including one due to R. Rocci from 1993. I thank M. Mohelkamp for these historical remarks.

Starting with $\sigma_4(Seg(\mathbb{P}A_1 \otimes \cdots \otimes \mathbb{P}A_n))$, there are exceptional limit points; namely consider $Seg(v_2(\mathbb{P}^1) \times \mathbb{P}^0 \times \cdots \times \mathbb{P}^0) \subset Seg(\mathbb{P}A_1 \otimes \cdots \otimes \mathbb{P}A_n)$. Any four points lying on $Seg(v_2(\mathbb{P}^1) \times \mathbb{P}^0 \times \cdots \times \mathbb{P}^0)$ will be linearly dependent. Exceptional limit points turn out to be important—an exceptional limit in $\sigma_5(Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C))$ is used in Bini's approximate algorithm to multiply

2×2 matrices with an entry zero, and an exceptional limit in $\sigma_6(Seg(\mathbb{P}A\times \mathbb{P}B\times \mathbb{P}C))$ is used in Schönhage's approximate algorithm to multiply 3×3 matrices using 21 multiplications; see §11.2.

Theorem 10.10.2.6 ([**44**, Thm. 3.3])**.** *The largest rank of a tensor in* $\mathbb{C}^2\otimes \mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^2$ *is* 4.

The proof is by examining the image of the map $\mathbb{P}^1\to\mathbb{P}(\mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^2)$. If the image is not contained in the hypersurface given by the hyperdeterminant, one is done immediately, as then it must contain two points of rank two. Otherwise the author studies lines in the hyperdeterminantal hypersurface and derives an explicit normal form.

Part 3

Applications

Chapter 11

The complexity of matrix multiplication

This book began with a discussion of Strassen's algorithm for 2×2 matrices, which showed that $\mathbf{R}(M_{m,m,m}) = \mathcal{O}(n^{2.86})$ and mentioned that the current world record is $\mathbf{R}(M_{m,m,m}) = \mathcal{O}(n^{2.38})$. The search for upper and lower bounds for the rank and border rank of matrix multiplication has been discussed throughout. This chapter discusses several results regarding matrix multiplication in a geometric context, in the hope that a geometric perspective will lead to advancements in the state of the art. However, it begins, in §11.1, by briefly mentioning how such algorithms are used in practice, and with pointers to the literature on implementation. In §11.2, subtle "approximate algorithms" due to D. Bini and A. Schönhage are presented. Schönhage found an algorithm to multiply 3×3 matrices, within an error of any prescribed ϵ, using only 21 multiplications. A different geometric approach to finding efficient algorithms, due to H. Cohn and C. Umans, uses the representation theory of finite groups. This program is outlined in §11.3. For completeness, but without any motivation or explanation (because I have none), §11.4 presents J. Laderman's expression for multiplying 3×3 matrices using 23 multiplications. Lower bounds for border rank have been discussed already throughout this book. Regarding lower bounds for rank, a geometric proof of the best asymptotic lower bound on the rank of matrix multiplication, due to Bläser, is given in §11.5 and stated below. Another lower bound for rank, which is better than Bläser's bound for small m, is the Brockett-Dobkin theorem stated below. The Brockett-Dobkin theorem is discussed in §11.6. The chapter concludes with a natural geometric interpretation of the notion of *multiplicative complexity* in §11.7, a notion that appears in the complexity literature.

For the convenience of the reader, here are the main results:

Theorem 11.0.2.7 (Lickteig [**220**, §3.8.3]). $\underline{\mathbf{R}}(M_{m,m,m}) \geq \frac{3m^2}{2} + \frac{m}{2} - 1$.

Theorem 11.0.2.8 (Bläser [**28**]). $\mathbf{R}(M_{m,m,m}) \geq \frac{5}{2}m^2 - 3m$.

Theorem 11.0.2.9 (Brockett-Dobkin [**41**]). $\mathbf{R}(M_{m,m,m}) \geq 2m^2 - 1$

Corollary 11.0.2.10 (Winograd [**334**]). $\mathbf{R}(M_{2,2,2}) = 7$.

Theorem 11.0.2.11 ([**201**]). $\underline{\mathbf{R}}(M_{2,2,2}) = 7$.

I do not include a proof of Theorem 10.0.2.11 because I am dissatisfied with the existing proof, which is a case-by-case approach that follows Baur's proof of the Brockett-Dobkin theorem presented in §11.6.2, applied to the different components of the boundary of $\sigma_6(Seg(\mathbb{P}^3\times\mathbb{P}^3\times\mathbb{P}^3))$. Although the boundary is not known explicitly, I group all potential boundary components into possible cases (of types of limits) and treat each potential case. I hope a new, more geometric, proof will be available soon.

Theorem 11.0.2.12 (Bläser [**29**]). $\mathbf{R}(M_{3,3,3}) \geq 19$.

Since the proof is essentially by quantifier elimination and does not appear to use geometry, the reader is referred to the original article.

11.1. "Real world" issues

If one replaces the phrase "scalar multiplications" with the phrase "arithmetic operations" in the definition, the exponent of matrix multiplication ω is unchanged; see [**54**, Prop. 15.1]. Nevertheless, in actual implementation, one would like to reduce the number of arithmetic operations if possible.

A variant of Strassen's algorithm due to Paterson-Winograd uses 15 addition/subtractions instead of the 18 used in Strassen's, so it is often used in implementation.

In actual computations, total memory usage and the pattern of memory accesses are very important. Strassen's algorithm uses more memory than the traditional one. Nevertheless, Strassen's algorithm is actually practical for implementations and gives considerable savings.

For example, for square matrices, in the implementation discussed in [**171**], Strassen's algorithm is used to reduce to matrices of size 12×12, after which one switches to the standard algorithm.

For further reading on implementation see [**54, 107, 31, 171**] and the references therein.

11.2. Failure of the border rank version of Strassen's conjecture

The presentation in this section follows [**47**]. Before giving an actual example, I present a near counterexample to the border rank version of Strassen's additivity conjecture (see §5.7) that captures the essential idea, which is that one chooses $M \subset \mathbb{P}(A{\otimes}B{\otimes}C)$ such that $M \cap Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)$ contains more than $\dim M + 1$ points. Then taking $\dim M + 2$ points in the intersection, one can take any point in the sum of the $\dim M + 2$ tangent spaces (see [**208**, §10.1]).

11.2.1. Example of Bini et al. An "approximate algorithm" for multiplying 2×2 matrices where the first matrix has a zero in the $(2, 2)$ slot is presented in [**26**]. The algorithm corresponds to a point of $\sigma_5(Seg(\mathbb{P}^2 \times \mathbb{P}^3 \times \mathbb{P}^3))$ of the above nature. Namely, consider $Seg(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1) \subset \mathbb{P}^7$. Any $\mathbb{P}^4$ will intersect $Seg(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)$ in at least $\deg(Seg(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)) = 6$ points. Even better, the following five points on the Segre span a $\mathbb{P}^3$. Let the three $\mathbb{C}^2$'s respectively have bases a_1, a_2, b_1, b_2, c_1, c_2. Write

$$x_1 = a_1{\otimes}b_1{\otimes}c_1,\ x_2 = a_2{\otimes}b_2{\otimes}c_2,\ x_3 = a_1{\otimes}b_1{\otimes}(c_1{+}c_2),\ x_4{=}a_2{\otimes}(b_1{+}b_2){\otimes}c_2.$$

The lines $\langle x_1, x_3\rangle$ and $\langle x_2, x_4\rangle$ are contained in the Segre, so there are two lines worth of points of intersection of the Segre with the $\mathbb{P}^3$ spanned by these four points, but to use [**208**, §10.1] to be able to get any point in the span of the tangent spaces to these four points, we need a fifth point that is not in the span of any three points. Consider

$$x_5 = -x_1 - x_2 + x_3 + x_4 = (a_1 + a_2){\otimes}b_1{\otimes}c_2$$

which is not in the span of any three of the points.

For the reduced matrix multiplication operator

$$\begin{aligned} &a^1_2{\otimes}b^2_1{\otimes}c^1_1 + a^1_2{\otimes}b^2_2{\otimes}c^2_1 + a^2_1{\otimes}b^1_1{\otimes}c^1_2 + a^2_2{\otimes}b^2_1{\otimes}c^1_2 + a^2_1{\otimes}b^1_2{\otimes}c^2_2 + a^2_2{\otimes}b^2_2{\otimes}c^2_2 \\ &\in (\mathbb{C}^2{\otimes}\mathbb{C}^{2*}){\otimes}(\mathbb{C}^2{\otimes}\mathbb{C}^{2*}){\otimes}(\mathbb{C}^2{\otimes}\mathbb{C}^{2*}),\end{aligned}$$

the relevant points are

$$x_1{=}a^1_2{\otimes}b^1_2{\otimes}c^1_2,\ x_2{=}a^2_1{\otimes}b^1_1{\otimes}c^1_1,\ x_3{=}a^1_2{\otimes}b^1_2{\otimes}(c^1_1{+}c^1_2),\ x_4{=}a^2_1{\otimes}(b^1_1{+}b^1_2){\otimes}c^1_1.$$

Take

$$\begin{aligned} x'_1 &= a^1_1{\otimes}b^1_2{\otimes}c^1_2 + a^1_2{\otimes}b^2_2{\otimes}c^1_2 - a^1_2{\otimes}b^2_1{\otimes}c^1_2, \\ x'_2 &= a^1_1{\otimes}b^1_1{\otimes}c^1_1 + a^2_1{\otimes}b^1_1{\otimes}c^2_1 - a^2_1{\otimes}b^1_1{\otimes}c^2_2, \\ x'_3 &= a^1_2{\otimes}b^2_1{\otimes}(c^1_1 + c^1_2), \\ x'_4 &= a^2_1{\otimes}(b^1_1 + b^1_2){\otimes}c^2_2, \\ x'_5 &= 0. \end{aligned}$$

The matrix multiplication operator for the partially filled matrices is $x'_1 + x'_2 + x'_3 + x'_4$. The fact that we did not use any of the initial points is not surprising as the derivatives can always be altered to incorporate the initial points.

While this is not an example of the failure of BRPP, it illustrates the method that is used in Schönhage's example, which is more complicated, because it arranges that all first order terms cancel so one can use second order data.

Remark 11.2.1.1. This reduced matrix multiplication operator has much of the symmetry group of the usual one. Write $A = U^*{\otimes}V$, $B = V^*{\otimes}W$, $C = W^*{\otimes}U$, with $U, V, W = \mathbb{C}^2$. The identity component of the 7-dimensional group that acts effectively on expressions is $SL(W) \times P_U \times P_V$, where

$$P_U = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \subset SL(U)$$

and P_V is similar (only transposed).

11.2.2. Schönhage's example. Recall Strassen's additivity conjecture from §5.7: Given $T_1 \in A_1{\otimes}B_1{\otimes}C_1$ and $T_2 \in A_2{\otimes}B_2{\otimes}C_2$, if one considers $T_1 + T_2 \in (A_1 \oplus A_2){\otimes}(B_1 \oplus B_2){\otimes}(C_1 \oplus C_2)$, where each $A_j{\otimes}B_j{\otimes}C_j$ is naturally included in $(A_1 \oplus A_2){\otimes}(B_1 \oplus B_2){\otimes}(C_1 \oplus C_2)$, then $\mathbf{R}(T_1 + T_2) = \mathbf{R}(T_1) + \mathbf{R}(T_2)$.

Theorem 11.2.2.1 (Schönhage [**282**]). *Let $\mathbf{c}_1 = \mathbf{a}_1\mathbf{b}_1$, $\mathbf{a}_2 = \mathbf{b}_2 = (\mathbf{a}_1 - 1)(\mathbf{b}_1 - 1)$ and $\mathbf{c}_2 = 1$. Identify $C_1 \simeq (A_1{\otimes}B_1)^*$ and $A_2 \simeq B_2^*$. Take*

$$T_1 : A_1^* \times B_1^* \to (A_1{\otimes}B_1)^*,$$
$$T_2 : A_2^* \times A_2 \to \mathbb{C}$$

to be respectively the tensor product and contraction map. (Or in bases, respectively the product of a column vector with a row vector, and the product of a row vector with a column vector.) Then $\underline{\mathbf{R}}(T_1+T_2) \leq \mathbf{a}_1\mathbf{b}_1+1$.

By Exercise 3.1.3.3, $\underline{\mathbf{R}}(T_1) = \mathbf{a}_1\mathbf{b}_1$ and $\underline{\mathbf{R}}(T_2) = \mathbf{a}_2$, so Schönhage's theorem gives a counterexample to the border rank version of Strassen's conjecture discussed in §5.7.

Proof. We need to find a useful limit of secant planes. Take $\mathbf{a}_1\mathbf{b}_1 + 1$ points on $Seg(\mathbb{P}A_1 \times \mathbb{P}B_1 \times \mathbb{P}C_2) = Seg(\mathbb{P}^{\mathbf{a}_1-1} \times \mathbb{P}^{\mathbf{b}_1-1} \times \mathbb{P}^0) \subset \mathbb{P}^{\mathbf{a}_1\mathbf{b}_1-1}$. These points must be linearly dependent. To be explicit, let $(a_i), (b_s)$ respectively be bases of A_1, B_1, $1 \leq i \leq \mathbf{a}_1$, $1 \leq s \leq \mathbf{b}_1$. Let c be a basis of C_2. Take the points $x_{is} = a_i{\otimes}b_s{\otimes}c$ and $x_0 = -(\sum_i a_i){\otimes}(\sum_s b_s){\otimes}c$ so $\sum_{i,s} x_{is} + x_0 = 0$.

As discussed in §10.8, using one derivative it is possible to obtain any point of

$$\begin{aligned} &\sum \hat{T}_{x_{is}} Seg(\mathbb{P}(A_1 \oplus A_2) \times \mathbb{P}(B_1 \oplus B_2) \times \mathbb{P}(C_1 \oplus C_2)) \\ &= \sum_{i,s} a_i{\otimes}b_s{\otimes}(C_1 \oplus C_2) + \sum_i a_i{\otimes}(B_1 \oplus B_2){\otimes}c + \sum_s (A_1 \oplus A_2){\otimes}b_s{\otimes}c. \end{aligned}$$

This space contains

$$A_1 \otimes B_1 \otimes C_1 \oplus A_1 \otimes B_1 \otimes C_2 \oplus A_1 \otimes B_2 \otimes C_2 \oplus A_2 \otimes B_1 \otimes C_2,$$

so one could recover T_1 with just one derivative, but $T_2 \in A_2 \otimes B_2 \otimes C_2$ is not possible. However, more can be gained by taking a second derivative. Let $(a_{uq}), (b_{uq})$ be bases of A_2, B_2 respectively, $1 \leq u \leq \mathbf{a}_1 - 1$, $1 \leq q \leq \mathbf{b}_1 - 1$. Choose curves $x_{is}(t)$ with $x'_{is} := x'_{is}(0)$ as follows:

$$\begin{aligned}
x'_{uq} &= a_{uq} \otimes b_q \otimes c + a_u \otimes b_{uq} \otimes c \in A_2 \otimes B_1 \otimes C_2 \oplus A_1 \otimes B_2 \otimes C_2, \\
x'_{u\mathbf{b}_1} &= a_u \otimes \Big(- \sum_q b_{uq} \Big) \otimes c \qquad \in A_1 \otimes B_2 \otimes C_2, \\
x'_{\mathbf{a}_1 q} &= \Big(- \sum_u a_{uq} \Big) \otimes b_q \otimes c \qquad \in A_2 \otimes B_1 \otimes C_2, \\
x'_{\mathbf{a}_1 \mathbf{b}_1} &= 0, \\
x'_0 &= 0.
\end{aligned}$$

Observe that, as required, the sum of these terms is zero, so following §10.8, we take second derivatives. Choose

$$\begin{aligned}
x''_{uq} &= a_{uq} \otimes b_{uq} \otimes c + a_u \otimes b_q \otimes c_{uq} \in A_2 \otimes B_2 \otimes C_2 \oplus A_1 \otimes B_1 \otimes C_1, \\
x''_{u\mathbf{b}_1} &= a_u \otimes b_{\mathbf{b}_1} \otimes c_{u\mathbf{b}_1} \qquad \in A_1 \otimes B_1 \otimes C_1, \\
x''_{\mathbf{a}_1 q} &= a_{\mathbf{a}_1} \otimes b_q \otimes c_{\mathbf{a}_1 q} \qquad \in A_1 \otimes B_1 \otimes C_1, \\
x''_{\mathbf{a}_1 \mathbf{b}_1} &= a_{\mathbf{a}_1} \otimes b_{\mathbf{b}_1} \otimes c_{\mathbf{a}_1 \mathbf{b}_1} \qquad \in A_1 \otimes B_1 \otimes C_1, \\
x''_0 &= 0.
\end{aligned}$$

Note that the sum $\sum_{is} x''_{is} + x''_0$ is indeed $T_1 + T_2$. The first term is all of T_2 plus most of T_1, and the rest of the terms contribute the missing terms of T_1. □

For readers who prefer an expression in terms of curves, the relevant curve is:

$$\begin{aligned}
T(t) = &\sum_{u,q} (a_u + t a_{uq}) \otimes (b_q + t b_{uq}) \otimes (c + t^2 c_{uq}) \\
&+ \sum_u a_u \otimes \Big(b_{\mathbf{b}_1} + t \Big(- \sum_q b_{uq} \Big) \Big) \otimes (c + t^2 c_{u\mathbf{b}_1}) \\
&+ \sum_q \Big(a_{\mathbf{a}_1} + t \Big(- \sum_u a_{uq} \Big) \Big) \otimes b_q \otimes (c + t^2 c_{\mathbf{a}_1 q}) \\
&+ a_{\mathbf{a}_1} \otimes b_{\mathbf{b}_1} \otimes (c + t^2 c_{\mathbf{a}_1 \mathbf{b}_1}) - \Big(\sum_i a_i \Big) \otimes \Big(\sum_s b_s \Big) \otimes c.
\end{aligned}$$

11.2.3. Application: $\underline{\mathbf{R}}(M_{3,3,3}) \leq 21$. First consider the matrix multiplication of partially filled matrices:

$$M_1 : \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \times \begin{pmatrix} * & * & * \\ * & 0 & 0 \\ * & 0 & 0 \end{pmatrix} \to \begin{pmatrix} * & * & * \\ * & 0 & 0 \\ * & 0 & 0 \end{pmatrix}.$$

Choosing bases $(\alpha^i_j), (\beta^i_j), (\gamma^i_j)$, the expression as a tensor is

$$\begin{aligned} M_1 = &\alpha^1_1 \otimes (\beta^1_1 \otimes \gamma^1_1 + \beta^1_2 \otimes \gamma^2_1 + \beta^1_3 \otimes \gamma^3_1) \\ &+ (\alpha^1_2 \otimes \beta^2_1 + \alpha^1_3 \otimes \beta^3_1)\gamma^1_1 + (\alpha^2_2 \otimes \beta^2_1 + \alpha^2_3 \otimes \beta^3_1)\gamma^1_2 + (\alpha^3_2 \otimes \beta^2_1 + \alpha^3_3 \otimes \beta^3_1)\gamma^1_3, \end{aligned}$$

Write $M_1 = T_1 + T_2$, where

$$\begin{aligned} T_1 &= (\alpha^1_2 \otimes \beta^2_1 + \alpha^1_3 \otimes \beta^3_1)\gamma^1_1 + (\alpha^2_2 \otimes \beta^2_1 + \alpha^2_3 \otimes \beta^3_1)\gamma^1_2 \\ &\quad + (\alpha^3_2 \otimes \beta^2_1 + \alpha^3_3 \otimes \beta^3_1)\gamma^1_3 : \mathbb{C}^2 \times \mathbb{C}^3 \to \mathbb{C}^6, \\ T_2 &= \alpha^1_1 \otimes (\beta^1_1 \otimes \gamma^1_1 + \beta^1_2 \otimes \gamma^2_1 + \beta^1_3 \otimes \gamma^3_1) : \mathbb{C}^3 \times \mathbb{C}^3 \to \mathbb{C}. \end{aligned}$$

Schönhage's algorithm shows that $\underline{\mathbf{R}}(M_1) \leq 7$.

Now consider

$$M_2 : \begin{pmatrix} 0 & * & * \\ * & * & * \\ 0 & * & * \end{pmatrix} \times \begin{pmatrix} * & * & * \\ 0 & * & 0 \\ 0 & * & 0 \end{pmatrix} \to \begin{pmatrix} * & * & * \\ 0 & * & 0 \\ 0 & * & 0 \end{pmatrix},$$

$$M_3 : \begin{pmatrix} 0 & * & * \\ 0 & * & * \\ * & * & * \end{pmatrix} \times \begin{pmatrix} * & * & * \\ 0 & 0 & * \\ 0 & 0 & * \end{pmatrix} \to \begin{pmatrix} * & * & * \\ 0 & 0 & * \\ 0 & 0 & * \end{pmatrix},$$

and observe that $M_{3,3,3} = M_1 + M_2 + M_3$.

Remark 11.2.3.1. Schönhage, following Pan, presents other approximating algorithms, including one that uses derivatives up to order 20 [**282**, §8]. In these algorithms one deals with the sum of three tensors $T_j : A^*_j \otimes B^*_j \to C_j$ which are all the sums of pairs of tensor products, respectively $\mathbb{C}^k \times (\mathbb{C}^n \oplus \mathbb{C}^n) \to \mathbb{C}^k \otimes (\mathbb{C}^n \oplus \mathbb{C}^n)$, $(\mathbb{C}^k \oplus \mathbb{C}^k) \times \mathbb{C}^n \to (\mathbb{C}^k \oplus \mathbb{C}^k) \otimes \mathbb{C}^n$, and $(\mathbb{C}^k \oplus \mathbb{C}^k \times (\mathbb{C}^n \oplus \mathbb{C}^n) \to \mathbb{C}^k \otimes \mathbb{C}^n$. (In the last case one takes two tensor products and adds the sum. In the first two cases one takes two tensor products and regards the sum as lying in the sum of two spaces.)

For these curves, he takes all the $2(k+1)(n+2)$ limit points to lie on a $\mathbb{P}^{k-1} \times \mathbb{P}^{k-1} \times \mathbb{P}^{k-1}$, but in fact only uses $k+1$ distinct points.

11.3. Finite group approach to upper bounds

H. Cohn and C. Umans [**94**] have proposed an approach to constructing explicit algorithms for matrix multiplication using the discrete Fourier transform and the representation theory of finite groups.

11.3.1. Preliminaries. Let G be a finite group and $\mathbb{C}[G]$ its group algebra; see §6.2. The discrete Fourier transform (DFT) $D : \mathbb{C}[G] \to \mathbb{C}^{|G|}$ is an invertible linear map that transforms multiplication in $\mathbb{C}[G]$ to block diagonal multiplication of vectors in $\mathbb{C}^{|G|}$. (See, e.g., [**54**] for an exposition of the DFT.)

Proposition 11.3.1.1. *As an algebra,*

$$\mathbb{C}[G] \simeq Mat_{d_1\times d_1}(\mathbb{C}) \times \cdots \times Mat_{d_k\times d_k}(\mathbb{C}),$$

where G has k irreducible representations and the dimension (character) of the j-th is d_j. The relevant block diagonal multiplication mentioned above is by these $d_j \times d_j$ blocks.

To prove Proposition 11.3.1.1, recall Wedderburn's theorem:

Theorem 11.3.1.2 (Wedderburn's theorem)**.** *Let $\mathcal{A}$ be a finite-dimensional simple algebra over a ring R. Then for some $n \in \mathbb{N}$, $\mathcal{A}$ is isomorphic to the ring of $n \times n$ matrices whose entries are in a division ring over R.*

In particular, if R is an algebraically closed field $\mathbb{F}$ (e.g., $\mathbb{C}$), then $\mathcal{A}$ is isomorphic to $Mat_{n\times n}(\mathbb{F})$.

For a proof see, e.g., [**143**, p. 128].

Recall from §6.2 that the multiplicity of an irreducible representation of a finite group in $\mathbb{C}[G]$ corresponds to the number of the elements in the corresponding conjugacy class. Now apply Schur's lemma (see §6.1) to conclude.

Exercise 11.3.1.3: Verify Proposition 11.3.1.1 directly for $\mathfrak{S}_3$.

11.3.2. Transforming the matrix multiplication problem. The idea of the Cohn-Umans approach is as follows: To multiply $Mat_{n\times m}\times Mat_{m\times p} \to Mat_{n\times p}$, one first bijectively maps bases of each of these three spaces into subsets of some finite group G. The subsets are themselves formed from three subsets S_1, S_2, S_3 of cardinalities n, m, p. The sets S_i must have a disjointness property, called the *triple product property* in [**94**]: if $s_1s_2s_3 = \mathrm{Id}$, with $s_i \in S_i$, then each $s_i = \mathrm{Id}$. Then the maps are to the subsets ${S_1}^{-1}S_2$, ${S_2}^{-1}S_3$, ${S_1}^{-1}S_3$.

The triple product property enables one to read off matrix multiplication from multiplication in the group ring.

They then show, if ω is the exponent of matrix multiplication, that, if one can find such a group and subsets, then

$$(nmp)^{\frac{\omega}{3}} \leq d^{\omega-2}|G|,$$

where d is the largest dimension of an irreducible representation of G. So one needs to find groups that are big enough to support triples satisfying the triple product property but as small as possible and with largest character as small as possible.

In [**93**] the authors give explicit examples which recover $\omega < 2.41$ and state several combinatorial and group-theoretic conjectures, which, if true, would imply $\omega = 2$.

The asymptotic examples are complicated to state. Here is an explicit example that beats the standard algorithm. Let

$$H = \mathbb{Z}_n \times \mathbb{Z}_n \times \mathbb{Z}_n =: H_1 \times H_2 \times H_3$$

and let

$$G = (H \times H) \ltimes \mathbb{Z}_2,$$

where $\mathbb{Z}_2$ acts by switching the factors. Write elements of G in the form $(a,b)z^i$, where $a, b \in H$ and $i \in \{0,1\}$. Write $H_4 = H_1$. Let $S_1, S_2, S_3 \subset G$ be defined by

$$S_i := \{(z,b)z^j \mid a \in H_i \backslash \{0\},\ b \in H_{i+1} \backslash \{0\},\ j \in \{0,1\}\}.$$

Then (S_1, S_2, S_3) satisfy the triple product property; see [**93**, Lem. 2.1]. Here $|S_i| = 2n(n-1)$, and setting $n = 17$ (the optimal value), they obtain $\omega < 2.9088$.

If we write $d = e^{f(n)}$ and $|G| = e^{g(n)}$, where G supports subsets with the triple product property to enable $n \times n$ matrix multiplication, then $n^\omega \leq (e^{f(n)})^{\omega-2} e^{g(n)}$ and one obtains

$$\omega \leq 2\frac{f(n) - \frac{1}{2}g(n)}{f(n) - \log(n)}.$$

11.4. $\mathbf{R}(M_{3,3,3}) \leq 23$

Verify that we have the following expression for $M_{3,3,3} : A \times B \to C$, where $A, B, C \simeq Mat_{3\times 3}$ (this expression is due to Laderman [**199**]):

$$
\begin{aligned}
M_{3,3,3} =\, & c^1_2 \otimes (a^1_1 + a^1_2 + a^1_3 - a^2_1 - a^2_2 - a^3_2 - a^3_3) \otimes b^2_2 \\
& + (c^2_1 + c^2_2) \otimes (a^1_1 - a^2_1) \otimes (-b^1_2 + b^2_2) \\
& + c^2_1 \otimes a^2_2 \otimes (-b^1_1 + b^1_2 + b^2_1 - b^2_2 - b^2_3 - b^3_1 + b^3_3) \\
& + (c^2_1 + c^1_2 + c^2_2) \otimes (-a^1_1 + a^2_1 + a^2_2) \otimes (b^1_1 - b^1_2 + b^2_2) \\
& + (c^1_2 + c^2_2) \otimes (a^2_1 + a^2_2) \otimes (-b^1_1 + b^1_2) \\
& + (c^1_1 + c^2_1 + c^3_1 + c^1_2 + c^2_2 + c^1_3 + c^3_3) \otimes a^1_1 \otimes b^1_1 \\
& + (c^3_1 + c^1_3 + c^3_3) \otimes (-a^1_1 + a^3_1 + a^3_2) \otimes (b^1_1 - b^1_3 + b^2_3) \\
& + (c^3_1 + c^3_3) \otimes (-a^1_1 + a^3_1) \otimes (b^1_3 - b^2_3) \\
& + (c^1_3 + c^3_3) \otimes (a^3_1 + a^3_2) \otimes (-b^1_1 + b^1_3) \\
& + c^1_3 \otimes (a^1_1 + a^1_2 + a^1_3 - a^2_2 - a^2_3 - a^3_1 - a^3_2) \otimes b^2_3 \\
& + c^3_1 \otimes a^3_2 \otimes (-b^1_1 + b^1_3 + b^2_1 - b^2_2 - b^2_3 - b^3_1 + b^3_2) \\
& + (c^3_1 + c^1_2 + c^3_2) \otimes (-a^1_3 + a^3_2 + a^3_3) \otimes (b^2_2 + b^3_1 - b^3_2) \\
& + (c^3_1 + c^3_2) \otimes (a^1_3 - a^3_3) \otimes (b^2_2 - b^3_2) \\
& + (c^1_1 + c^2_1 + c^3_1 + c^1_2 + c^3_2 + c^1_3 + c^2_3) \otimes a^1_3 \otimes b^3_1 \\
& + (c^1_2 + c^3_2) \otimes (a^3_2 + a^3_3) \otimes (-b^3_1 + b^3_2) \\
& + (c^2_1 + c^1_3 + c^2_3) \otimes (-a^1_3 + a^2_2 + a^2_3) \otimes (b^2_3 + b^3_1 - b^3_3) \\
& + (c^2_1 + c^2_3) \otimes (a^1_3 - a^2_3) \otimes (b^2_3 - b^3_3) \\
& + (c^1_3 + c^2_3) \otimes (a^2_2 + a^2_3) \otimes (-b^3_1 + b^3_3) \\
& + c^1_1 \otimes a^1_2 \otimes b^2_1 \\
& + c^2_2 \otimes a^2_3 \otimes b^3_2 \\
& + c^2_3 \otimes a^2_1 \otimes b^1_3 \\
& + c^3_2 \otimes a^3_1 \otimes b^1_2 \\
& + c^3_3 \otimes a^3_3 \otimes b^3_3.
\end{aligned}
$$

11.5. Bläser's $\frac{5}{2}$-Theorem

The best asymptotic lower bound on the rank of matrix multiplication is the following:

Theorem 11.5.0.1 (M. Bläser [**28**]). $\mathbf{R}(M_{m,m,m}) \geq \frac{5}{2}m^2 - 3m.$

Here is a proof of Bläser's theorem (from [**201**]) that uses Strassen's equations in the form of Theorem 3.8.2.4, which is implicit, but hidden, in Bläser's original proof.

Lemma 11.5.0.2. *Let U be a vector space and let $P \in S^dU^*\backslash 0$. Let $u_1, \dots, u_n$ be a basis of U. Then there exists a subset $u_{i_1}, \dots, u_{i_s}$ of cardinality $s \leq d$ such that $P|_{\langle u_{i_1}, \dots, u_{i_s} \rangle}$ is not identically zero.*

Proof. Let $c^j u_j \in U$ denote an arbitrary element. Consider $P|_U$ as a function of the c^j's. It consists of terms $p_{i_1,\dots,i_d} c^{i_1} \cdots c^{i_d}$ with $i_1 \leq \cdots \leq i_d$. Take, e.g., the first nonzero monomial appearing; it can involve at most d of the c^j's. Then P restricted to the span of the corresponding u_j's will be nonzero. □

Lemma 11.5.0.3. *Given any basis of $Mat^*_{m\times m}$, there exists a subset of at least $m^2 - 3m$ basis vectors that annihilate elements* $\mathrm{Id}, x, y \in Mat_{m\times m}$ *such that $[x, y] := xy - yx$ has maximal rank m.*

Proof. Let $A = Mat_{m\times m} \simeq U^* \otimes W$. Fixing a basis of A^* is equivalent to fixing its dual basis of A. By Lemma 11.5.0.2 with $P = \det$, we may find a subset S_1 of at most m elements of our basis of A with some $z \in Span(S_1)$ with $\det(z) \neq 0$. Use $z : U \to W$ to identify $U \simeq W$, which enables us to now consider A as an algebra with z playing the role of the identity element.

Now let $a \in A$ be generic. Then the map $\mathrm{ad}(a) : A \to A$, $x \mapsto [a, x]$ will have a one-dimensional kernel. By letting $P = \mathrm{ad}(a)^*(\det)$ and applying Lemma 11.5.0.2 again, we may find a subset S_2 of our basis of cardinality at most m such that there is an element $x \in A$ such that $\mathrm{ad}(a)(x)$ is invertible. Note that $\mathrm{ad}(x) : A \to A$ also is such that there are elements y with $\mathrm{ad}(x)(y)$ invertible. Thus we may apply Lemma 11.5.0.2 a third time to find a cardinality at most m subset S_3 of our basis such that $\mathrm{ad}(x)(y)$ is invertible. Now in the worst possible case our three subsets are of maximal cardinality and do not intersect, in which case we have a cardinality $m^2 - 3m$ subset of our dual basis that annihilates $z = \mathrm{Id}, x, y$ with $\mathrm{rank}([x, y]) = m$. □

Proof of Theorem 11.5.0.1. Let ϕ denote an expression of $M_{m,m,m}$ as a sum of r rank one tensors. Since $\mathrm{Lker}(M) = 0$ (i.e., $\forall a \in A\backslash 0$, $\exists b \in B$ such that $M(a, b) \neq 0$), we may write $\phi = \psi_1 + \psi_2$ with $\mathbf{R}(\psi_1) = m^2$, $\mathbf{R}(\psi_2) = r - m^2$ and $\mathrm{Lker}(\psi_1) = 0$. Now consider the m^2 elements of A^* appearing in ψ_1. Since they span A^*, by Lemma 11.5.0.3 we may choose a subset of $m^2 - 3m$ of them that annihilate Id, x and y, where x, y are such that $[x, y]$ has full rank. Let ϕ_1 denote the sum of all monomials in ψ_1 whose A^* terms annihilate Id, x, y, so $\mathbf{R}(\phi_1) \geq m^2 - 3m$. Let $\phi_2 = \psi_1 - \phi_1 + \psi_2$.

Now apply Theorem 3.8.2.4 with $T = \phi_2$, $\alpha = \mathrm{Id}$, $\alpha_1 = M(x)$, $\alpha_2 = M(y)$. If $x = x^i_j u^j \otimes w_i$, then $M(x) = x^i_j (u^j \otimes v_k) \otimes (w_i \otimes v^k) : \mathbb{C}^{m^2} \to \mathbb{C}^{m^2}$ is of rank m^2 and so is $[M(x), M(y)] = [x,y]^i_j(u^j \otimes v_k)\otimes(w_i \otimes v^k)$. Hence $\underline{\mathbf{R}}(\phi_2) \geq \frac{1}{2}\dim[M(x), M(y)] + m^2 = \frac{3}{2}m^2$ and thus $\mathbf{R}(\phi_1+\phi_2) \geq \frac{5}{2}m^2 - 3m$. □

11.6. The Brockett-Dobkin Theorem

After some preliminaries on the algebra of $n \times n$ matrices, I give a proof of the Brockett-Dobkin Theorem 11.6.2.1.

11.6.1. Preparation for the proof: The algebra of $n \times n$ matrices. An *algebra* $\mathcal{A}$ is a vector space with a multiplication compatible with the linear structure. All algebras we study will be associative. Inside an algebra one has *left ideals*, where $I_L \subset \mathcal{A}$ is a left ideal if it is a linear subspace satisfying $au \in I_L$ for all $a \in \mathcal{A}$ and all $u \in I_L$. Similarly, a linear subspace $I_R \subset A$ is a *right ideal* if $va \in I_R$ for all $a \in \mathcal{A}$ and all $v \in I_R$. An ideal is *two-sided* if it is both a left ideal and a right ideal. An algebra is *simple* if it has no nontrivial two sided ideals.

Now let V be a $\mathbf{v}$-dimensional vector space and consider the algebra $\mathcal{A} = \mathrm{End}(V)$ of endomorphisms of V. This algebra is isomorphic to the algebra $Mat_{\mathbf{v}\times\mathbf{v}}$.

Given a subspace $U \subset V$ consider the subsets

$$R_U := \{f \in End(V) \mid f(V) \subseteq U\},$$

and

$$L_U := \{h \in End(V) \mid U \subseteq \ker h\}.$$

Exercises 11.6.1.1:

1. Show that R_U is a right ideal and L_U is a left ideal.
2. If $\dim U = k$, determine $\dim R_U$ and $\dim L_U$.
3. If $W \subset V$ is another subspace of dimension k, then show that there exists $a \in End(V)$ such that $aR_U = R_W$ and $b \in End(V)$ such that $L_U b = L_W$.
4. Show that we may choose a basis of V to identify $End(V) \simeq Mat_{\mathbf{v}\times\mathbf{v}}$ in such a way that R_U consists of the matrices whose first $\mathbf{v}-k$ rows are zero. Similarly, another basis will enable us to identify L_U with the set of matrices whose first k columns are zero.

Note that $L_U R_U = 0$, i.e., if $a \in L_U$ and $b \in R_U$, then $ab : V \to V$ is the zero map.

Proposition 11.6.1.2. *Let V be a $\mathbf{v}$-dimensional vector space.*

(1) *All left (resp. right) ideals of $End(V)$ are of the form L_U (resp. R_U) for some U as above.*

(2) *The dimensions of the nontrivial ideals of $End(V)$ are $\mathbf{v}, 2\mathbf{v}, \ldots,$ and $(\mathbf{v}-1)\mathbf{v}$.*

(3) *For any left (resp. right) ideal I of dimension $k\mathbf{v}$, there is a complementary left (resp. right) ideal I' of dimension $(\mathbf{v}-k)\mathbf{v}$ such that we have the vector space decomposition $End(V) = I \oplus I'$.*

(4) *No nonzero left (resp. right) ideal is contained in a proper nonzero right (resp. left) ideal.*

(5) *Any left (resp. right) ideal is contained in a maximal left (resp. right) ideal. (By part (2), the maximal ideal is of dimension $\mathbf{v}(\mathbf{v}-1)$.)*

Proof. All the assertions are immediate consequences of the first assertion. To prove the first, let R be a proper right ideal and choose $a \in R$ of maximal rank. Note that a is not of full rank (otherwise $R = End(V)$). Let $U = \operatorname{image} a \subset V$, so $R \supseteq R_U$. But now say $a' \in R$ is such that $\operatorname{image}(a') \not\subseteq R$. Then all but a finite number of linear combinations of a and a' will have rank greater than a, a contradiction. □

In fact, the above theorem applies to all simple algebras over $\mathbb{C}$ thanks to the Wedderburn Theorem 11.3.1.2.

11.6.2. Proof of the Brockett-Dobkin Theorem.

Theorem 11.6.2.1 (Brockett-Dobkin [**41**]). $\mathbf{R}(M_{m,m,m}) \geq 2m^2 - 1$.

The following proof is a coordinate-free version of a proof due to Baur and published in [**54**].

Proof. Let $\mathbf{a} = \mathbf{b} = \mathbf{c} = m^2$ and consider matrix multiplication as a bilinear map $A \times B \to C$. We assume that ϕ is an expression of M as a sum of $2\mathbf{a}-2$ rank one tensors and obtain a contradiction. We may write $\phi = \phi_1 + \phi_2$ with $\mathbf{R}(\phi_j) \leq a - 1$ and assume moreover that $\dim(\mathrm{Lker}\phi_1) = 1$ (consider the terms of A appearing in ϕ). Since $\phi_2 \in Sub_{\mathbf{a}-1,\mathbf{a}-1,\mathbf{a}-1}$, there exists a nonzero $b \in B$ such that $\phi_2(A, b) = 0$, and thus $Ab \subseteq \operatorname{image}(\phi_1)$. Since we also have $\phi_1 \in Sub_{\mathbf{a}-1,\mathbf{a}-1,\mathbf{a}-1}$, Ab does not coincide with C and thus is a nontrivial left ideal. Recall that the minimal dimension of an ideal is m and $\mathbf{a}-m$ is the maximal dimension. I claim we may write $\phi_1 = \phi_{11} + \phi_{12}$ with $\mathbf{R}(\phi_{11}) \leq m-1$, $\mathbf{R}(\phi_{12}) \leq \mathbf{a}-m$ and furthermore that $\operatorname{image}(\phi_{11}) \subset Ab$. This last condition is possible because, since $\dim Ab > m-1$, at least m of the elements of B^* appearing in the expression for ϕ_1 must evaluate to be nonzero on b. We take any subset of $m-1$ of these elements, and (by our hypothesis that $\dim \mathrm{Lker}\, \phi_1 = 1$) since the elements of A^* appearing in the expression of ϕ_1 are all linearly independent, we see that $\operatorname{image}(\phi_{11}) \subset Ab$.

I claim that $\mathrm{Rker}(\phi_{12} + \phi_2) = 0$. Otherwise let b' be in its annihilator, and we would have an ideal Ab' with $\dim Ab' = m - 1$, a contradiction.

Let $\tilde{\phi} = \phi_{12} + \phi_2$, $\mathbf{R}(\tilde{\phi}) \leq 2\mathbf{a} - m - 1$. Let $L \supseteq Ab$ denote a maximal ideal containing Ab. Then (by the same reasoning as the claim above) we may split $\tilde{\phi} = \tilde{\phi}_1 + \tilde{\phi}_2$ with $\mathbf{R}(\tilde{\phi}_1) \leq \mathbf{a} - 1$ and $\mathbf{R}(\tilde{\phi}_2) \leq \mathbf{a} - 1$ and moreover $\mathrm{Rker}(\tilde{\phi}_1 |_{A\times L}) = 0$; i.e., letting $U = \mathrm{Rker}\,\tilde{\phi}_1$, we have $B = U \oplus L$.

Now let $a \in A$ be such that $a \in \mathrm{Lker}(\tilde{\phi}_2)$ and consider the ideal aA. Let $x \in B$; then $x = x_1 + x_2$ with $x_1 \in L$ and $x_2 \in U$. Consider $ax = ax_1 + ax_2$. Then $ax_1 \in L$, as L is a left ideal, but $ax_2 = \phi_{11}(a, x_2) \in Ab$, so $aA \subseteq L$, a contradiction as a left ideal cannot be contained in a right ideal. □

A consequence is:

Corollary 11.6.2.2 (Winograd [**334**]). $\mathbf{R}(M_{2,2,2}) = 7$.

11.7. Multiplicative complexity

The *multiplicative complexity* of $T \in A{\otimes}B{\otimes}C$ is its rank considered as an element of $(A \oplus B){\otimes}(A \oplus B){\otimes}C$. (This definition differs from those in the literature, e.g., [**54**, p. 352], but is equivalent.) J. Hopfcroft and L. Kerr [**167**] gave an explicit class of examples where the multiplicative complexity is smaller than rank when working over $\mathbb{F}_2$: the multiplication of $p \times 2$ matrices by $2 \times n$ matrices. For example, for 2×2 matrices by 2×3 matrices defined over $\mathbb{F}_2$, they showed that the rank is 11, but gave an explicit algorithm in terms of 10 multiplications. Here is their algorithm expressed as a tensor in $(A \oplus B){\otimes}(A \oplus B){\otimes}C$:

$$\begin{aligned}
M_{2,2,3} = {} & \frac{1}{2}(a^1_1 + b^2_1){\otimes}(a^1_2 + b^1_1){\otimes}(c^1_1 - c^2_1) \\
& + \frac{1}{2}(a^1_1 + b^2_2){\otimes}(a^1_2 + b^1_2){\otimes}(c^1_2 + c^2_1 + c^2_3) \\
& + \frac{1}{2}(a^1_1 + b^2_3){\otimes}(a^1_2 + b^1_3){\otimes}(c^1_3 - c^2_3) + (a^2_1 + b^2_1){\otimes}(a^2_2 + b^1_1){\otimes}c^2_1 \\
& + \frac{1}{2}(a^2_1 + b^2_2){\otimes}(a^2_2 + b^1_2){\otimes}(-c^2_1 + c^2_2 - c^2_3) + (a^2_1 + b^2_3){\otimes}(a^2_2 + b^1_3){\otimes}c^2_3 \\
& + \frac{1}{2}(a^1_1 - b^2_1){\otimes}(-a^1_2 + b^1_1){\otimes}(c^1_1 + c^2_1) \\
& + \frac{1}{2}(a^1_1 - b^2_2){\otimes}(-a^1_2 + b^1_2){\otimes}(c^1_2 - c^2_1 - c^2_3) \\
& + \frac{1}{2}(a^1_1 - b^2_3){\otimes}(-a^1_2 + b^1_3){\otimes}(c^1_3 + c^2_3) \\
& + \frac{1}{2}(a^2_1 - b^2_2){\otimes}(-a^2_2 + b^1_2){\otimes}(c^2_1 + c^2_2 + c^2_3).
\end{aligned}$$

Note that this algorithm is valid in any characteristic. In [**6**] it is shown that the rank is 11 over any field.

This definition of multiplicative complexity gives an immediate proof of (14.8) in [**54**]:

Proposition 11.7.0.3. *For a tensor $T \in A \otimes B \otimes C$, the multiplicative complexity of T is at most the rank of T, which is at most twice the multiplicative complexity of T.*

To see this, note that $(A \oplus B) \otimes (A \oplus B) \otimes C = A \otimes B \otimes C \oplus A \otimes B \otimes C \oplus A \otimes A \otimes C \oplus B \otimes B \otimes C$, and so any expression for T in $(A \oplus B)^{\otimes 2} \otimes C$ of rank r projects to an expression for T of rank at most $2r$ in $A \otimes B \otimes C$.

Remark 11.7.0.4. It might also be natural to consider expressions of $T \in A \otimes B \otimes C$ in $(A \oplus B \oplus C)^{\otimes 3}$. In any case, the savings would be at best by a factor of 6 by the same reasoning as in the previous paragraph.

Chapter 12

Tensor decomposition

In applications one is often handed a tensor and asked i) to determine its attributes (e.g., its rank), and ii) once an attribute such as rank, say r, has been determined, to perform the CP decomposition, i.e., to rewrite the tensor as a sum of r rank one elements. Such decompositions date back to Hitchcock in 1927; see [**164, 165**]. Two situations where one needs to determine the rank and then find the CP decomposition have already been discussed in the Introduction: fluorescence spectroscopy and blind source separation. A special case of source separation in wireless communication is discussed in more detail in §12.2.

Tensor decomposition goes under various names such as PARAFAC, for "parallel factor analysis", *canonical polyadic decomposition*, *CANDECOMP*, for "canonical decomposition", and *CP decomposition*, which was suggested by P. Comon and which may be interpreted as an abbreviation of *canonical polyadic*, as an acronym combining CANDECOMP and PARAFAC, or as the initials of the individual who suggested it.

For certain applications, one would like to be assured that a given CP decomposition of a tensor is unique. (As was already mentioned, if such a decomposition determines a region of the brain of a patient causing epileptic seizures to be removed, one does not want a choice of several possible regions to operate on.) The second part of this chapter, §§12.3–12.5 is a discussion of tests for uniqueness, including Kruskal's celebrated test, and explicit decomposition algorithms.

This chapter begins with a discussion of cumulants in §12.1, following up on the discussion in the Introduction. Next, in §12.2, there are two different approaches to *blind deconvolution* in wireless communication. The first uses a CP decomposition, which, as the reader will recall, is related to secant varieties of Segre varieties. The second uses what De Lathauwer calls a *block term decomposition*, which is related to secant varieties of subspace varieties.

The next three sections discuss the existence of unique or almost unique CP decompositions: the failure of uniqueness is reviewed in §12.3, and a property, r-NWD, that assures uniqueness with probability one, is discussed. A concise proof of Kruskal's theorem is given in §12.5. Several exact decomposition algorithms are presented in §12.4.

The phrase that a property holds "with probability one" in this chapter means that the set of tensors that fail to have the stated property is of measure zero (with respect to any measure that respects the vector space structure); more precisely, the set where the property fails is a proper subvariety.

Regarding things not covered in this chapter, nonuniqueness is also discussed in [**110**], and a decomposition algorithm that uses elimination theory is proposed in [**35**].

Remark 12.0.0.5. In this chapter, I generally ignore noise. What is really received is a tensor $\tilde{T}$ that is T plus a "small" generic tensor, where the small tensor comes from the noise. Noise is taken into account when one performs the tensor decomposition by some particular algorithm, which is outside the scope of this book.

12.1. Cumulants

12.1.1. Blind source separation. Recall the problem of blind source separation (BSS) [**98, 96**] discussed in §1.3.3. Say $y^1, \dots, y^m$ are functions of a variable t (which will be time in all applications). Such are called *stochastic processes*. Say we expect that there are exactly $r < m$ statistically independent functions (see Definition 1.3.2.2) from which the y^j are constructed and we would like to find them. In the example of pirate radio from §1.3.3, the statistically independent functions would be obtained from the radio sources and the y^j would be obtained from the measured signals. That is, we have an equation of the form:

$$y = Ax + v. \tag{12.1.1}$$

Here A is a fixed $m \times r$ matrix, $v = v(t) \in \mathbb{R}^m$ is a vector-valued function representing the noise, and $x(t) = (x^1(t), \dots, x^r(t))^T$ represents the statistically independent functions of t. ("T" denotes transpose.) In the literature y is referred to as the *observation vector*, x is the *source vector*, v is the *additive noise*, and A is the *mixing matrix*.

The assumptions of §1.3.3 are valid when there are a large number of independent contributions to the noise because of the central limit theorem (which says that a sum of independent random variables will tend to a Gaussian). Otherwise [**98**, p. 326] they can be weakened by requiring that the v^i be a linear combination of independent components (rather than assuming the components themselves to be independent as in §1.3.3). This can be weakened further to just requiring the components of v to have small variance.

One would like to recover $x(t)$, plus the matrix A, from knowledge of the function $y(t)$ alone. The x^j are like eigenvectors, in the sense that they are only well defined up to scale and permutation, so "recover" means modulo this ambiguity.

The key to (approximately) recovering A, x from y will be the symmetric tensor decomposition of the associated cumulants.

12.1.2. Moments and cumulants. Recall the *moments*

$$m_x^{i_i,\dots,i_p} := E\{x^{i_1}\cdots x^{i_p}\} = \int_{\mathbb{R}} x^{i_1}\cdots x^{i_p}\,d\mu.$$

Moments transform well under a linear map, but badly under a transformation such as (12.1.1), e.g.,

$$m_y^{ij} = m_v^{ij} + m_v^i A_s^j m_x^s + m_v^j A_s^i m_x^s + A_s^i A_t^j m_x^{st}, \tag{12.1.2}$$

$$\begin{aligned} m_y^{ijk} &= m_v^{ijk} + m_v^{ij} A_s^k m_x^s + m_v^{ik} A_s^j m_x^s + m_v^{jk} A_s^i m_x^s \\ &\quad + m_v^i A_s^j A_t^k m_x^{st} + m_v^j A_s^i A_t^k m_x^{st} + m_v^k A_s^i A_t^j m_x^{st}. \end{aligned} \tag{12.1.3}$$

This has the consequence that the moments are not effective for keeping track of statistical independence. This motivates the introduction of cumulants, which are additive on statically independent functions.

At this point it will be useful to adopt symmetric tensor notation. Let $e_1, \dots, e_r$ be the basis of $\mathbb{R}^r$ where the x^s are coordinates, and let $f_1, \dots, f_m$ be the basis of $\mathbb{R}^m$ where the y^j are coordinates. Write (do not forget that the summation convention is in use!) $m_{1,x} = m_x^s e_s \in \mathbb{R}^r$, $m_{2,x} := m_x^{st} e_s e_t \in S^2\mathbb{R}^r$, etc., and similarly for y. Here the $e_{s_1}\cdots e_{s_q}$, with $1 \le s_1 \le \cdots \le s_q \le r$, form a basis of $S^q\mathbb{R}^r$.

With this notation, rewrite (12.1.2) as

$$\begin{aligned} m_{2,y} &= m_{2,v} + m_v(Am_{1,x}) + (A\cdot m_{2,x}), \\ m_{3,y} &= m_{3,v} + m_{2,v}(Am_{1,x}) + m_v(A\cdot m_{2,x}) + (A\cdot m_{3,x}), \end{aligned}$$

where

$$(A\cdot m_{p,x})^{s_1,\dots,s_p} = A_{s_1}^{i_1}\cdots A_{s_p}^{i_p} m_{p,x}^{s_1,\dots,s_p},$$

and if $u \in S^p\mathbb{R}^r$, $v \in S^q\mathbb{R}^r$, recall that the symmetric tensor product $uv \in S^{p+q}\mathbb{R}^r$ is

$$(uv)^{i_1,\dots,i_{p+q}} = \frac{1}{(p+q)!}\sum u^I v^J,$$

where the summation is over $I = i_1, \dots, i_p$ with $i_1 \leq \cdots \leq i_p$, $u^I = u^{i_1}\cdots u^{i_p}$, and analogously for J.

Ignoring statistics, one possible motivation for defining cumulants would be to have polynomials in the moments that transform well under affine linear transformations. There is a standard trick for doing this. First, form a generating function for the moments, which is a formal series in variables $\xi_1, \dots, \xi_{\mathbf{v}}$:

$$M(\xi) = E\{\exp(\xi_i x^i)\} = \sum_I \frac{1}{|I|!}\xi_I m^I,$$

that is, the moment m^I is the coefficient of ξ_I in the expansion. Assemble the to be defined cumulants into a formal series,

$$K(\xi) = \sum_I \frac{1}{|I|!}\xi_I \kappa^I.$$

To have the desired behavior under affine linear transformations, we define

$$K(\xi) := \log(M(\xi)) \tag{12.1.4}$$

because $\log(ab) = \log(a) + \log(b)$. Heree we follow [**232**, §2.2].

Cumulants satisfy the essential property

- *Independence*: If $x(t), u(t)$ are statistically independent, then $\kappa_p(x+u) = \kappa_p(x) + \kappa_p(u)$.

They also satisfy:

- *Affine linearity*: Let $f : \mathbb{R}^{\mathbf{v}} \to \mathbb{R}^{\mathbf{w}}$ be an affine linear map $x \mapsto Ax + u$, where $u \in \mathbb{R}^{\mathbf{v}}$ is a fixed vector; then $\kappa_p(Ax+u) = A\kappa_p(x)$ for $p > 1$, and
- *Annihilation of Gaussians*: $\kappa^p(v) = 0$ for all $p > 2$ if v is normally distributed.

Here a *normally distributed*, i.e., *Gaussian*, function is one of the form

$$g(t) = \frac{1}{\sqrt{2\pi\kappa_g}}e^{-\frac{(t-m_g)^2}{2\kappa_g^2}},$$

where $m_g = E\{g\}$ is the mean and $\kappa_g = m_{g^2} - (m_g)^2$ is the variance.

Exercise 12.1.2.1: Verify the above properties for cumulants.

Returning to the BSS problem, ignoring the small κ_v^{ijk}, we also have

$$\kappa_y^{ijk} = A_s^i A_t^j A_u^k \kappa_x^{stu} = A_s^i A_s^j A_s^k \kappa_x^{sss},$$

i.e.,

$$\kappa_{3,y} = A \cdot \kappa_{3,x} \in S^3\mathbb{R}^m,$$

and, if we need it,

$$\kappa_{4,y} = A \cdot \kappa_{4,x} \in S^4\mathbb{R}^m,$$

etc. Unlike decomposing a quadratic form, a symmetric tensor of order three (or higher) on $\mathbb{R}^m$, if it admits such a decomposition as a sum of $r < \frac{m^2+3m+2}{6}$ third powers, that decomposition is generally unique up to trivialities, i.e., reordering the factors, and, in our case, rescaling the individual factors (since we have the added functions x_s).

The higher order cumulants are of the form

$$\kappa_p(x) := m_p(x) + \sum_\lambda C_{p,\lambda} m_{\lambda_1}(x) \cdots m_{\lambda_r}(x),$$

where the sum is over partitions λ of p and arranged so that the sum transforms linearly. The $C_{p,\lambda}$ have been determined explicitly; see [**278**, Thm. 6.2]. The case $p = 3$ was given in (1.3.1).

12.2. Blind deconvolution of DS-CMDA signals

This section reports on the papers [**249**] and [**109**]. In wireless communication, one type of system is *direct-sequence code-division multiple access* (DS-CDMA) signals.

Signals (e.g., mobile phone) are sent out from R sources and received by I antennae. The signals arrive mixed, and the goal is to recover the individual signals, just as in the cocktail party problem discussed in §1.3.

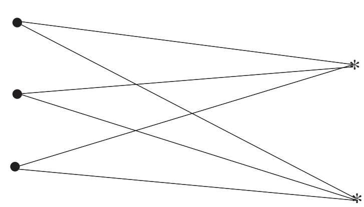


Figure 12.2.1. $R = 3$ users send signals to $I = 2$ antennae.

12.2.1. A first attempt. Consider a situation where there are R sources of the signals (*users* in the literature), and each transmits K symbols in a message, say the r-th user wants to send out $\vec{s_r} = (s_{1r}, \ldots, s_{Kr})$. (These symbols can be thought of, e.g., as complex numbers of some prescribed range of values.) There are I antennae receiving the signals.

Remark 12.2.1.1. There are typical "constellations" of values for the symbols s_{kr} that are used in telecommunication. With *Binary Phase Shift Keying* (BPSK), the values will be either $+1$ or -1. With *Quadrature amplitude modulation* QAM-4, they will be $\pm 1 \pm i$. Sometimes they are just known to have constant modulus (every user transmits complex values with fixed but unknown modulus and arbitrary phase).

By the time the signals arrive at the i-th antenna, there has been some distortion in intensity and phase of the signal (called *fading/gain*), so what is actually received at the i-th antennae are the IK quantities $a_{ir}s_{kr}$, where a_{ir} is the fading/gain between user r and antenna element i.

What is received in total in this scenario is a tensor $T \in \mathbb{C}^I \otimes \mathbb{C}^K$ of rank R with

$$T_{ik} = \sum_{r=1}^{R} a_{ir} s_{kr}.$$

The goal would be to recover the signals $\vec{s_r}$, $1 \leq r \leq R$, from this tensor (matrix) by decomposing it into a sum of rank one tensors (matrices). **The Problem**: As the reader knows well by now, such a decomposition is never unique when $R > 1$. **The fix**: Introduce "codes" to "spread" the signals that each user sends out to obtain a three-way tensor which will generally have a unique decomposition. Such spreading is common in telecommunications, e.g., redundancy is introduced to compensate for errors caused by signals being corrupted in transmission.

12.2.2. CP decomposition in the absence of ICI and ISI. Here we assume that transmission is line of sight—the signals travel directly from the source to the antennae.

The noiseless/memoryless data model for multiuser DS-CDMA is as follows: $\vec{s_r}$ and the a_{ir} are as above, but instead of the r-th user transmitting $\vec{s_r}$, it introduces redundancy by means of a code, called a *spreading sequence*, which is fixed for each user, say $\vec{c_r} = (c_{1r}, \dots, c_{J'r})$. (The choice of index labeling is to conform with the notation in [**109**]. The range J' will later be modified to J.) The symbol s_{kr} is transmitted at several frequencies; the result is simultaneous transmission of J' quantities $(s_{kr}c_{1r}, \dots, s_{kr}c_{J'r})$, called *chips*, at time k. The total transmission over $J'K$ units of time of the r-th user is the $J'K$ chips $c_{j'r}s_{kr}$, $1 \leq j' \leq J'$, $1 \leq k \leq K$. (J' is called the *spreading gain*.) The i-th antenna receives the KJ' quantities $\sum_r a_{ir}c_{j'r}s_{kr}$, $1 \leq j' \leq J'$, $1 \leq k \leq K$, where, as above, a_{ir} is the fading/gain between user r and antenna element i. What is received in total is a tensor $T \in \mathbb{C}^I \otimes \mathbb{C}^{J'} \otimes \mathbb{C}^K$ of rank R with

$$T_{ij'k} = \sum_{r=1}^{R} a_{ir} c_{j'r} s_{kr},$$

and the goal is to recover the signals $\vec{s_r}$, $1 \leq r \leq R$, from this tensor.

Remark 12.2.2.1. In the scenario described here, fixing k and r, the chips $s_{kr} c_{jr}$ are transmitted in succession. In other scenarios, they may be transmitted simultaneously, but at different frequencies.

At this point the reader knows that this decomposition can be done, at least approximately. In particular, the decomposition will be unique as long as the tensors are sufficiently general and r is sufficiently small. For how small, see §§12.3.4 and 12.5. In practice an alternating least squares (ALS) algorithm is often used; see, e.g., [**98**, p. 538].

In the articles [**249**] and [**109**], it is assumed that the spreading gain is either known or estimated. However, this is not strictly necessary in theory because of the uniqueness of tensor decomposition. Similarly, the articles assume the number of users R is known, but this is not strictly necessary either, as there are tests for the rank of a tensor (or the subspace rank in the case of [**109**]).

In this model, there are three types of diversity that are exploited: temporal diversity (K symbols are transmitted) spatial diversity (an array of I antennae is employed) and code diversity (every user has his own code vector $\vec{c_r}$ of length J').

Remark 12.2.2.2. Had there just been one antenna, there would be the same problem as above, namely a matrix to decompose instead of a 3-way tensor. Some CMDA systems function with just one antenna, in which case to enable communication the code vectors need to be communicated to the receiver.

Remark 12.2.2.3. In "real life" situations, there are several different possible scenarios. If the senders are trying to communicate with the receiver, then one can assume the c_{jr} are known quantities. In so-called "noncooperative" situations they must be determined. In noncooperative situations, one may need to determine R, perform the decomposition, and from that recover the $\vec{s_r}$. In cooperative situations R is known and one proceeds directly to the decomposition. The decomposition is aided further as the $\vec{c_r}$ are known as well.

Remark 12.2.2.4. In commercial situations often there are very few antennae, as they are expensive.

Interference. *Interchip interference* (ICI) occurs when the signal follows several paths to the antennae (e.g., bouncing off buildings or hills). It will occur unless there is line of sight propagation.

Intersignal interference (ISI) occurs when the time delay from different paths the signal takes causes a copy of, e.g., s_{kr} to arrive at the same time as $s_{k+1,r}$.

If at worst some $c_{j'r}s_{kr}$ is delayed by some paths so that it arrives simultaneously with some $c_{\tilde{j}'r}s_{k+L',r}$, one says "ISI occurs over at most L' symbols".

ICI over L' chips causes ISI over $\lceil \frac{L'}{J'} \rceil$ symbols, as explained below.

Remark 12.2.2.5. In the articles under discussion, there is a strong hypothesis made on the interference, namely that it occurs close to the users and far from the antennae, so the impulse response is the same for all the antennae.

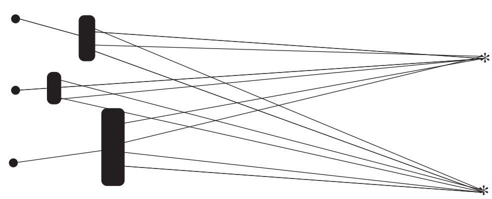


Figure 12.2.2. $R = 3$ users send signals to $I = 2$ antennae with interference.

If the chip $(1, 0, \ldots, 0)$ (called an *impulse*) is transmitted, the i-th antenna receives a signal

$$\vec{g} = (g_1, g_2, \ldots, g_{L'})$$

(called the *impulse response*) because the signal travels different paths to arrive at the antenna, so differently scaled versions of the impulse arrive at different moments in time. Here the impulse response is assumed to be finite, i.e., the signal dies out eventually.

For this paragraph, suppress the r-index in two-index subscripts: say the r-th user sends out $\vec{s}_r = (s_1, \ldots, s_K)$ spread by $\vec{c}_r$, i.e., it sends the vector

$$(s_1c_1, s_1c_2, \ldots, s_1c_{J'}, s_2c_1, s_2c_2, \ldots, s_Kc_{J'}).$$

What is received is a vector of length $KJ' + L'$:
(12.2.1)

$$\begin{aligned}
&(s_1c_1g_1, s_1(c_1g_2 + c_2g_1), s_1(c_1g_3 + s_1c_2g_2 + c_3g_1), \dots, s_1c_{J'}g_{L'}, 0, \dots, 0) \\
&\quad + (0, \dots, 0, s_2c_1g_1, s_2(c_1g_2 + s_1c_2g_1), \\
&\qquad s_2(c_1g_3 + s_1c_2g_2 + c_3g_1), \dots, s_2c_{J'}g_{L'}, 0, \dots, 0) \\
&\quad \vdots \\
&\quad + (0, \dots, 0, s_Kc_1g_1, s_K(c_1g_2 + s_1c_2g_1), \\
&\qquad s_K(c_1g_3 + s_1c_2g_2 + c_3g_1), \dots, s_Kc_{J'}g_{L'}),
\end{aligned}$$

where in the second term in the summation there are J' initial zeros, the third has $2J'$ initial zeros, and in the last, there are KJ' initial zeros.

Note that if $L' \geq J'$, the *symbols* will interfere with each other; that is, a chip containing s_1 will arrive at the same time as a chip containing s_2. In the next subsection, a way to avoid this problem by adding zeros is described, as proposed in [**249**]. In §12.2.4, a different way to avoid this problem is proposed that does not require adding zeros.

12.2.3. CP decomposition with ICI but no ISI. Now assume there is ICI over at most L' chips. To eliminate ICI, the spreading codes $c_{j'r}$ may be modified by, e.g., adding L' zeros at the end, so one has a vector of length $J := L' + J'$.

Note that $(c_1, \dots, c_{J'}) = c_1(1, 0, 0, \dots, 0) + c_2(0, 1, 0, \dots, 0) + \cdots$, so the *effective code* taking into account the impulse response is

$$\begin{aligned}
\vec{h}_r :&= c_1(\vec{g}, 0, 0, \dots, 0) + c_2(0, \vec{g}, 0, \dots, 0) + \cdots \\
&= \sum_{l=1}^{L'} g_{l,r}c_{j-l+1,r}, \quad 1 \leq j \leq J = L' + J'.
\end{aligned}$$

Here c_{ur} is taken to be zero when $u \leq 0$ or $u > J'$. The i-th antenna receives

$$T_{ijk} := \sum_{r=1}^{R} a_{ir}h_{jr}s_{kr},$$

and the total information received is a tensor $T \in \mathbb{C}^I \otimes \mathbb{C}^J \otimes \mathbb{C}^K$.

One then may recover the vectors $\vec{s}_r$ up to the above-mentioned ambiguity as discussed above.

12.2.4. Block term decomposition in the presence of ICI and ISI. In the above technique, it was necessary to add L' trailing zeros to the code. As long as L' is small and the situation is cooperative, this is not a problem, but if L' is large, it can be inefficient. Moreover, in noncooperative situations, users might not be inclined to modify their codes for the sake of Big Brother. What follows is a method to recover the symbols s_{kr} without the codes having these added zeros, called *block term decomposition* (or BTD for short). It is thus more efficient, and applicable to situations where the receivers have no control over the codes.

Continuing the above notation, but *without assuming that L' trailing zeros have been added to the code*, the signal received by the i-th antenna is more mixed: To conform with the notation of [**109**], let each user emit $\tilde{K}$ symbols. Also, in this subsection $J' = J$ (because no trailing zeros are added).

Thus r vectors (12.2.1) are transmitted (with K replaced by $\tilde{K}$) and the goal is to recover the $s_{\tilde{k}r}$ from what is received at the antennae. First observe that a CP decomposition is not useful—the tensor received will in general have rank greater than r.

To illustrate, let $R = L' = I = J = 2$ and $\tilde{K} = 3$, so $T \in \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^3$. Let $\vec{h}_r \in \mathbb{C}^{L'+J-1}$ be the convolution of $\vec{c}_r$ and $\vec{g}_r$ as before. (In this example, block term decomposition will not be unique.) Write

$$T = \vec{a}_1 \otimes \begin{pmatrix} h_{11}s_{11} & h_{11}s_{21} + h_{31}s_{11} & h_{11}s_{31} + h_{31}s_{21} & h_{31}s_{31} \\ h_{21}s_{11} & h_{21}s_{21} & h_{21}s_{31} & 0 \end{pmatrix}$$
$$+ \vec{a}_2 \otimes \begin{pmatrix} h_{12}s_{12} & h_{12}s_{22} + h_{32}s_{12} & h_{12}s_{32} + h_{32}s_{22} & h_{32}s_{32} \\ h_{22}s_{12} & h_{22}s_{22} & h_{22}s_{32} & 0 \end{pmatrix},$$

where there are no relations among the constants $h_{\alpha,r}, s_{\tilde{k},r}$. Although the two matrices are received as vectors, since $\tilde{K}$ is known, one may write them as matrices. Such a tensor will generally have rank three. Furthermore, were one to perform the CP decomposition, the $\vec{s}_r$'s would be mixed together. (The index convention here for the $\vec{h}_r$ is different from that in [**109**].)

In (12.2.1) the quantities are arranged in a vector of length 6; here they are put into 3×2-matrices, where the order sends

$$(1,2,3,4,5,6,7,8) \to \begin{pmatrix} 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 \end{pmatrix}.$$

This is justified as long as we know the integer J.

We may write the matrices as products of matrices H_r, S_r, where

$$H_r = \begin{pmatrix} h_{1r} & h_{3r} \\ h_{2r} & 0 \end{pmatrix}, \quad S_r = \begin{pmatrix} s_{1r} & s_{2r} & s_{3r} & 0 \\ 0 & s_{1r} & s_{2r} & s_{3r} \end{pmatrix}.$$

In general, the tensor $T \in \mathbb{C}^I \otimes \mathbb{C}^J \otimes \mathbb{C}^K$ will admit a *block term decomposition* into a sum of r elements of $\hat{S}ub_{1,L,L}(\mathbb{C}^I \otimes \mathbb{C}^J \otimes \mathbb{C}^K)$, where we know the $J \times K$ matrix (x^r_{jk}) has rank at most L because the matrix S_r does.

One uses a modified ALS algorithm to obtain the r matrices (x^r_{jk}). Once they are obtained, because one has *a priori* knowledge of the factorization into H_rS_r where both H_r, S_r have special form, one can finally recover the vectors $\vec{s}_r$.

Set $K = \tilde{K} + L - 1$ and $L = J + L' - 1$. We obtain matrices

$$S_r := \begin{pmatrix} s_{12} & s_{2r} & \cdots & s_{\tilde{K}r} & 0 & \cdots & 0 \\ 0 & s_{12} & s_{2r} & \cdots & s_{\tilde{K}r} & 0 & \cdots \\ & & \vdots & & & & \\ 0 & \cdots & 0 & s_{12} & s_{2r} & \cdots & s_{\tilde{K}r} \end{pmatrix} \in \mathbb{C}^L {\otimes} \mathbb{C}^K,$$

$$H_r := \begin{pmatrix} h_{1r} & h_{J+1,r} & h_{2J+1,r} & \cdots & & & h_{J+L'-2,r} & h_{J+L'-1,r} \\ h_{2r} & h_{J+2,r} & h_{2J+2,r} & \cdots & & & h_{J+L'-1,r} & 0 \\ \vdots & & & & \iddots & \iddots & \vdots & \vdots \\ h_{Jr} & h_{2J,r} & h_{3J,r} & \cdots & h_{J+L'-1,r} & 0 & \cdots & 0 \end{pmatrix}$$
$$\in \mathbb{C}^J {\otimes} \mathbb{C}^L,$$

$$x^r = H_rS_r \in \mathbb{C}^J {\otimes} \mathbb{C}^K,$$

and the tensor $T \in \mathbb{C}^I {\otimes} \mathbb{C}^J {\otimes} \mathbb{C}^K$ with entries

$$T_{ijk} = \sum_{r=1}^{R} a_{ir} {\otimes} x^r_{jk}.$$

Exercise 12.2.4.1: Given a 2×4 matrix x^r as above, explicitly solve for the s_{jr} in terms of the entries of x^r. ⊚

In general H will be rectangular, which is not a problem if it has a left inverse. In practice one takes the unknown matrix X to be the Moore-Penrose inverse of H and uses approximation methods (least squares) to recover $\vec{s}$, exploiting the Toeplitz structure of S.

12.3. Uniqueness results coming from algebraic geometry

12.3.1. Nonuniqueness of expressions for general polynomials and tensors. By the Alexander-Hirschowitz theorem (3.2.2.4 or 5.4.1.1), a general form of degree d in $\mathbf{v}$ variables has at most a finite number of presentations as a sum of s powers of linear forms in all the cases not on their list when $\frac{\binom{\mathbf{v}+d-1}{d}}{\mathbf{v}}$ is an integer. For all other cases of general polynomials, there are parameters worth of expressions of the polynomial. However, unless a secant variety is defective, for any $[p] \in \sigma_r(v_d(\mathbb{P}^{\mathbf{v}-1}))_{general}$ when r is less then the filling dimension, there is at most a finite number of presentations as a sum of r d-th powers of linear forms.

Remark 12.3.1.1. When there are parameters worth of presentations, the number of parameters the presentation depends on is

$$r\mathbf{v} - 1 - \min\left\{\binom{\mathbf{v}+d-1}{d} - 1, \dim \sigma_r(v_d(\mathbb{P}^{\mathbf{v}-1}))\right\}.$$

Parameter spaces of algebraic varieties are often themselves algebraic varieties, and one can ask the more ambitious question of describing the parameter spaces. Work on this was done by Mukai, e.g., [**236**], and continued by Iliev and Ranestad [**179, 177, 178**], and Ranestad and Schreyer [**273**] (this last reference is a survey of what was known up to that time).

12.3.2. Nonuniqueness for points on a tangent line to $v_d(\mathbb{P}^1)$. Let $\phi \in S^d\mathbb{C}^2$ be such that $\mathbf{R}_S(\phi) = d$. Then $\phi = x^{d-1}y$ for some $x, y \in \mathbb{C}^2$. In other words, ϕ lies on a tangent line to $v_d(\mathbb{P}^1)$ (see §9.2.2). Then any hyperplane containing ϕ that is not tangent to $v_d(\mathbb{P}^1)$ will be spanned by exactly d points of $v_d(\mathbb{P}^1)$, so there is a $(d-1)$-parameter family of expressions of ϕ as a sum of d d-th powers (parametrized by an open subset of $G(d, S^d\mathbb{C}^2) = \mathbb{P}^{d-1}$).

12.3.3. Unique up to finite decompositions. I remind the reader of the following result mentioned in Chapter 3:

Proposition 12.3.3.1. *Assume that the rank of a tensor (resp. symmetric rank of a symmetric tensor $T \in S^dV$) is r, which is less than the generic rank (see Theorem 3.1.4.3 for a list of generic ranks).*

(1) *If T is symmetric, then with probability one there are finitely many decompositions of T into a sum of r (symmetric) rank one tensors unless $d = 2$, where there are always infinitely many decompositions when $r > 1$.*
(2) *If $T \in \mathbb{C}^{\mathbf{v}}\otimes\mathbb{C}^{\mathbf{v}}\otimes\mathbb{C}^{\mathbf{v}}$, then as long as $\mathbf{v} \neq 3$, with probability one there are a finite number of decompositions into a sum of r rank one tensors.*

12.3.4. Uniqueness of decompositions with probability one: NWD.

Definition 12.3.4.1. A projective variety $X \subset \mathbb{P}V$ is called k-NWD, *non-weakly k-defective*, if a general hyperplane tangent to X at k points is tangent to X only at a finite number of points, and otherwise it is *weakly k-defective.*

By Terracini's lemma in §5.3, if $\sigma_k(X)$ is defective, then X is weakly k-defective, but the converse is not necessarily true.

Theorem 12.3.4.2 ([**87**])**.** *If $X \subset \mathbb{P}V$ is r-NWD and r is less than the generic rank, then for general $[v] \in \sigma_r(X)$, there exists a unique expression*

$$v = \sum_{i=1}^{r} w_i$$

with $[w_i] \in X$.

See [**87**] for the proof, which is based on a variant of Terracini's lemma in §5.3.

The Veronese varieties which are weakly defective have been classified.

Theorem 12.3.4.3 (Chiantini-Ciliberto-Mella-Ballico [**86, 233, 14**])**.** *The weakly k-defective varieties $v_d(\mathbb{P}^n)$ are the triples (k,d,n):*

(i) the k-defective varieties, namely $(k,2,n)$, $k = 2,\dots,\binom{n+2}{2}$, $(5,4,2)$, $(9,4,3)$, $(14,4,4)$, $(7,3,4)$,

and

(ii) $(9,6,2)$, $(8,4,3)$.

Thus for any other (d,k,n) with k less than generic, a general point of $\sigma_k(v_d(\mathbb{P}^n))$ has a unique decomposition as a sum of k d-th powers.

Regarding tensors, there is the following result.

Theorem 12.3.4.4 ([**88**])**.** *Let $\mathbf{a} \leq \mathbf{b} \leq \mathbf{c}$. Then $Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)$ is r-NWD for $r \leq \frac{\mathbf{ab}}{16}$. More precisely, if $2^\alpha \leq \mathbf{a} < 2^{\alpha+1}$ and $2^\beta \leq \mathbf{b} < 2^{\beta+1}$, then $Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)$ is $2^{\alpha+\beta-2}$-NWD.*

In particular, let $T \in A \otimes B \otimes C$. If $\underline{\mathbf{R}}(T) \leq \frac{\mathbf{ab}}{16}$, then with probability one, T has a unique CP decomposition into a sum of $\underline{\mathbf{R}}(T)$ rank one terms. More precisely, say $2^\alpha \leq \mathbf{a} < 2^{\alpha+1}$ and $2^\beta \leq \mathbf{b} < 2^{\beta+1}$; then if $\underline{\mathbf{R}}(T) \leq 2^{\alpha+\beta-2}$, then with probability one, T has a unique CP decomposition into a sum of $\underline{\mathbf{R}}(T)$ rank one terms.

Compare this with Kruskal's theorem which, e.g., if $\mathbf{a} = \mathbf{b} = \mathbf{c}$, only assures a unique CP decomposition up to roughly $\frac{3\mathbf{a}}{2}$, compared with the $\frac{\mathbf{a}^2}{16}$ above. Also note that if $\mathbf{a} = \mathbf{b} = \mathbf{c} > 3$, then one always has $\underline{\mathbf{R}}(T) \leq \lceil \frac{\mathbf{a}^3-1}{3\mathbf{a}-2} \rceil$, and if $\mathbf{R}(T) \leq \lceil \frac{\mathbf{a}^3-1}{3\mathbf{a}-2} \rceil - 1$, then with probability one T has at most a finite number of decompositions into a sum of $\underline{\mathbf{R}}(T)$ rank one tensors.

While the property of being r-NWD is hard to check in practice (one needs to write down a general hyperplane tangent to points), the following *a priori* stronger property is easier to verify:

Definition 12.3.4.5. Let $X \subset \mathbb{P}V$ be an irreducible variety and let $p_1,\dots,p_r$ be r general points on X. Then X is *not tangentially weakly r-defective*, or

r-NTWD for short, if for any $x \in X$ such that $\hat{T}_x X \subset \langle \hat{T}_{p_1} X, \dots, \hat{T}_{p_r} X \rangle$, one has $x = p_i$ for some i.

Then essentially by definition:

Proposition 12.3.4.6 ([**88, 87**]). *r-TNWD implies r-NWD.*

12.3.5. Symmetric tensors of low rank have unique decompositions. The behavior of symmetric tensors of low rank is markedly different from general tensors of low rank. For a general tensor in $A_1 \otimes \cdots \otimes A_n$ of rank say n, its border rank can be anywhere between 2 and n. This contrasts the following result:

Theorem 12.3.5.1 ([**47**]). *Let $p \in S^d V$. If $\mathbf{R}_S(p) \le \frac{d+1}{2}$, i.e., the symmetric tensor rank of p is at most $\frac{d+1}{2}$, then $\mathbf{R}_S(p) = \underline{\mathbf{R}}_S(p)$ and the expression of p as a sum of $\mathbf{R}_S(p)$ d-th powers is unique (up to trivialities).*

12.4. Exact decomposition algorithms

In this section I present several exact decomposition algorithms for tensors and symmetric tensors. There is a general principle that in any situation where one has equations for secant varieties, there are exact methods to help obtain an exact decomposition. I discuss a few such just to give a flavor of the subject.

Recall that an improved version of Sylvester's algorithm was already presented in §3.5.3.

In this section I generally follow the presentation in [**254**].

12.4.1. Catalecticant algorithm for recovering the Waring decomposition of $P \in S^d V$. We can imitate the Sylvester algorithm in higher dimensions as follows: Let $P \in S^d V$. Consider the polarization $P_{\delta,d-\delta} : S^\delta V^* \to S^{d-\delta} V$. Say $P = x_1^d + \cdots + x_r^d$ for some $x_j \in V$. Then, $P_{\delta,d-\delta} = x_1^\delta \otimes x_1^{d-\delta} + \cdots + x_r^\delta \otimes x_r^{d-\delta}$, so for each $\overline{Q} \in \ker P_{\delta,d-\delta}$, $\overline{Q}(x_j^\delta) = 0$, i.e., $Q(x_j) = 0$ thinking of Q as a homogeneous polynomial of degree δ. (Here I use an overline when considering Q as a multiliner form and remove the overline when considering it as a homogeneous polynomial.)

Thus if $\bigcap_{Q \in \ker P_{\delta,d-\delta}} \mathrm{Zeros}(Q) \subset \mathbb{P}V$ is a finite collection of points, we have a list of candidates for terms appearing in the Waring decomposition of P. Assuming one can compute $\bigcap_{Q \in \ker P_{\delta,d-\delta}} \mathrm{Zeros}(Q) \subset \mathbb{P}V$, one can then find the true decomposition by linear algebra.

Catalecticant algorithm.

(1) Take $\delta = \lceil \frac{d}{2} \rceil$ and compute $\ker P_{\delta,d-\delta}$. Note that $\underline{\mathbf{R}}_S(P) \ge \mathrm{rank} P_{\delta,d-\delta}$.

(2) Compute $Z_{P,\delta} := \bigcap_{Q\in\ker P_{\delta,d-\delta}} \mathrm{Zeros}(Q) \subset \mathbb{P}V$. If $Z_{P,\delta}$ is not a finite set, stop—the algorithm will not work. Otherwise, write $Z_{P,\delta} = \{[x_1], \dots, [x_s]\}$ with $x_j \in V$ and continue.

(3) Solve the $\binom{\mathbf{v}+\delta-1}{\delta}$ linear equations $\sum c_j x_j^d = P$ for the s unknowns c_j. Then $P = \sum_j c_j x_j^d$. Note that if $\mathbf{R}_S(P) < s$, some of the terms will not appear.

Exercise 12.4.1.1: Decompose $81v_1^4+17v_2^4+626v_3^4-144v_1v_2^2v_3+216v_1^3v_2-108v_1^3v_3+216v_1^2v_2^2+54v_1^2v_3^2+96v_1v_2^3-12v_1v_3^3-52v_2^3v_3+174v_2^2v_3^2-508v_2v_3^3+72v_1v_2v_3^2-216v_1^2v_2v_3$ into a sum of fourth powers. ⊚

When will this algorithm work? Here is a partial answer:

Theorem 12.4.1.2 ([**254**], slightly improving [**172**])**.** *Let $P \in S^dV$ be a general element of $\sigma_r(v_d(\mathbb{P}V))$, i.e., $\mathbf{R}_S(P) = r$ and P is general among symmetric tensors of rank r.*

If $d = 2\delta$ is even and $\mathbf{R}_S(P) \le \binom{\mathbf{v}-1+\delta}{\delta} - \mathbf{v}$ or if $d = 2\delta+1$ is odd and $\mathbf{R}_S(P) \le \binom{\mathbf{v}-2+\delta}{\delta-1}$, then $Z_{P,\delta} := \bigcap_{Q\in\ker P_{\delta,d-\delta}} \mathrm{Zeros}(Q) \subset \mathbb{P}V$ consists of r points and the catalecticant algorithm produces the unique Waring decomposition of P.

If $d = 2\delta$ is even and $\mathbf{R}_S(P) \le \binom{\mathbf{v}-1+\delta}{\delta} - \mathbf{v} + 1$, the algorithm will still work but $Z_{P,\delta}$ might consist of more than $\mathbf{R}_S(P)$ points when $\mathbf{v} > 3$.

Example 12.4.1.3 (A modified algorithm for solving a general cubic equation)**.** The standard algorithm for solving a cubic equation $f(x) = ax^3 + bx^2 + cx + d$ is probably Cardano's. Another algorithm, which uses the discrete Fourier transform is due to Lagrange. Here is yet another, based on tensor decomposition. Work with the homogenized polynomial $P(x, y) = ax^3 + bx^2y + cxy^2 + dy^3$. Apply the catalecticant algorithm to obtain $P = x_1^3 + x_2^3$, where x_1, x_2 are linear combinations of x, y. Set $\omega = e^{\frac{2\pi i}{3}}$. Then the roots are the roots of the system of linear equations $x_1 = -\omega^a x_2$ with $a = 0, 1, 2$.

12.4.2. Young flattening algorithm for recovering the Waring decomposition of $P \in S^dV$. Let $P \in S^dV$. Recall the Young flattening map from (3.10.1):

$$YF_{d,\mathbf{v}}(P) : S^\delta V^* \otimes \Lambda^a V \to S^\delta V \otimes \Lambda^{a+1} V \tag{12.4.1}$$

whose minors of size $\binom{\mathbf{v}-1}{\lceil \frac{\mathbf{v}}{2} \rceil} r$ give equations for $\sigma_r(v_d(\mathbb{P}V))$. A recipe for writing down a matrix representing $YF_{d,\mathbf{v}}(P)$ in bases was given in §3.10.2. This particular Young flattening is often called a *Koszul flattening.*

Consider $M \in \ker(YF_{d,\mathbf{v}}(P))$ as a map $S^\delta V \to \Lambda^a V$ and restrict this map to the rank one elements. Following [**254**], we say that $[v] \in \mathbb{P}V$ is an *eigenline* of M if $M(v^\delta)\wedge v=0$. This usage coincides with the classical definition in linear algebra when $\delta=a=1$ and has become standard whenever $a=1$.

Note that when $P=x_1^d+\cdots+x_r^d$, each $[x_j]$ is an eigenline for M. Thus if there are a finite number of eigenlines, one can recover the Waring decomposition of P if one can compute the eigenlines.

Theorem 12.4.2.1 ([**254**, Thm. 3.4], generalizing [**73**] (case $a=1$))**.** *For a general $M : S^\delta V\to\Lambda^a V$, the number of eigenlines for M is*

(1) *Infinite when $a=0$ or $\mathbf{v}$ and $\mathbf{v}>2$.*
(2) *Zero for $2\le a\le \mathbf{v}-3$.*
(3) *δ for $a=0,2$ and $\mathbf{v}=2$.*
(4) *$\frac{\delta^{\mathbf{v}}-1}{\delta-1}$ for $a=1$.*
(5) *$\frac{(\delta+1)^{\mathbf{v}}+(-1)^{\mathbf{v}-1}}{\delta+2}$ for $a=\mathbf{v}-2$.*

For the proof, see [**254**].

Example 12.4.2.2 (Decomposing a general quintic in three variables)**.** Let $V=\mathbb{C}^3$ and let $P\in S^5V$. Consider $YF_{2,\mathbf{v}}(P) : S^2V{\otimes}V^*\to V^*{\otimes}S^2V$. Applied to a rank one element v^5, $YF_{2,\mathbf{v}}(v^5)(M)$ is the map

$$w\mapsto \Omega(M(v^2)\wedge v\wedge w)v^2,$$

where $\Omega\in\Lambda^3V^*$ is a fixed volume form. By Theorem 12.4.2.1, if P is general, a general element M in the kernel of the linear map $YF_{2,\mathbf{v}}(P)$ will have seven eigenlines $\{[x_1],\dots,[x_7]\}$, and the decomposition of P may be recovered by solving the system of linear equations $P=\sum_{j=1}^7 c_jx_j^5$ for the coefficients c_j.

Example 12.4.2.3 (Decomposing a general cubic in four variables: the Sylvester Pentahedral Theorem)**.** Let $V=\mathbb{C}^4$ and let $P\in S^3V$. Then $YF_{2,\mathbf{v}}(P) : V{\otimes}\Lambda^2V^*\to V^*{\otimes}V$ and by Theorem 12.4.2.1, if P is general, a general element M in the kernel of the linear map $YF_{2,\mathbf{v}}(P)$ will have five eigenlines $\{[x_1],\dots,[x_5]\}$, and the decomposition of P may be recovered by solving the system of linear equations $P=\sum_{j=1}^5 c_jx_j^4$ for the coefficients c_j.

When d is even, the catalecticant algorithm works better than the Young flattening algorithm, so, it is natural to ask when the Young flattening algorithm will work for odd d. Here is a partial answer:

Theorem 12.4.2.4 ([**254**])**.** *Assume $\mathbf{v}\ge 4$ and write $d=2\delta+1$. Let $P\in S^dV$ be a general element of symmetric rank r. Write $Z_{P,YF}$ for the set of common eigenlines for* $\ker(YF_{\delta,\mathbf{v}}(P))$.

(1) *If* $\mathbf{v}$ *is odd and* $r \le \binom{\delta+\mathbf{v}-1}{\delta}$, *then* $Z_{P,YF}$ *consists of* r *points and the Young flattening algorithm gives the unique Waring decomposition of* P.

(2) *If* $\mathbf{v}$ *is even and* $r \le \binom{\delta+\mathbf{v}-1}{\delta}$, *take the intersection of the eigenlines of* $\ker(YF_{\delta,\mathbf{v}}(P))$ *and* $\mathrm{image}(YF_{\delta,\mathbf{v}}(P))^{\perp}$. *Then using these eigenlines, the Young flattening algorithm produces the unique Waring decomposition of* P.

There is also a slight extension of the theorem in the case $\mathbf{v} = 4$; see [**254**].

12.4.3. Exact decomposition algorithms for triple tensor products. Assume for simplicity that $\mathbf{a} \le \mathbf{b} \le \mathbf{c}$. Let $T \in A{\otimes}B{\otimes}C$ have rank r which is less than maximal, and let T be generic among tensors of its rank. The only cases where T is known to fail to have a unique decomposition as a sum of r rank one tensors are: the unbalanced cases $\mathbf{c} \ge (\mathbf{a}-1)(\mathbf{b}-1)+2$ and $r = (\mathbf{a}-1)(\mathbf{b}-1)+1$, and the isolated cases of $r = 6$ in $\mathbb{C}^4{\otimes}\mathbb{C}^4{\otimes}\mathbb{C}^4$ (where there are exactly two decompositions) and $r = 8$ in $\mathbb{C}^3{\otimes}\mathbb{C}^6{\otimes}\mathbb{C}^6$ (where it is not known how many decompositions exist).

Theorem 12.4.3.1 ([**88**])**.** *The three cases above are the only cases that fail to have a unique decomposition when* $\mathbf{c} \le 8$.

Proposition 12.4.3.2. *A general element of* $\mathbb{C}^2{\otimes}\mathbb{C}^{\mathbf{b}}{\otimes}\mathbb{C}^{\mathbf{b}}$ *has a unique decomposition as a sum of* $\mathbf{b}$ *rank one elements. An algorithm for obtaining the decomposition is given in the proof.*

Proof. Let $T \in A{\otimes}B{\otimes}C = \mathbb{C}^2{\otimes}\mathbb{C}^{\mathbf{b}}{\otimes}\mathbb{C}^{\mathbf{b}}$ and consider $\mathbb{P}T(B^*) \subset \mathbb{P}(A{\otimes}C)$. It will generally have dimension $\mathbf{b}-1$ and so will intersect the Segre variety $Seg(\mathbb{P}A \times \mathbb{P}C) = Seg(\mathbb{P}^1 \times \mathbb{P}^{\mathbf{v}-1})$ in $\deg(Seg(\mathbb{P}^1 \times \mathbb{P}^{\mathbf{v}-1})) = \mathbf{v}$ points, say $[a_1{\otimes}c_1], \dots, [a_{\mathbf{v}}{\otimes}c_{\mathbf{v}}]$. Then $T = a_1{\otimes}b_1{\otimes}c_1 + \cdots + a_{\mathbf{v}}{\otimes}b_{\mathbf{v}}{\otimes}c_{\mathbf{v}}$ for some unique $b_1, \dots, b_{\mathbf{v}} \in B$. (One can use T to solve for these points or use $T(C^*) \cap Seg(\mathbb{P}A \times \mathbb{P}B)$.) $\square$

Exercise 12.4.3.3: Show that a general element of $\mathbb{C}^3{\otimes}\mathbb{C}^3{\otimes}\mathbb{C}^5$ has exactly six decompositions into a sum of five rank one tensors.

12.5. Kruskal's theorem and its proof

Consider a tensor $T \in A{\otimes}B{\otimes}C$. Say we have an expression

$$T = u_1{\otimes}v_1{\otimes}w_1 + \cdots + u_r{\otimes}v_r{\otimes}w_r \tag{12.5.1}$$

and we want to know if the expression is unique.

Remark 12.5.0.4 (Remark on terminology)**.** In the tensor literature one often says that an expression as above is *essentially unique* rather than unique because one can rescale u_i, v_i, w_i by numbers that multiply to be one, and permute the order in the summation. However, I use the word unique, because there is a unique set of points on the Segre that express T.

12.5.1. An obvious necessary condition for uniqueness. Recall that for the tensor product of two vector spaces, an expression as a sum of r elements is never unique unless $r = 1$. Thus an obvious necessary condition for uniqueness is that we cannot be reduced to a two-factor situation. For example, an expression of the form

$$T = a_1{\otimes}b_1{\otimes}c_1 + a_1{\otimes}b_2{\otimes}c_2 + a_3{\otimes}b_3{\otimes}c_3 + \cdots + a_r{\otimes}b_r{\otimes}c_r,$$

where each of the sets $\{a_i\}, \{b_j\}, \{c_k\}$ is linearly independent, is not unique because of the first two terms. In other words, if we consider for (12.5.1) the sets $\mathcal{S}_A = \{[u_i]\} \subset \mathbb{P}A$, $\mathcal{S}_B = \{[v_i]\} \subset \mathbb{P}B$, and $\mathcal{S}_C = \{[w_i]\} \subset \mathbb{P}C$, then uniqueness implies that each of them consists of r distinct points.

12.5.2. Kruskal rank: general linear position.

Definition 12.5.2.1. Let $\mathcal{S} = \{x_1, \ldots, x_p\} \subset \mathbb{P}W$ be a set of points. We say that the points of $\mathcal{S}$ are in 2-general linear position if no two points coincide; they are in 3-general linear position if no three lie on a line, and more generally, they are in r*-general linear position* if no r of them lie in a $\mathbb{P}^{r-2}$. The *Kruskal rank* of $\mathcal{S}$, $k_{\mathcal{S}}$, is defined to be the maximum number r such that the points of $\mathcal{S}$ are in r-general linear position.

Remark 12.5.2.2 (Remark on the tensor literature)**.** If one chooses a basis for W so that the points of $\mathcal{S}$ can be written as columns of a matrix (well defined up to rescaling columns), then $k_{\mathcal{S}}$ will be the maximum number r such that all subsets of r column vectors of the corresponding matrix are linearly independent. This was Kruskal's original definition.

12.5.3. Statement and proof of the theorem.

Theorem 12.5.3.1 (Kruskal [**195**])**.** *Let $T \in A{\otimes}B{\otimes}C$. Say T admits an expression $T = \sum_{i=1}^{r} u_i{\otimes}v_i{\otimes}w_i$. Let $\mathcal{S}_A = \{[u_i]\}$, $\mathcal{S}_B = \{[v_i]\}$, $\mathcal{S}_C = \{[w_i]\}$. If*

$$r \leq \frac{1}{2}(k_{\mathcal{S}_A} + k_{\mathcal{S}_B} + k_{\mathcal{S}_C}) - 1, \tag{12.5.2}$$

then T has rank r and its expression as a rank r tensor is essentially unique.

More generally, the following result holds.

Theorem 12.5.3.2 ([**290**])**.** *Let $T \in A_1{\otimes} \cdots \otimes A_n$ and say T admits an expression $T = \sum_{i=1}^{r} u_i^1{\otimes} \cdots \otimes u_i^n$. Let $\mathcal{S}_{A_k} = \{[u_i^k]\}$. If*

$$\sum_{k=1}^{n} k_{\mathcal{S}_{A_k}} \geq 2r + n - 1, \tag{12.5.3}$$

then T has rank r and its expression as a rank r tensor is unique.

Remark 12.5.3.3. In the special case $r \leq \dim A_i$ for some of the A_i, less restrictive hypotheses are possible. See [**298**] for the latest in this direction and [**297**] for a summary of previous work.

Above, we saw that a necessary condition for uniqueness is that $k_{\mathcal{S}_A}, k_{\mathcal{S}_B}, k_{\mathcal{S}_C} \geq 2$.

Exercise 12.5.3.4: Show that if (12.5.2) holds, then $k_{\mathcal{S}_A}, k_{\mathcal{S}_B}, k_{\mathcal{S}_C} \geq 2$. ⊚

If $\mathbf{a} = \mathbf{b} = \mathbf{c}$ and T has multilinear rank $(\mathbf{a}, \mathbf{a}, \mathbf{a})$, i.e., T does not live in any subspace variety, then it is easy to see that such an expression is unique when $r = \mathbf{a}$. Kruskal's theorem extends uniqueness to $\mathbf{a} \leq r \leq \frac{3}{2}\mathbf{a} - 1$ under additional hypotheses.

The key to the proof of Kruskal's theorem is the following lemma:

Lemma 12.5.3.5 (Permutation lemma)**.** *Let $\mathcal{S} = \{p_1, \dots, p_r\}$ and $\tilde{\mathcal{S}} = \{q_1, \dots, q_r\}$ be sets of points in $\mathbb{P}W$ and assume that no two points of $\mathcal{S}$ coincide (i.e., that $k_{\mathcal{S}} \geq 2$) and that $\langle \tilde{\mathcal{S}} \rangle = W$. If all hyperplanes $H \subset \mathbb{P}W$ that have the property that they contain at least $\dim(H) + 1$ points of $\tilde{\mathcal{S}}$ also have the property that $\#(\mathcal{S} \cap H) \geq \#(\tilde{\mathcal{S}} \cap H)$, then $\mathcal{S} = \tilde{\mathcal{S}}$.*

Remark 12.5.3.6 (Remark on the tensor literature)**.** If one chooses a basis for W and writes the two sets of points as matrices $M, \tilde{M}$, then the hypothesis can be rephrased (in fact this was the original phrasing) as to say that all $x \in \mathbb{C}^{\mathbf{w}}$ such that the number of nonzero elements of the vector ${}^t\tilde{M}x$ is less than $r - \operatorname{rank}(\tilde{M}) + 1$ also have the property that the number of nonzero elements of the vector ${}^t\tilde{M}x$ is at most the number of nonzero elements of the vector ${}^tMx$. To see the correspondence, the vector x should be thought of as a point of W^* giving an equation of H, and zero elements of the vector ${}^t\tilde{M}x$ correspond to columns that pair with x to be zero, i.e., that satisfy an equation of H, i.e., points that are contained in H.

Remark 12.5.3.7. There is a slight discrepancy in hypotheses from the original here: I have assumed that $\langle \tilde{\mathcal{S}} \rangle = W$, so $\operatorname{rank}(\tilde{M}) = \mathbf{w}$. This is all that is needed by Proposition 3.1.3.1. Had I not assumed this, there would be trivial cases to eliminate at each step of the proof.

12.5.4. Proof of Kruskal's theorem.

Proof of the permutation lemma. First note that if one replaces "hyperplane" by "point", then the lemma follows immediately as the points of $\mathcal{S}$ are distinct. The proof will proceed by induction going from hyperplanes to points. Assume that $(k+1)$-planes M that have the property that they contain at least $k+2$ points of $\tilde{\mathcal{S}}$ also have the property that

$\#(\mathcal{S}\cap M) \geq \#(\tilde{\mathcal{S}}\cap M)$. We will show that the same holds for k-planes. Fix a k-plane L containing $\mu \geq k+1$ points of $\tilde{\mathcal{S}}$, and let $\{M_\alpha\}$ denote the set of $k+1$ planes containing L and at least $\mu+1$ elements of $\tilde{\mathcal{S}}$. We have

$$\#(\tilde{\mathcal{S}}\cap L) + \sum_\alpha \#(\tilde{\mathcal{S}}\cap (M_\alpha \backslash L)) = r,$$
$$\#(\mathcal{S}\cap L) + \sum_\alpha \#(\mathcal{S}\cap (M_\alpha \backslash L)) \leq r.$$

The first line holds because every point of $\tilde{\mathcal{S}}$ not in L is in exactly one M_α, and the second holds because every point of $\mathcal{S}$ not in L is in at most one M_α. Now

$$\sum_\alpha \#(\tilde{\mathcal{S}}\cap M_\alpha) = \sum_\alpha \#(\tilde{\mathcal{S}}\cap (M_\alpha \backslash L)) + \#\{M_\alpha\}\#(\tilde{\mathcal{S}}\cap L)$$

and similarly for $\mathcal{S}$, and we have by induction for each α,

$$\#(\mathcal{S}\cap M_\alpha) \geq \#(\tilde{\mathcal{S}}\cap M_\alpha).$$

Putting these together gives the result. $\square$

Proof of Kruskal's Theorem. Given decompositions

$$T = \sum_{j=1}^r u_j \otimes v_j \otimes w_j = \sum_{j=1}^r \tilde{u}_j \otimes \tilde{v}_j \otimes \tilde{w}_j$$

of length r with the first expression satisfying the hypotheses of the theorem, we want to show that they are the same. (Note that if there were a decomposition of length, e.g., $r-1$, we could construct from it a decomposition of length r by replacing $\tilde{u}_1\otimes\tilde{v}_1\otimes\tilde{w}_1$ by $\frac{1}{2}\tilde{u}_1\otimes\tilde{v}_1\otimes\tilde{w}_1 + \frac{1}{2}\tilde{u}_1\otimes\tilde{v}_1\otimes\tilde{w}_1$, so uniqueness of the length r decomposition implies the rank is r.) I first show that $\mathcal{S}_A = \tilde{\mathcal{S}}_A$, $\mathcal{S}_B = \tilde{\mathcal{S}}_B$, $\mathcal{S}_C = \tilde{\mathcal{S}}_C$. By symmetry it is sufficient to prove the last statement. By the permutation lemma it is sufficient to show that if $H \subset \mathbb{P}C$ is a hyperplane such that $\#(\tilde{\mathcal{S}}_C \cap H) \geq \mathbf{c}-1$, then $\#(\mathcal{S}_C \cap H) \geq \#(\tilde{\mathcal{S}}_C \cap H)$ because by Exercise 12.5.3.4 we already know that $k_{\mathcal{S}_C} \geq 2$.

Let $S = \#(\mathcal{S}_C \not\subset H)$ and write $\sum u_j\otimes v_j\otimes w_j = \sum_{\sigma=1}^S u_\sigma\otimes v_\sigma\otimes w_\sigma + \sum_\mu u_\mu\otimes v_\mu\otimes w_\mu$, where we have reordered our indices so that the first S of the $[w]$'s do not lie on H.

I show that $\#(\tilde{\mathcal{S}}_C \not\subset H) \geq \#(\mathcal{S}_C \not\subset H) = S$. Let $h \in \hat{H}^\perp$ be nonzero. We have

$$\#(\tilde{\mathcal{S}}_C \not\subset H) \geq \operatorname{rank}(T(h))$$

and $T(h) = \sum_{\sigma=1}^{S} u_\sigma \otimes v_\sigma \otimes w_\sigma(h)$, where none of the $w_\sigma(h)$'s are zero. Hence

$$\begin{aligned}\operatorname{rank}(T(h)) &= \min\{\dim\langle u_\sigma\rangle, \dim\langle v_\sigma\rangle\} \\ &\geq \dim\langle u_\sigma\rangle + \dim\langle v_\sigma\rangle - S.\end{aligned}$$

Now by the definition of the Kruskal rank, $\dim\langle u_\sigma\rangle \geq \min\{S, k_{\mathcal{S}_A}\}$ and $\dim\langle v_\sigma\rangle \geq \min\{S, k_{\mathcal{S}_B}\}$.

So far we have

$$\#(\tilde{\mathcal{S}}_C \not\subset H) \geq \min\{S, k_{\mathcal{S}_A}\} + \min\{S, k_{\mathcal{S}_B}\} - S. \tag{12.5.4}$$

Thus, we would be done if we knew that $S \leq \min\{k_{\mathcal{S}_A}, k_{\mathcal{S}_B}\}$.

Finally use our hypothesis in the form

$$r - k_{\mathcal{S}_C} + 1 \leq k_{\mathcal{S}_A} + k_{\mathcal{S}_B} - r + 1, \tag{12.5.5}$$

which combined with the assumption $\#(\tilde{\mathcal{S}}_C \cap H) \geq \mathbf{c} - 1$ gives

$$\#(\tilde{\mathcal{S}}_C \not\subset H) \leq r - (\mathbf{c} - 1) \leq r - (k_{\mathcal{S}_C} - 1) \geq k_{\mathcal{S}_A} + k_{\mathcal{S}_B} - r + 1. \tag{12.5.6}$$

Exercise 12.5.4.1: Show that (12.5.4) and (12.5.6) together imply $S \leq \min\{k_{\mathcal{S}_A}, k_{\mathcal{S}_B}\}$ to finish the argument. ⊚

Now that we have shown that $\mathcal{S}_A = \tilde{\mathcal{S}}_A$, etc., say we have two expressions

$$\begin{aligned}T &= u_1 \otimes v_1 \otimes w_1 + \cdots + u_r \otimes v_r \otimes w_r, \\ T &= u_1 \otimes v_{\sigma(1)} \otimes w_{\tau(1)} + \cdots + u_r \otimes v_{\sigma(r)} \otimes w_{\tau(r)}\end{aligned}$$

for some $\sigma, \tau \in \mathfrak{S}_r$. First observe that if $\sigma = \tau$, then we are reduced to the two-factor case, which is easy.

Exercise 12.5.4.2: Show that if $T \in A \otimes B$ of rank r has expressions $T = a_1 \otimes b_1 + \cdots + a_r \otimes b_r$ and $T = a_1 \otimes b_{\sigma(1)} + \cdots + a_r \otimes b_{\sigma(r)}$, then $\sigma = \mathrm{Id}$.

So assume that $\sigma \neq \tau$; then there exists a smallest $j_0 \in \{1, \dots, r\}$ such that $\sigma(j_0) =: s_0 \neq t_0 := \tau(j_0)$. I claim that there exist subsets $S, T \subset \{1, \dots, r\}$ with the following properties:

- $s_0 \in S$, $t_0 \in T$,
- $S \cap T = \emptyset$,
- $\#(S) \leq r - k_{\mathcal{S}_B} + 1$, $\#(T) \leq r - k_{\mathcal{S}_C} + 1$, and
- $\langle v_j \mid j \in S^c\rangle =: \hat{H}_S \subset B$, $\langle w_j \mid j \in T^c\rangle =: \hat{H}_T \subset C$ are hyperplanes.

Here $S^c = \{1, \dots, r\} \backslash S$.

To prove the claim take a hyperplane $\hat{H}_T \subset C$ containing w_{s_0} but not containing w_{t_0}. The set of indices of the w_j contained in $\hat{H}_T$ is at least $k_{\mathcal{S}_C} - 1$, ensuring the cardinality bound for T. Now consider the linear space $\langle v_t \mid t \in T\rangle \subset B$. Since $\#(T) \leq r - k_{\mathcal{S}_C} + 1 \leq k_{\mathcal{S}_B} - 1$ (the last inequality holds because $k_{\mathcal{S}_A} \leq r$), adding any vector of $\mathcal{S}_B$ to $\langle v_t \mid t \in T\rangle$

would increase its dimension, in particular, $v_{s_0} \notin \langle v_t \mid t \in T\rangle$. Thus there exists a hyperplane $\hat{H}_S \subset B$ containing $\langle v_t \mid t \in T\rangle$ and not containing v_{s_0}. Let S be the set of indices of the v_j contained in $\hat{H}_S$. Then S, T have the desired properties.

Now by construction $T|_{H_S{}^{\perp} \times H_T{}^{\perp}} = 0$, which implies that there is a nontrivial linear relation among the u_j for the j appearing in $S \cap T$, but this number is at most $\min(r - k_{\mathcal{S}_B} + 1, r - k_{\mathcal{S}_C} + 1)$, which is less than $k_{\mathcal{S}_A}$. $\square$

Remark 12.5.4.3. There were several inequalities used in the proof that were far from sharp. Kruskal proves versions of his theorem with weaker hypotheses designed to be more efficient regarding the use of the inequalities.

Remark 12.5.4.4. The proof above is essentially Kruskal's. It also resembles the proof in [**299**] except that the proof there is in coordinates. The reduction from a 16 page proof to the 3 page proof above is mostly due to writing statements invariantly rather than in coordinates. A completely different short proof appears in [**276**].

Chapter 13

P v. NP

This chapter discusses geometric approaches to $\mathbf{P} \neq \mathbf{NP}$ and its algebraic variants. It begins, in §13.1, with an introduction to the complexity issues that will be discussed. Many problems in complexity theory are stated in terms of graphs, so §13.2 discusses graphs and polynomials that arise in the study of graphs. In §13.3, the algebraic complexity classes $\mathbf{VP}, \mathbf{VNP}$ and variants are defined. What is known about the permanent and determinant regarding these algebraic classes is discussed in §13.4, including the classical problem of expressing perm_m as the determinant of a larger matrix. These polynomials also play roles in probability and statistics. In Valiant's theory only two polynomials, the permanent and determinant, play a central role, but it would be useful to have a larger class of polynomials to work with. Two such, the immanants defined by Littlewood, and the α-permanent family, are discussed in §13.5. The immanants conjecturally interpolate in complexity from the determinant to the permanent. §13.6 is an introduction to the Geometric Complexity Theory (GCT) program proposed by K. Mulmuley and M. Sohoni. This program is in some sense a border rank version of Valiant's conjecture. A geometric approach to L. Valiant's holographic algorithms, which were put in a tensor framework by J. Cai, is presented in §13.9. To prepare for the study of holographic algorithms, a parametric description of (a large open subset of) the Grassmannian and spinor varieties is given in §13.8. This is a topic of interest in its own right. Sections 13.8 and 13.9 may be read independently of the rest of the chapter except for §13.1.

General references for this chapter are [**56**], [**54**, Chap. 21], [**139**], and [**227**]. The beginning of the introduction follows [**294, 56**], and §§13.3, 13.4 generally follow [**139**].

13.1. Introduction to complexity

13.1.1. A brief history. Hilbert's 10th problem asked to find a systematic procedure to determine if a polynomial with integral coefficients in several variables has an integral solution, which may be the origin of *computability theory*. (Matiyasevich [**231**] proved in 1970 that such an algorithm could not exist.) The question of

what is *feasibly computable*, that is, can be computed (on, say, a Turing machine) in a number of steps that is polynomial in the size of the input data,

was asked as early as in the 1950s by Shannon, von Neumann, Yablonskii, and others. See the quotations provided at the end of [**294**]. A constant theme of the questions from early on was if there was an alternative to "brute force searches" for problems such as the traveling salesman problem.

Roughly speaking, a *problem* in complexity theory is a class of expressions to evaluate (e.g., count the number of four colorings of a planar graph). An *instance* of a problem is a particular member of the class (e.g., count the number of four colorings of the complete graph with four vertices). **P** is the class of problems that admit an algorithm that solves any instance of it in a number of steps that depends polynomialy on the size of the input data. One says that such problems "admit a polynomial time solution". The class **P** was defined by Cobham [**92**], Edmonds [**118**], and Rabin [**269**]. **NP** is the class of problems where a proposed solution to an instance can be positively checked in polynomial time. The famous conjecture of Cook, Karp, and Levin is $\mathbf{P} \neq \mathbf{NP}$.

Another perspective on **P** and **NP** mentioned in Chapter 1 that goes back to Gödel, is that **NP** is meant to model intuition, or theorem proving, and **P** is meant to model proof checking, where coming up with a proof is supposed to be difficult, but checking a proposed proof, easy.

Some efficient algorithms are not the obvious ones, for example, using Gaussian elimination to compute the determinant, which avoids evaluating a huge expression. The question is if there is such a "trick" to avoid brute force searches in general.

A problem P is *hard* for a complexity class **C** if all problems in **C** can be *reduced* to P (i.e., there is an algorithm to translate any instance of a problem in **C** to an instance of P with comparable input size). A problem P is *complete* for **C** if it is hard for **C** and if $P \in \mathbf{C}$. Much of complexity theory revolves around finding problems complete for a given class. When discussing algebraic complexity classes, instead of dealing with problems, one deals with sequences of polynomials and one uses the same terminology of hardness and completeness with respect to the sequence. The reduction of one problem to another is replaced by requiring that one sequence of polynomials may be realized as a linear projection of another. L. Valiant defined classes **VP** and **VNP** as algebraic analogs of **P** and **NP**. The sequence (perm_n) is **VNP**-complete. The sequence $(\det_n)$ is in **VP**, but it is not known if it is **VP**-complete. Since the determinant is such a special polynomial, a complexity class was invented for which it is complete, namely, the class $\mathbf{VP}_{ws}$ defined in §13.3.3.

I only discuss **P** v. **NP** in its original formulation in the context of holographic algorithms, as there the geometry of tensors plays a prominent role. Most of this chapter is on algebraic variants of **P** v. **NP**. These algebraic variants are expected to be easier to prove. For example, separating the classes associated to $(\det_n)$ and (perm_n) would be a consequence of $\mathbf{P} \neq \mathbf{NP}$, and if these classes were to coincide, there would be a spectacular collapse of complexity classes beyond showing that $\mathbf{P} = \mathbf{NP}$.

Holographic algorithms are related to evaluations of pairings $V \times V^* \to \mathbb{C}$, while the algebraic analogs deal with evaluations of polynomials. I discuss these two types of evaluations in the next two subsections.

13.1.2. Vector space pairings. For each n, let V_n be a complex vector space and assume that $\dim(V_n)$ grows exponentially fast with n. Consider the problem of evaluating the pairing of the vector space with its dual:

$$\begin{aligned} V_n \times V_n^* &\to \mathbb{C}, \\ (v, \alpha) &\mapsto \langle \alpha, v \rangle. \end{aligned} \tag{13.1.1}$$

As explained in §13.9.1, all problems in **P** and **NP** can be reexpressed as computing such pairings. It is known that general pairings (13.1.1) require on the order of $\dim(V_n)$ arithmetic operations to perform. However, if V_n has additional structure and α, v are in a "special position" with respect to this structure, the pairing may be evaluated faster. A trivial example would be if V_n were equipped with a basis and v were restricted to be a linear combination of only the first few basis vectors. I will be concerned with more subtle examples such as the following: Let $V_n = \Lambda^n \mathbb{C}^{2n}$; then inside V_n are the decomposable vectors (the cone over the Grassmannian $G(n, \mathbb{C}^{2n})$, or in other words, the closure of the set of vectors of minors of $n \times n$ matrices in the space $\bigoplus_j \Lambda^j \mathbb{C}^n \otimes \Lambda^j \mathbb{C}^n$; see §13.8), and if α, v are decomposable, the pairing $\langle \alpha, v \rangle$ can be evaluated in polynomial time in n (Proposition 13.8.1.3). From a geometric perspective, this is one of the key ingredients to L. Valiant's *holographic algorithms* discussed in §13.9. For n large, the codimension of the Grassmannian is huge, so it would seem highly unlikely that any interesting problem could have α, v so special. However, small Grassmannians are of small codimension. This leads to the second key ingredient to holographic algorithms. On the geometric side, if $[v_1] \in G(k_1, W_1)$ and $[v_2] \in G(k_2, W_2)$, then $[v_1 \otimes v_2] \in G(k_1 k_2, W_1 \otimes W_2)$. Thus, if the vectors α, v can be thought of as being built out of vectors in smaller spaces, there is a much better chance of success. Due to the restrictive nature of **NP** problems, this is exactly what occurs. The third key ingredient is that there is some flexibility in how the small vector spaces are equipped with the additional structure, and I show (Theorem 13.9.3.3) that even for **NP**-complete problems there is sufficient flexibility to allow everything to work up to this point. The difficulty occurs when one tries to tensor together the small vector spaces in a way that is simultaneously compatible for V_n and V_n^*, although the "only" problem that can occur is one of signs; see §13.9.4.

13.1.3. Evaluation of polynomials. The second type of evaluation I will be concerned with is that of sequences of (homogeneous) polynomials, $p_n \in S^{d(n)}\mathbb{C}^{\mathbf{v}(n)}$, where the degree $d(n)$ and the number of variables $\mathbf{v}(n)$ are required to grow at least linearly and at most polynomially with n. These are the objects of study in Valiant's algebraic theory. (Valiant does not require homogeneous polynomials, but any polynomial can be replaced with a homogeneous polynomial by adding an extra variable; see §2.6.5.) A generic such sequence is known to require an exponential (in n) number of arithmetic operations to evaluate and one would like to characterize the sequences where the evaluation can be done in a number of steps polynomial in n. Similar to the trivial example in §13.1.2 above, there are sequences such as $p_n = x_1^{d(n)} + \cdots + x_{\mathbf{v}(n)}^{d(n)}$ where it is trivial to see that there is a polynomial time evaluation, but there are other, more subtle examples, such as $\det_n \in S^n\mathbb{C}^{n^2}$ where the fast evaluation may be attributed to a group action, as discussed in §1.4. Similar to the remark above regarding holographic algorithms and signs, if one changes the signs in the expression of the determinant, e.g., to all plus signs, to obtain the permanent, one arrives at a **VNP**-hard sequence, where **VNP** is Valiant's algebraic analogue of **NP** defined in §13.3.

A first observation is that if a polynomial is easy to evaluate, then any specialization of it is also easy to evaluate. From a geometer's perspective, it is more interesting to look at the zero sets of the polynomials, to get sequences of hypersurfaces in projective spaces. If a polynomial is easy to evaluate, the polynomial associated to any linear section of its zero set is also easy to evaluate. This leads to Valiant's conjecture that the permanent sequence (perm_m) cannot be realized as a linear projection of the determinant sequence $(\det_n)$ unless n grows faster than any polynomial in m (Conjecture 13.4.4.2). (Another reason to look at the sequence $(\det_n)$ is that it is hard with respect to formula size, i.e., the class $\mathbf{VP}_e$; see §13.3.1.) The best result on this conjecture is due to T. Mignon and N. Ressayre [**235**], who use local differential geometry. While the local differential geometry of the $\det_n$-hypersurface is essentially understood (see Theorem 13.4.6.1), a major difficulty in continuing their program is to distinguish the local differential geometry of the perm_m-hypersurface from that of a generic hypersurface. (Such a distinction will be necessary for breaking the so-called "natural proof barrier" [**274**].) Furthermore, the determinant hypersurface is so special it may be difficult to isolate exactly which of its properties are the key to it having a fast evaluation.

From the geometric point of view, a significant aesthetic improvement towards approaching Valiant's conjecture is the *Geometric complexity theory* (GCT) program proposed by K. Mulmuley and M. Sohoni. Instead of regarding the determinant itself, one considers its GL_{n^2}-orbit closure in $\mathbb{P}(S^n\mathbb{C}^{n^2})$ and similarly for the permanent. One now compares two algebraic varieties that are invariant under a group action. In §13.6, I give an overview of the GCT program.

13.2. Polynomials in complexity theory, graph theory, and statistics

In this section I discuss several natural polynomials that arise in complexity theory.

Given $P \in S^dV^*$, let

$$G(P) := \{g \in GL(V) \mid P(g \cdot x) = P(x)\ \forall x \in V\}$$

denote the group preserving P.

Consider the determinant of a linear map $f : E \to F$, where E, F are n-dimensional vector spaces (cf. §2.6.12). Note that $\det = \Omega_E \otimes \Omega_F$ for some $\Omega_E \in \Lambda^n E^*$, $\Omega_F \in \Lambda^n F$ which are well defined up to reciprocal rescaling. Recall from §8.2.5 that $G(\det_n) = (SL(E) \times SL(F))/\mu_n \rtimes \mathbb{Z}_2$.

13.2.1. More exercises on the Pfaffian.

Recall the Pfaffian from §2.7.4:

$$\begin{aligned}\mathrm{Pf}(x^i_j) &= \frac{1}{2^n n!} \sum_{\sigma \in \mathfrak{S}_{2n}} \mathrm{sgn}(\sigma) x^{\sigma(1)}_{\sigma(2)} \cdots x^{\sigma(2m-1)}_{\sigma(2m)} \\ &= \frac{1}{2^n} \sum_{\sigma \in \mathcal{P}} \mathrm{sgn}(\sigma) x^{\sigma(1)}_{\sigma(2)} \cdots x^{\sigma(2m-1)}_{\sigma(2m)},\end{aligned}$$

where $\mathcal{P} \subset \mathfrak{S}_{2m}$ consists of the permutations such that $\sigma(2i-1) < \sigma(2i)$ for all $0 \le i \le m$.

(1) Write $(\sigma(1), \sigma(2), , \dots, \sigma(2m-1), \sigma(2m)) = (i_1, j_1, i_2, j_2, \dots, i_m, j_m)$. Arrange $1, \dots, 2m$ on a circle. Draw lines from i_s to j_s, let cr denote the number of crossings that occur (so, for example, for the identity there are none, for $\sigma = (23)$ there is one, etc.). Show that $\mathrm{sgn}(\sigma) = (-1)^{cr}$.

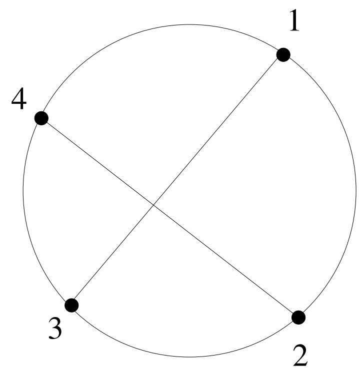


Figure 13.2.1. One crossing for $\sigma = (2,3)$.

(2) Show that cr depends just on the set of pairs $(i_1, j_1), , \dots, (i_m, j_m)$, so that $cr(S)$ makes sense for any partition S of $2m$ into a set of unordered pairs. If one assumes $i_s < j_s$ for all s, then $cr(S) = \#\{(r,s) \mid i_r < i_s < j_r < j_s\}$.

(3) Show that

$$\mathrm{Pf}(x) = \sum_{S \in \mathcal{S}} (-1)^{cr(S)} x^{i_1}_{j_1} \cdots x^{i_m}_{j_m},$$

where $\mathcal{S}$ is the set of all partitions of $2k$ into a set of unordered pairs $(i_1, \dots, i_m), (j_1, \dots, j_m)$.

13.2.2. Graphs. A *graph* $G = (V, E)$ is an ordered pair consisting of a set of *vertices* $V = V(G)$ and a set of *edges* $E = E(G)$, which is a subset of the unordered pairs of vertices. A *bipartite graph* is a graph where $V = V_1 \sqcup V_2$ and $E \subset V_1 \times V_2$. A *weighted graph* is a graph equipped with a function $w : E \to \mathbb{C}$.

Recall from §1.4.2 that the permanent counts perfect matchings of a bipartite graph. If one allows weighted graphs, one can form a weighted incidence matrix whose nonzero entries are the weights of the edges, and then the permanent counts the weighted perfect matchings.

13.2.3. The perfect matching polynomial/hafnian. If x is the skew-symmetric incidence matrix of a (weighted) graph, then define $\mathrm{PerfMat}(x)$ to be the number of perfect matchings of the graph.

As the permanent is to the determinant, the perfect matching polynomial is to the Pfaffian:

$$\text{PerfMat}(x^i_j) = \frac{1}{2^n n!} \sum_{\sigma \in \mathfrak{S}_{2n}} x^{\sigma(1)}_{\sigma(2)} \cdots x^{\sigma(2n-1)}_{\sigma(2n)} = \frac{1}{2^n} \sum_{\sigma \in \mathcal{P}} x^{\sigma(1)}_{\sigma(2)} \cdots x^{\sigma(2n-1)}_{\sigma(2n)}. \tag{13.2.1}$$

Here $\mathcal{P}$ is as in (2.7.1). This polynomial is sometimes called the *hafnian*. (The name hafnian was introduced by Caianiello in [**66**].)

Exercise 13.2.3.1: Show that PerfMat indeed counts perfect matchings.

13.2.4. The Hamiltonian cycle polynomial. A *cycle cover* of a directed graph is a union of cycles traversing the edges and visiting each node exactly once. Given a directed graph with n vertices, we substitute 1 for x^i_j if (i,j) is an edge of G into perm_n (and zero elsewhere) to obtain the number of cycle covers of G. If G is a weighted graph and we instead substitute in the weights, we obtain the number of weighted cycle covers. Let $\text{HC}_n(x)$ denote the number of weighted cycle covers consisting of a single cycle, called *Hamiltonian circuits*.

Then

$$\text{HC}_n(x) = \sum_{\sigma \in \{n\text{-cycles}\} \subset \mathfrak{S}_n} x^1_{\sigma(1)} \cdots x^n_{\sigma(n)}, \tag{13.2.2}$$

and is called the *Hamiltonian cycle polynomial*.

Exercise 13.2.4.1: Show that $\text{HC}_n(x)$ is indeed represented by the polynomial (13.2.2).

Exercise 13.2.4.2: Show that the group preserving HC_n contains $\mathfrak{S}_n$ acting diagonally, transpose, and $C_L \times C_R$, where C_L (resp. C_R) is the group of cyclic permutation matrices acting on the left (resp. right), as well as pairs of diagonal matrices with reciprocal determinants acting on the left and right.

13.3. Definitions of VP, VNP, and other algebraic complexity classes

In this section I define the complexity classes $\mathbf{VP}_e$, **VP**, $\mathbf{VP}_{ws}$, **VNP**, and their closures.

13.3.1. $\mathbf{VP}_e$. An elementary measure of the complexity of a polynomial p is as follows: given an expression for p, count the total number of additions plus multiplications present in the expression, and then take the minimum over all possible expressions—this minimum is denoted $E(p)$ and called the *expression size* of p. The class $\mathbf{VP}_e$ is the set of sequences of polynomials (p_n) such that $E(p_n)$ is bounded by a polynomial in n.

Aside 13.3.1.1. There are two types of multiplications, multiplying a variable by a variable, and a variable by a scalar. As far as complexity classes are concerned, it does not matter whether or not one counts the multiplications of a scalar times a variable because the variable multiplications and additions bound the number of multiplications by a scalar.

Example 13.3.1.2.

$$p_n(x, y) = x^n + nx^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \binom{n}{3}x^{n-3}y^3 + \cdots + y^n.$$

This expression for p_n involves $n(n+1)$ multiplications and n additions, but one can also write

$$p_n(x, y) = (x + y)^n,$$

which uses $n - 1$ multiplications and n additions.

It turns out that the expression size is too naïve a measurement of complexity, as considering Example 13.3.1.2, we could first compute $z = x + y$, then $w = z^2$, then w^2, etc., until the exponent is close to n, for a significant savings in computation.

13.3.2. Arithmetic circuits and the class VP. It is useful to define the class **VP** in terms of *arithmetic circuits*.

Definition 13.3.2.1. An *arithmetic circuit* C is a finite, acyclic, directed graph with vertices of in-degree 0 or 2 and exactly one vertex of out-degree 0. The vertices of in-degree 0 are labelled by elements of $\mathbb{C} \cup \{x_1, \ldots, x_n\}$, and those of in-degree 2 are labelled with $+$ or $*$. (The vertices of in-degree 2 are called *computation gates*.) The *size* of C is the number of vertices. From a circuit C, one can construct a polynomial p_C in the variables $x_1, \ldots, x_n$.

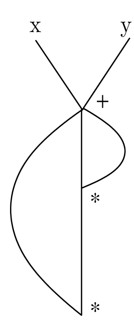


Figure 13.3.1. Circuit for $(x + y)^3$.

Definition 13.3.2.2. The class **VP** is the set of sequences (p_n) of polynomials of degree $d(n)$ in $\mathbf{v}(n)$ variables, where $d(n), \mathbf{v}(n)$ are bounded by polynomials in n and such that there exists a sequence of circuits (C_n) of polynomialy bounded size such that C_n computes p_n.

Remark 13.3.2.3. The condition that $\mathbf{v}(n)$ is bounded by a polynomial makes **VP** more like an arithmetic analog of the class **NC** than of **P**. For the definition of the class **NC**, see, e.g., [**261**].

If C is a tree (i.e., all out-degrees are at most one), then the size of C equals the number of $+$'s and $*$'s used in the formula constructed from C, so $E(p)$ is the smallest size of a tree circuit that computes p. The class $\mathbf{VP}_e$ is thus equivalently the set of sequences (p_n) such that there exists a sequence (C_n) of tree circuits, with the size of C_n bounded by a polynomial in n, such that C_n computes p_n.

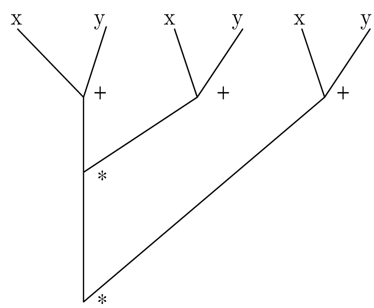


Figure 13.3.2. Tree circuit for $(x+y)^3$.

Exercise 13.3.2.4: Let f be a polynomial of degree d. Then $E(f) \geq d-1$.

A polynomial $p(y_1, \dots, y_m)$ is a *projection* of $q(x_1, \dots, x_n)$ if there exist constants a_i^s, c_i such that setting $x_i = a_i^s y_s + c_i$ gives $p(y_1, \dots, y_m) = q(a_1^s y_s + c_1, \dots, a_n^s y_s + c_n)$. Geometrically, if one homogenizes the polynomials by adding variables y_0, x_0, one can study the zero sets in projective space. Then p is a projection of q if and only if , letting P, Q denote the corresponding homogeneous polynomials, $\mathrm{Zeros}(P) \subset \mathbb{CP}^m$ is a linear section of $\mathrm{Zeros}(Q) \subset \mathbb{CP}^n$. This is because if one considers a projection map $V \to V/W$, then $(V/W)^* \simeq W^\perp \subset V^*$.

An essential property of the class **VP** is that it is closed under linear projections:

Proposition 13.3.2.5. *If $\pi_n : \mathbb{C}^{\mathbf{v}(n)} \to \mathbb{C}^{\mathbf{v}'(n)}$ is a sequence of linear projections, and a family (p_n) is in* **VP**, *then the family $(\pi_n \circ p_n)$ is in* **VP**.

For a proof see, e.g., [**54**, Chap. 21] or [**139**].

13.3.3. The determinant and the class $\mathbf{VP}_{ws}$. A famous example of a sequence in **VP** is $(\det_n)$, despite its apparently huge expression size. While it is known that $(\det_n) \in \mathbf{VP}$, it is not known whether or not it is **VP**-complete. On the other hand, it is known that $(\det_n)$ is $\mathbf{VP}_e$-hard, although it is not known whether or not $(\det_n) \in \mathbf{VP}_e$. (It is generally conjectured that $(\det_n) \notin \mathbf{VP}_e$.) When complexity theorists and mathematicians are confronted with such a situation, what else do they do other than make another definition?

The following definition is due to S. Toda [**310**], and refined by G. Maloud and N. Portier [**227**]:

Definition 13.3.3.1. The class $\mathbf{VP}_{ws}$ is the set of sequences (p_n) where $\deg(p_n)$ is bounded by a polynomial and such that there exists a sequence of circuits (C_n) of polynomialy bounded size such that C_n represents p_n, and such that at any multiplication vertex, the component of the circuit of one of the two edges coming in is disconnected from the rest of the circuit by removing the multiplication vertex. Such a circuit is called *weakly skew.*

Theorem 13.3.3.2 ([**227**, Prop. 5 and Thm. 6])**.** *The sequence* $(\det_n)$ *is* $\mathbf{VP}_{ws}$*-complete.*

That $(\det_n) \in \mathbf{VP}_{ws}$ will follow from §13.4.2. For the other direction, see [**227**].

13.3.4. Aside VP v. P. The class **VP** was defined by Valiant as an algebraic analog of the class **P**. Briefly, let $\Sigma^n := \mathbb{F}_2^n \oplus \mathbb{F}_2^{n-1} \oplus \cdots \oplus \mathbb{F}_2$ and let $\Sigma^* = \bigcup_{n\geq 0} \Sigma^n$. The class **P** may be defined as the set of subsets $L \subset \Sigma^*$ with the property that the characteristic function $\chi_L : \Sigma^* \to \{0, 1\}$ is computable by a deterministic Turing machine M in polynomial time; i.e., there exists a function $t : \mathbb{N} \to \mathbb{N}$, bounded by a polynomial, such that for all $n \in \mathbb{N}$ and all $x \in \Sigma^n$, M computes $\chi_L(x)$ in at most $t(n)$ steps.

The above Turing machine M can be viewed as computing a sequence (f_n) of functions $f_n : \Sigma^n \to \mathbb{F}_2$, where $f_n = \chi_{L\cap\Sigma^n}$.

Every such f_n is represented by a polynomial $F_n \in \mathbb{F}_2[x_1, \ldots, x_n]$ of degree at most one in each indeterminate, in particular $\deg(F_n) \leq n$.

Differences between **P** and **VP** are: for a family of polynomials to be in **VP**, one does not require the family to be uniformly described (as the F_n are uniformly described by a Turing machine), in **VP** one is allowed to work over arbitrary fields instead of just $\mathbb{F}_2$, and **VP** restricts degree. For simplicity, in this book I restrict attention to the field $\mathbb{C}$.

13.3.5. VNP. The class **VNP** essentially consists of polynomials whose coefficients can be determined in polynomial time. More precisely, consider a sequence $h = (h_n)$ of (not necessarily homogeneous) polynomials in variables $x_1, \dots, x_n$ of the form

$$h_n = \sum_{e \in \{0,1\}^n} g_n(e) x_1^{e_1} \cdots x_n^{e_n}, \tag{13.3.1}$$

where $(g_n) \in \mathbf{VP}$. Define **VNP** to be the set of all sequences that are projections of sequences of the form h ([**54**, §21.2]).

Conjecture 13.3.5.1 (Valiant, [**313**]). **VP** $\neq$ **VNP**.

Theorem 13.3.5.2 (Valiant, [**313, 320**]). *Let $f_n \in S^{d(n)}\mathbb{C}^{\mathbf{v}(n)}$ be a sequence. The following are equivalent:*

(1) $(f_n) \in \mathbf{VNP}$.

(2) *(f_n) is in* **VNP** *with the additional requirement that the sequence (g_n) of* (13.3.1) *is in* $\mathbf{VP}_e$.

(3) *There exist $(g_n) \in \mathbf{VP}$ and a function $m : \mathbb{N} \to \mathbb{N}$ bounded by a polynomial, such that for all n,*

$$f_n(x_1, \dots, x_n) = \sum_{e \in \{0,1\}^{m-n}} g_{m(n)}(x_1, \dots, x_n, e_{n+1}, \dots, e_{m(n)}).$$

(4) *As in* (3) *but with $(g_m) \in \mathbf{VP}_e$.*

See, e.g, [**54**, §21.2], [**139**] for a proof.

Aside 13.3.5.3 (**VNP** v. **NP**). The relation of **VNP** with **NP** is as follows: Given $L \subset \Sigma^*$, by definition, $L \in \mathbf{NP}$ if there exist $L' \in \mathbf{P}$ and a p-bounded function $t : \mathbb{N} \to \mathbb{N}$ such that

$$\chi_L(x) = \bigvee_{e \in \Sigma^{t(n)}} \chi_{L'}(x, e).$$

Replacing $x \in \Sigma^n$ by an n-tuple of variables, and "or"($\bigvee$) with summation, yields (3) of Theorem 13.3.5.2.

13.3.6. Closures of algebraic complexity classes. In what follows it will be useful to take closures, just as with the study of rank it was often convenient to work with border rank. Hence the following definition:

Definition 13.3.6.1 ([**50**]). Let **C** be a complexity class defined in terms of a measure $L_{\mathbf{C}}(p)$ of complexity of polynomials, where a sequence (p_n) is in **C** if $L_{\mathbf{C}}(p_n)$ grows polynomialy. For $p \in S^d\mathbb{C}^v$, write $\overline{L_{\mathbf{C}}}(p) \leq r$ if p is in the Zariski closure of the set $\{q \in S^d\mathbb{C}^v \mid L_{\mathbf{C}}(q) \leq r\}$, and define the class $\overline{\mathbf{C}}$ to be the set of sequences (p_n) such that $\overline{L_{\mathbf{C}}}(p_n)$ grows polynomially.

Remark 13.3.6.2. The class $\overline{\mathbf{VP}_e}$ was described geometrically in terms of successive joins and multiplicative joins of projective space in [**212**].

13.4. Complexity of perm_n and $\det_n$

13.4.1. Complexity of perm_n.

Proposition 13.4.1.1 ([**312**]). $(\mathrm{perm}_n), (\mathrm{HC}_n) \in \mathbf{VNP}$.

Proof. Let $\mathcal{W}_n \subset Mat_{n\times n}$ denote the set of permutation matrices. Note that

$$\mathrm{perm}_n(x) = \sum_{P\in\mathcal{W}_n} \prod_{j=1}^{n} (Px)^j_j.$$

To conclude, it would be sufficient to find a sequence (g_n) in $\mathbf{VP}_e$, $g_n : Mat_{n\times n}(\{0,1\}) \to \mathbb{F}_2$ such that $g(M) = 1$ if M is a permutation matrix and is zero otherwise, as then

$$\mathrm{perm}_n(x) = \sum_{M\in Mat_{n\times n}(\{0,1\})} g_n(M) \prod_{j=1}^{n} (Mx)^j_j.$$

Consider

$$g_n(M) := \Big\{ \prod_{\substack{1\leq i,j,k,l\leq n \\ (i,j)\neq(k,l),\, i=k \text{ or } j=l}} (1 - M^i_j M^k_l) \Big\} \Big\{ \prod_{1\leq i\leq n} \sum_{1\leq j\leq n} M^i_j \Big\}.$$

The first term in parentheses vanishes if and only if in some row or column of M more than a single 1 occur. If the first factor does not vanish, the second is zero if and only if some row has all entries equal to zero. Thus $g(M) = 1$ if and only if M is a permutation matrix and is zero otherwise.

To show that $(g_n) \in \mathbf{VP}_e$, note that the expression for the second term involves n multiplications and n^2 additions and the expression for the first term involves less than $2n^4$ multiplications and n^4 additions, so $(g_n) \in \mathbf{VP}_e$.

For the case of HC, multiply g_n by a polynomial that is zero unless a given permutation matrix is an n-cycle. The polynomial $\epsilon_n(M) = (1 - M^1_1)(1 - (M^2)^1_1) \cdots (1 - (M^{n-1})^1_1)$ will do because it clearly has small expression size, and a permutation is an n-cycle if and only if an element returns to its starting slot only after n applications. Here the element is the $(1, 1)$ entry of M. $\square$

13.4.2. A division-free algorithm for the determinant. Recall the symmetric functions e_j, p_j from §6.11. Let $f : \mathbb{C}^n \to \mathbb{C}^n$ be a linear map, with eigenvalues (counting multiplicity) $\lambda_1, \dots, \lambda_n$. Then $\det(f)$ is the product of the eigenvalues $e_n(\lambda) = \lambda_1 \cdots \lambda_n$. There is an easy way to compute the sum of the eigenvalues $p_1(\lambda) = \lambda_1 + \cdots + \lambda_n$, namely $\mathrm{trace}(f)$, which involves no multiplications. Similarly, $p_2(\lambda) = \lambda_1^2 + \cdots + \lambda_n^2 = \mathrm{trace}(f^2)$, which involves less than n^3 multiplications. We can compute $p_1(\lambda), \dots, p_n(\lambda)$ using less than n^4 multiplications. But now, we can recover the $e_j(\lambda)$ from the $p_i(\lambda)$, in particular $e_n(\lambda) = \det(f)$, at a small cost.

Thus we have a division-free algorithm for computing the determinant using $O(n^4)$ multiplications. This algorithm was published in [**104**].

Exercise 13.4.2.1: Show that the product of two $n \times n$ matrices can be computed by a weakly skew circuit of polynomial size in n. Conclude that $(\det_n) \in \mathbf{VP}_{ws}$.

13.4.3. $\mathbf{VP}_e$ hardness of det, perm.

Theorem 13.4.3.1 (Valiant [**313**], Liu-Regan [**223**]). *Every $f \in \mathbb{C}[x_1, \dots, x_n]$ of expression size u is both a projection of $\det_{u+1}$ and perm_{u+1}.*

The $u+1$ result is due to Liu-Regan. Earlier, Valiant had proved the result for $u+2$.

Proof. Here is a proof for $u+3$ that captures the flavor of the arguments. Given a circuit C_f for f, define a six-tuple $G(C_f) = (V, E, s, t, \lambda, \epsilon)$, where (V, E) is a directed graph with one input node s and one output node t in which every path has a length mod 2 which is congruent to $\epsilon \in \{0, 1\}$ and $\lambda : E \to \mathbb{C} \cup \{x_1, \dots, x_n\}$ is a weight function.

$G(C_f)$ is defined recursively in the construction of C_f.

If $C_f \in \mathbb{C} \cup \{x_1, \dots, x_n\}$, then $G(C_f)$ has two vertices s, t with an edge (s, t) joining them and $\lambda((s, t)) = C_f$. ($\epsilon = 1$).

Now work inductively. Assume that $G_1 = G(C_{f_1})$, $G_2 = G(C_{f_2})$ and define $G(C_f)$, where $C_f = C_{f_1} \omega C_{f_2}$ with $\omega \in \{+, *\}$, as follows:

If $C_f = C_{f_1} * C_{f_2}$, take $G = G_1 \sqcup G_2/(t_1 \equiv s_2)$, and set $s = s_1$, $t = t_2$, and $\epsilon \equiv \epsilon_1 + \epsilon_2 \bmod 2$.

If $C_f = C_{f_1} + C_{f_2}$, there are three subcases:

(i) If $C_{f_1}, C_{f_2} \in \mathbb{C} \cup \{x_1, \dots, x_n\}$, start with $G_1 \sqcup G_2/(t_1 \equiv t_2)$ and add a new vertex to be s, and two edges $(s, s_1), (s, s_2)$. Set $\epsilon = 0$.

(ii) If $\epsilon_1 = \epsilon_2$, then $G = G_1 \sqcup G_2/(s_1 \equiv s_2, t_1 \equiv t_2)$; set $\epsilon = \epsilon_1$.

(iii) If $\epsilon_1 \neq \epsilon_2$, then to $G = G_1 \sqcup G_2/(s_1 \equiv s_2)$ add a new edge (t_2, t_1) and set $t = t_1$ $\epsilon = \epsilon_1$.

Example 13.4.3.2. $f(x) = x_1x_2x_3 + x_4x_5x_6$:

$$f(x) = \det \begin{pmatrix} 0 & x_1 & 0 & x_4 & 0 \\ 0 & 1 & x_2 & 0 & 0 \\ x_3 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_5 \\ x_6 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Continuing with the proof, at most one new node was added at each step, except at the initial step, and there are $u+2$ steps, so G has at most $u+3$ nodes.

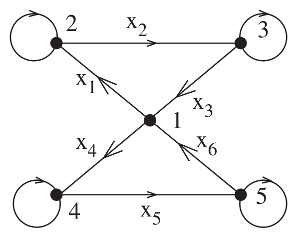


Figure 13.4.1. Weighted directed graph G' associated to $f = x_1x_2x_3 + x_4x_5x_6$.

Let $G(s,t)$ denote the set of paths from s to t in G. For each path $\pi \in G(s,t)$, let $\lambda(\pi)$ denote the product of the weights of all the edges traversed in π. Then

$$f = \sum_{\pi \in G(s,t)} \lambda(\pi).$$

Recall from §13.2.2 that perm_n counts cycle covers of a directed graph. The graph G can be modified to a graph G' so that the paths from s to t in G correspond to the cycle covers of G' as follows: If $\epsilon = 0$, let G' have the same vertices as G, and if $\epsilon = 1$, let $V(G') = V(G)/(s=t)$. In either case, add self-loops $v \to v$ to all vertices not equal to s, t.

The cycle covers of G' are in 1-1 correspondence with elements of $G(s,t)$, proving the assertion regarding the permanent with $|V'| \leq u + 3$. Note moreover that all the cycle covers are even, so $\det_{|V'|}$ works equally well. $\square$

Here is a matrix for $f = \mathrm{perm}_3$ expressed as a 13×13 determinant (or permanent) using the above algorithm:

$$\begin{pmatrix}
0 & x_1^1 & 0 & x_1^1 & 0 & x_2^1 & 0 & x_2^1 & 0 & x_3^1 & 0 & x_3^1 & 0\\
0 & 1 & x_2^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
x_3^3 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & x_3^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
x_2^3 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & x_3^2 & 0 & 0 & 0 & 0 & 0 & 0\\
x_1^3 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & x_1^2 & 0 & 0 & 0 & 0\\
x_3^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & x_1^2 & 0 & 0\\
x_2^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & x_2^2\\
x_1^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.$$

The idea behind the Liu-Regan algorithm is as follows. Their algorithm outputs matrices of the form

$$\begin{pmatrix} & \alpha & & a \\ -1 & & A & \\ & \ddots & & \beta \\ 0 & & -1 & \end{pmatrix}.$$

For example, the matrix for $\sum_{i=1}^n a_i x_i$ is

$$\begin{pmatrix} x_1 & x_2 & \dots & x_{n-1} & x_n \\ -1 & & & 0 & \frac{a_1}{a_n} \\ & -1 & & & \frac{a_2}{a_n} \\ & \ddots & & & \vdots \\ 0 & & & -1 & \frac{a_{n-1}}{a_n} \end{pmatrix},$$

and for $x_1 \cdots x_n$ one takes

$$\begin{pmatrix} x_1 & & & 0 \\ -1 & x_2 & & \\ & \ddots & \ddots & \\ 0 & & -1 & x_n \end{pmatrix}.$$

There are then recipes for glueing together polynomials, which is straightforward for products, but more complicated for sums.

Exercise 13.4.3.3: Verify that in fact perm_3 is a projection of $\det_7$ as was pointed out by B. Grenet:

$$\mathrm{perm}_3(x) = \det_7 \begin{pmatrix} 0 & x_{11} & x_{21} & x_{31} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & x_{33} & x_{23} & 0 \\ 0 & 0 & 1 & 0 & 0 & x_{13} & x_{33} \\ 0 & 0 & 0 & 1 & x_{13} & 0 & x_{23} \\ x_{22} & 0 & 0 & 0 & 1 & 0 & 0 \\ x_{32} & 0 & 0 & 0 & 0 & 1 & 0 \\ x_{12} & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

13.4.4. Determinantal complexity.

Definition 13.4.4.1. Let p be a polynomial. Define $dc(p)$, the *determinantal complexity* of p, to be the smallest integer such that p is an affine linear projection of $\det_{dc(p)}$. Define $\overline{dc}(p)$ to be the smallest integer δ such that there exists a sequence of polynomials p_t, of constant determinantal complexity δ for $t \neq 0$ and $\lim_{t\to 0} p_t = p$.

Conjecture 13.4.4.2 (Valiant, [**312**])**.** *$dc(\mathrm{perm}_m)$ grows faster than any polynomial in m. In other words, $\mathbf{VP}_{ws} \neq \mathbf{VNP}$.*

In [**279**] it was shown that $L_e(\mathrm{perm}_m) \le O(m^2 2^m)$, so combined with Theorem 13.4.3.1 this yields $dc(\mathrm{perm}_m) \le O(m^2 2^m)$.

Conjecture 13.4.4.3 (Mulmuley-Sohoni,[**244**]). *$\overline{dc}(\mathrm{perm}_m)$ grows faster than any polynomial in m. In other words, $\overline{\mathbf{VP}}_{ws} \neq \overline{\mathbf{VNP}}$.*

13.4.5. Differential invariants. In this subsection I define an infinitesimal analog of determinantal complexity. For readers who skipped §8.4, I summarize information about differential invariants needed for this section.

Let $X \subset \mathbb{P}^n$ and $Y \subset \mathbb{P}^m$ be varieties such that there is a linear space $L \simeq \mathbb{P}^m \subset \mathbb{P}^n$ such that $Y = X \cap L$.

Say $y \in Y = X \cap L$. Then the differential invariants of X at y will project to the differential invariants of Y at y. A definition of differential invariants adequate for this discussion (assuming X, Y are hypersurfaces) is as follows: choose local coordinates $(x^1, \dots, x^{n+1})$ for $\mathbb{P}^n$ at $x = (0, \dots, 0)$ such that $T_xX = \langle \frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n} \rangle$ and expand out a Taylor series for X:

$$x^{n+1} = r^2_{i,j} x^i x^j + r^3_{i,j,k} x^i x^j x^k + \cdots .$$

The zero set of $(r^2_{ij} dx^i \cdot dx^j, \dots, r^k_{i_1,\dots,i_k} dx^{i_1} \cdots dx^{i_k})$ in $\mathbb{P}T_xX$ is independent of choices. Write this zero set as $\mathrm{Zeros}(F_{2,x}(X), \dots, F_{k,x}(X))$. I will refer to the polynomials $F_{\ell,x}(X)$, although they are not well defined individually. For more details see, e.g., [**180**, Chap. 3].

One says that X can approximate Y to k-th order at $x \in X$ mapping to $y \in Y$ if one can project the differential invariants to order k of X at x to those of Y at y.

Write $dc_k(p)$ for the smallest number such that a general point of $\mathrm{Zeros}(p)$ can be approximated to order k by a determinant of size $dc_k(p)$, called the *k-th order determinantal complexity of p*. Since p is a polynomial, there exists a k_0 such that $dc_j(p) = dc(p)$ for all $j \ge k_0$.

In [**235**] it was shown that the determinant can approximate *any* polynomial in v variables to second order if $n \ge \frac{v}{2}$ and that perm_m is generic to order two, giving the lower bound $dc(\mathrm{perm}_m) \ge \frac{m^2}{2}$. The previous lower bound was $dc(\mathrm{perm}_m) \ge \sqrt{2}m$, due to J. Cai [**57**] building on work of J. von zur Gathen [**329**].

13.4.6. Differential invariants of $\{\det_n = 0\} \subset \mathbb{P}^{n^2-1}$. If $X \subset \mathbb{P}V$ is a quasi-homogeneous variety, i.e., a group G acts linearly on V and $X = \overline{G \cdot [v]}$ for some $[v] \in \mathbb{P}V$, then $T_{[v]}X$ is a $\mathfrak{g}([v])$-module, where $\mathfrak{g}([v])$ denotes the Lie algebra of the stabilizer of $[v]$ in G.

Let E, F be n-dimensional vector spaces and let $e^1, \dots, e^n$ be a basis of E^*, let $f^1, \dots, f^n$ be a basis of F^*, and let $v = e^1 \otimes f^1 + \cdots + e^{n-1} \otimes f^{n-1}$, so $[v] \in \mathrm{Zeros}(\det_n)$ and

$$\mathrm{Zeros}(\det_n) = \overline{SL(E) \times SL(F) \cdot [v]} \subset \mathbb{P}(E \otimes F).$$

Write $E' = v(F) \subset E^*$, $F' = v(E) \subset F^*$ and set $\ell_E = E^*/E'$, $\ell_F = F^*/F'$. Then, using v to identify $F' \simeq (E')^*$, one obtains $T_{[v]}\mathrm{Zeros}(\det_n) = \ell_E \otimes F' \oplus (F')^* \otimes F' \oplus (F')^* \otimes \ell_F$ as a $\mathfrak{g}([v])$-module. Write an element of $T_{[v]}\mathrm{Zeros}(\det_n)$ as a triple (x, A, y). In matrices,

$$v = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & 0 \end{pmatrix}, \quad T_{[v]} \sim \begin{pmatrix} A & y \\ x & 0 \end{pmatrix}.$$

Taking the $\mathfrak{g}([v])$-module structure into account, it is not hard to show:

Theorem 13.4.6.1 ([**212**]). *Let* $X = \mathrm{Zeros}(\det_n) \subset \mathbb{P}^{n^2-1} = \mathbb{P}(E \otimes F)$ *and* $v = e_1 \otimes f_1 + \cdots + e_{n-1} \otimes f_{n-1} \in X$. *With the above notation, the differential invariants of* X *at* $[v]$ *are*

$$\begin{aligned} \mathrm{Zeros}(F_{2,[v]}(X)) &= \mathrm{Zeros}(xy), \\ \mathrm{Zeros}(F_{2,[v]}(X), F_{3,x}(X)) &= \mathrm{Zeros}(xy, xAy), \\ &\vdots \\ \mathrm{Zeros}(F_{2,[v]}(X), \dots, F_{k,x}(X)) &= \mathrm{Zeros}(xy, xAy, \dots, xA^{k-2}y). \end{aligned}$$

In the language of §8.4, there exists a framing such that $F_k = xA^{k-2}y \in S^k T^*_{[v]} X$.

Since the permanent hypersurface is not quasi-homogeneous, its differential invariants are more difficult to calculate. It is even difficult to write down a general point in a nice way (that depends on m, keeping in mind that we are not concerned with individual hypersurfaces, but sequences of hypersurfaces). For example, the point on the permanent hypersurface chosen in [**235**]:

$$\begin{pmatrix} 1-m & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}$$

is not general as there is a finite group that preserves it. To get lower bounds it is sufficient to work with any point of the permanent hypersurface, but one will not know if the obtained bounds are sharp. To prove that $dc(\mathrm{perm}_m)$ grows at least exponentially with m, one might expect to improve the exponent by one at each order of differentiation. The following theorem shows that this does not happen at order three.

Theorem 13.4.6.2 ([**212**]). *Let p be a homogeneous polynomial in* $\mathbf{v}$ *variables, then $dc_3(p) \leq \mathbf{v} + 1$, and in particular $dc_3(\mathrm{perm}_m) \leq m^2 + 1$.*

Proof. The rank of F_2 for the determinant is $2(n-1)$, whereas the rank of F_2 for the permanent, and of a general homogeneous polynomial in q variables at a general point of its zero set is $q-2$, so one would need to use a projection to eliminate $(n-1)^2$ variables to agree to order two.

Thus it is first necessary to perform a projection so that the matrix A, which formerly had independent variables as entries, will now be linear in the entries of x, y, $A = A(x, y)$. Then the projected pair F_2, F_3 is not generic because it has two linear spaces of dimension $n-1$ in its zero set. This can be fixed by setting $y = L(x)$ for a linear isomorphism $L : \mathbb{C}^{n-1} \to \mathbb{C}^{n-1}$. At this point one has $F_2 = L(x)x$, $F_3 = L(x)A(x, L(x))x$. Take L to be the identity map, so the cubic is of the form $\sum_{i,j} x_i A_{ij}(x) x_j$, where the $A_{ij}(x)$ are arbitrary. This is an arbitrary cubic. $\square$

To better understand perm_m and its geometry, we need a more geometric definition of it. This is provided in the next section.

13.5. Immanants and their symmetries

Let V be a module for a group G. Then each instance of the trivial representation in the symmetric algebra of V^* gives rise to a G-hypersurface in $\mathbb{P}V$. If we have a sequence (V_n, G_n), this leads to a sequence of polynomials. For example, $\Lambda^n\mathbb{C}^n \otimes \Lambda^n\mathbb{C}^n$ gives rise to the determinant in $(\mathbb{C}^n \otimes \mathbb{C}^n)^*$ whose zero set is $\sigma_{n-1}(Seg(\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}))$.

There are many such natural sequences of polynomials, and it might be useful to study the complexity of more than just the determinant and permanent. In §8.3, the sequence $4DPascal_n \in S^n(\mathbb{C}^{n^4})$ was described, which is at least as hard to compute as the permanent. It has the advantage of having a reductive stabilizer that resembles the stabilizer of the determinant, so it might be easier to compare the geometry of $\det_n$ and $4DPascal_m$. In this section I describe other families of polynomials with stabilizers similar to the stabilizer of the permanent. I begin with a more geometric definition of the permanent which emphasizes its symmetries.

Let $V = E \otimes F$, with $\dim E = \dim F = n$. Consider the decomposition of S^nV, first as a $SL(E) \times SL(F)$-module, i.e., $\bigoplus_{|\pi|=n} S_\pi E \otimes S_\pi F$. There is a unique instance of the trivial representation, which gives rise to $\det_n$. Now consider subgroups of $SL(E) \times SL(F)$ and trivial representations that arise. Let $T_E \subset SL(E)$ denote the diagonal matrices (a torus). One could consider $T_E \times T_F$. Then in each $S_\pi E$, by definition, T_E acts trivially exactly on the SL-weight zero subspace, $(S_\pi E)_0$. For most factors, this is too big to determine a preferred polynomial but:

Exercise 13.5.0.3: Show that $\dim(S^n E)_0 = 1$.

13.5.1. The permanent. Consider the element $p_{(n)}(x) \in (S^n E)_0 \otimes (S^n F)_0$, which is unique up to scale.

Exercise 13.5.1.1: Show that $[p_{(n)}] = [\mathrm{perm}_n]$.

Equivalently, let E, F respectively have bases $e_1, \dots, e_n$ and $f_1, \dots, f_n$; then

$$\mathrm{perm}_n := \pi_S(e_1 \cdots e_n \otimes f_1 \cdots f_n) \in S^n E \otimes S^n F \subset S^n(E \otimes F), \tag{13.5.1}$$

where $\pi_S : (E^{\otimes n} \otimes F^{\otimes n}) \to S^n(E \otimes F)$ is the projection.

Exercise 13.5.1.2: Show that if $x \in E^* \otimes F^*$ is expressed as a matrix, then (13.5.1) agrees with the definition of §1.4.2, i.e., that

$$\mathrm{perm}_n(x) = \sum_{\sigma \in \mathfrak{S}_n} x^1_{\sigma(1)} \cdots x^n_{\sigma(n)}. \circledcirc$$

Let $N_E := T_E \rtimes \mathcal{W}_E$ denote the semidirect product of T_E with the permutation matrices (the normalizer of the torus). Using (13.5.1), it is clear that perm_n is invariant under the action of $N_E \times N_F \rtimes \mathbb{Z}_2$. In fact equality holds:

Theorem 13.5.1.3 ([**230**] (also see [**32**])).

$$G(\mathrm{perm}_m) = N_E \times N_F \rtimes \mathbb{Z}_2. \tag{13.5.2}$$

The idea of the proof is to study the group preserving the most singular subset of Zeros(perm_m), as $G(\mathrm{perm}_m)$ must stabilize this as well. (This is in analogy with the way one determines the stabilizer of $\det_n$, where the most singular locus is the Segre variety.) This subset turns out to be quite easy to work with, and the proof becomes straightforward.

Remark 13.5.1.4. The permanent appears often in probability and statistics: computing normalizing constants in non-null ranking models, point-process theory and computing moments of complex normal variables, testing independence with truncated data, etc. See [**113**] and the references therin.

13.5.2. Immanants. To find more polynomials invariant under a group action consider subgroups of $(N_E \times N_F) \rtimes \mathbb{Z}_2$.

I first discuss the weight zero subspaces in more detail. Note that $\mathcal{W}_E$ acts on the weight zero subspace of any $SL(E)$-module.

Exercise 13.5.2.1: Show that for any partition π, the space $(S_\pi E)_0$ is zero unless $\dim E$ divides $|\pi|$.

Proposition 13.5.2.2 ([**190**]). *Let $|\pi| = n$. The space $(S_\pi E)_0$ considered as a $\mathcal{W}_E$-module is the irreducible $\mathfrak{S}_n$-module $[\pi]$.*

Proof. If $e_1, \dots, e_n$ is a basis of E, then $(E^{\otimes n})_0$ has a basis $e_{\sigma(1)}\otimes \cdots \otimes e_{\sigma(n)}$ where $\sigma \in \mathfrak{S}_n$. Thus we may identify $(E^{\otimes n})_0$ with the regular representation of $\mathfrak{S}_n$. On the other hand, $(E^{\otimes n})_0 = (\bigoplus_{|\pi|=n}[\pi]\otimes S_\pi E)_0 = \bigoplus_{|\pi|=n}[\pi]\otimes(S_\pi E)_0$. Now we use the fact that the regular representation admits both a left and right action by $\mathfrak{S}_n$, and as such decomposes $\mathbb{C}[\mathfrak{S}_n] = \bigoplus_\pi[\pi]\otimes[\pi]^* = \bigoplus_\pi[\pi]\otimes[\pi]$ to conclude. $\square$

Corollary 13.5.2.3. *$\langle \mathrm{perm}_n, \det_n\rangle \subset S^n(E\otimes F)$ is a space of $S(N_E\times N_F)\rtimes \mathbb{Z}_2$-invariant polynomials in $S^n(\mathbb{C}^{n^2})$.*

Note that if we were to allow $N_E \times N_F$ above, the above result does not hold as $\det_n$ is not invariant under the action by a permutation with negative sign, although $[\det_n]$ is.

Identify $F \simeq E^*$ and consider the diagonal $\delta(\mathfrak{S}_n) \subset \mathcal{W}_E \times \mathcal{W}_F$. Then as a $\delta(\mathfrak{S}_n)$-module, we have the decomposition

$$(S_\pi E)_0\otimes(S_\pi F)_0 = [\pi]\otimes[\pi] = [\pi]^*\otimes[\pi],$$

which has a preferred trivial submodule—the identity map.

Following [**336, 190**], define $IM_\pi \in (S_\pi E)_0\otimes(S_\pi F)_0 \subset S^n(E\otimes F)$ to be the unique polynomial invariant under $(T_E\times T_F)\rtimes\delta(\mathfrak{S}_n)$ and such that, using the identification $F \simeq E^*$, $p_\pi(\mathrm{Id}) = 1$. IM_π is called an *immanant.* Note that IM_π is also invariant under the $\mathbb{Z}_2$ action given by taking transpose.

Exercise 13.5.2.4: Show that in bases, for $x = (x^i_j)$,

$$IM_\pi(x) = \sum_{\sigma\in\mathfrak{S}_n} \chi_\pi(\sigma)x^1_{\sigma(1)}\cdots x^n_{\sigma(n)}. \tag{13.5.3}$$

Here χ_π is the character of $[\pi]$. This is the original definition, due to Littlewood [**222**].

Note that the determinant and permanent are special cases of immanants.

13.5.3. A fast algorithm for $IM_{(2,1^{n-2})}$. In [**150**] the following fast algorithm was presented for $IM_2 := IM_{(2,1^{n-2})}$. First note that $\chi_{(2,1^{n-2})}(\sigma) = \mathrm{sgn}(\sigma)[F(\sigma) - 1]$, where $F(\sigma)$ is the number of elements fixed by the permutation σ. To see this note that $[(2, 1^{n-2})]$ is the tensor product of the sign representation with the standard representation on $\mathbb{C}^{n-1}$. Consider the trace of σ as an $n \times n$ permutation matrix; it is just the number of elements fixed by σ. But the trace of σ on the trivial representation is 1, hence the character of the standard representation on a permutation σ is the number of elements fixed by σ minus one.

Therefore

$$IM_2 := IM_{(2,1^{n-2})} = \sum_{\sigma\in\mathfrak{S}_n} \mathrm{sgn}(\sigma)[F(\sigma)-1]x^1_{\sigma(1)}\cdots x^n_{\sigma(n)}. \tag{13.5.4}$$

Exercise 13.5.3.1: Show that if $x \in Mat_{n\times n}$ is a matrix with all diagonal entries zero, then $IM_2(x) = -\det(x)$.

Now consider a matrix x of the form $x = x_0 + \lambda\,\mathrm{Id}$, where x_0 has all diagonal entries zero. I use an overline to denote the polarization of a polynomial (see §2.6.4):

$$\begin{aligned} IM_2(x) &= IM_2(x_0+\lambda\,\mathrm{Id}) \\ &= \overline{IM_2}(x_0^n) + n\lambda\overline{IM_2}(x_0^{n-1},\mathrm{Id}) \\ &\quad+ \binom{n}{2}\lambda^2\overline{IM_2}(x_0^{n-2},\mathrm{Id},\mathrm{Id}) + \cdots + \overline{IM_2}(\mathrm{Id}^n). \end{aligned}$$

The first term is just $-\det(x_0)$. Consider the second term, which is obtained from (13.5.4) by replacing, for each j, $x^j_{\sigma(j)}$ by $(\mathrm{Id})^j_{\sigma(j)} = \delta^j_{\sigma(j)}$, which means for each j one obtains the size $n-1$ immanant $IM_2((x_0)^{\hat{j}}_{\hat{j}}) = -\det((x_0)^{\hat{j}}_{\hat{j}})$. Thus the coefficient of λ is minus the sum of the size $n-1$ principal minors of x_0. But this is $e_{n-1}(\lambda)$. Continuing, each coefficient is, up to a constant independent of x, just $e_{n-j}(\lambda)$. But applying the same argument as in §13.4.2 to the $e_j(\lambda)$, we see that these can be computed using less than $O(n^4)$ multiplications.

Finally, recall that for any immanant, $IM_\pi(Dx) = IM_\pi(x)$ for each diagonal matrix D with determinant one. Using such a D, we may arrange that all diagonal entries of x are either equal or zero. The cost of such a multiplication is at worst $O(n^2)$.

Exercise 13.5.3.2: Show that having entries zero makes the algorithm even faster.

Strassen's theorem [**302**] that if there exists a circuit of polynomial size that uses division to compute a function of polynomial degree, then there also exists a circuit of polynomial size without division to compute the function, allows us to conclude:

Proposition 13.5.3.3. $IM_{(2,1^{n-2})} \in \mathbf{VP}$.

More generally, hook immanants of bounded width are in **VP**; see [**53**]. On the other hand, it is not known if these immanants are in $\mathbf{VP}_{ws}$.

13.5.4. Immanants close to the permanent are VNP-complete. Inspired by the work of Hartmann [**159**] and Barvinok [**18**], P. Bürgisser conjectured [**52**] that any family of immanant polynomials of polynomially growing width is **VNP**-complete. This conjecture has been proven in many cases, with the strongest results in [**45**]. A typical result is:

Theorem 13.5.4.1 ([**52**]). *For each fixed k, the sequence $(IM_{n-k,1^k})$ is* **VNP**-*complete.*

13.5.5. The α-permanent. Define

$$\mathrm{perm}_\alpha(x) := \sum_{\sigma\in\mathfrak{S}_n} \alpha^{cyc(\sigma)} x^1_{\sigma(1)} \cdots x^n_{\sigma(n)}, \tag{13.5.5}$$

where $cyc(\sigma)$ is the number of cycles in the cycle decomposition of σ. Since $\mathrm{perm}_\alpha \in S^n(E\otimes F)_0$, the weight zero subspace, it is clearly invariant under the action of $T_E \times T_F$. Moreover, since conjugation of a permutation σ by a permutation leaves the cycle type unchanged, the α-permanents have the same symmetry group as a general immanant.

The α-permanent appears as the coefficients of the multivariable Taylor expansion of

$$\det(\mathrm{Id} - DX)^{-\alpha},$$

where D is a diagonal matrix with variables $x_1, \ldots, x_n$ on the diagonal.

The α-permanent arises in statistics. It is the density function for a class of Cox processes called boson processes; see [**191**]. See [**322**] for a survey of its properties and uses.

13.6. Geometric complexity theory approach to $\overline{\mathbf{VP}_{ws}}$ v. $\overline{\mathbf{VNP}}$

I follow [**51**] for much of this section.

13.6.1. A geometric approach towards $\overline{\mathbf{VP}_{ws}} \neq \overline{\mathbf{VNP}}$. In a series of papers [**238**]–[**245**], K. Mulmuley and M. Sohoni outline an approach to **P** v. **NP** that they call the *Geometric Complexity Theory (GCT)* program. The GCT program translates the study of the hypersurfaces

$$\{\mathrm{perm}_m = 0\} \subset \mathbb{C}^{m^2} \quad \text{and} \quad \{\det_n = 0\} \subset \mathbb{C}^{n^2}$$

to a study of the orbit closures

$$\mathcal{P}erm^m_n := \overline{GL_{n^2} \cdot [\ell^{n-m}\,\mathrm{perm}_m]} \subset \mathbb{P}(S^n\mathbb{C}^{n^2}) \quad \text{and} \tag{13.6.1}$$

$$\mathcal{D}et_n := \overline{GL_{n^2} \cdot [\det_n]} \subset \mathbb{P}(S^n\mathbb{C}^{n^2}). \tag{13.6.2}$$

Here ℓ is a linear coordinate on $\mathbb{C}$, and one takes any linear inclusion $\mathbb{C} \oplus \mathbb{C}^{m^2} \subset \mathbb{C}^{n^2}$ to have $\ell^{n-m}\,\mathrm{perm}_m$ be a homogeneous degree n polynomial on $\mathbb{C}^{n^2}$. Note that if perm_m is an affine linear projection of $\det_n$, i.e., if $\ell^{n-m}\,\mathrm{perm}_m \in \mathrm{End}(\mathbb{C}^{n^2}) \cdot \det_n$, then $\mathcal{P}erm^m_n \subset \mathcal{D}et_n$. Mulmuley and Sohoni observe that Conjecture 13.4.4.3 is equivalent to:

Conjecture 13.6.1.1 ([**244**]). *There does not exist a constant $c \geq 1$ such that for sufficiently large m, $\mathcal{P}erm^m_{m^c} \subset \mathcal{D}et_{m^c}$.*

By the discussion in §13.4.4, $\mathcal{P}erm^m_n \subset \mathcal{D}et_n$ for $n = O(m^2 2^m)$.

Both $\mathcal{D}et_n$ and $\mathcal{P}erm^m_n$ are GL_{n^2}-varieties, so their ideals are GL_{n^2}-modules. In [**244**] it was proposed to solve Conjecture 13.6.1.1 by showing:

Conjecture 13.6.1.2 ([**244**]). *For all $c \geq 1$ and for infinitely many m, there exists an irreducible $GL_{m^{2c}}$-module appearing in $\mathcal{P}erm^m_{m^c}$, but not appearing in $\mathcal{D}et_{m^c}$.*

A program to prove Conjecture 13.6.1.2 is outlined in [**245**]. The paper [**51**] contains a detailed study of what mathematics would be needed to carry out the program as stated.

In this section I discuss three strategies to attack Conjecture 13.6.1.1: representation-theoretic study of the coordinate rings of the corresponding SL_{n^2}-orbits in affine space, (partially) resolving the extension problem from the GL-orbits to the GL-orbit closures, and direct geometric methods.

13.6.2. Coordinate rings and SL-orbits. Instead of considering $\mathcal{P}erm^m_n$ and $\mathcal{D}et_n$, one can consider the SL_{n^2}-orbit closures in affine space. The orbit $SL_{n^2} \cdot \det_n$ is already closed, so its coordinate ring, as an SL_{n^2}-module, can in principle be determined by representation theory alone, as explained below. $SL_{n^2} \cdot \ell^{n-m} \operatorname{perm}_m$ is not closed, but it is close enough to being closed that the coordinate ring of its closure, as an SL_{n^2}-module, can in principle be determined using representation theory. (This is called *partial stability* in [**245**].) In this subsection I explain how these procedures work and how this method can be combined with other techniques discussed below.

Here and in what follows, let $W = \mathbb{C}^{n^2} = E \otimes F$.

Coordinate rings of homogeneous spaces. The coordinate ring of a reductive group G has a left-right decomposition as a $(G-G)$-bimodule,

$$\mathbb{C}[G] = \bigoplus_{\lambda \in \Lambda^+_G} V^*_\lambda \otimes V_\lambda, \tag{13.6.3}$$

where V_λ denotes the irreducible G-module of highest weight λ and Λ^+_G indexes the irreducible (finite-dimensional) G-modules; see §16.1.4.

Let $H \subset G$ be a closed subgroup. The coordinate ring of the homogeneous space G/H is obtained by taking (right) H-invariants in (13.6.3), which gives rise to the (left) G-module decomposition

$$\mathbb{C}[G/H] = \mathbb{C}[G]^H = \bigoplus_{\lambda \in \Lambda^+_G} V^*_\lambda \otimes V^H_\lambda = \bigoplus_{\lambda \in \Lambda^+_G} (V^*_\lambda)^{\oplus \dim V^H_\lambda}. \tag{13.6.4}$$

The second equality holds because V_λ^H is a trivial (left) G-module. See [**192**, Thm. 3, Ch. II, §3], or [**268**, §7.3] for an exposition of these facts.

Example: $\mathbb{C}[GL(W)\cdot \det_n]$. Recall the stabilizer of $\det_n$ in $GL(W)$ from §13.2. Write $H = G(\det_n) = S(GL(E)\times GL(F))/\mu_n \rtimes \mathbb{Z}_2$ for this stabilizer.

By the discussion above, we need to determine

$$S_\pi(W)^H = S_\pi(E\otimes F)^H = \Big[\bigoplus_{\mu,\nu}(S_\mu E\otimes S_\nu F)^{\oplus k_{\pi\mu\nu}}\Big]^H,$$

where $k_{\pi\mu\nu}$ is the Kronecker coefficient; see §6.5.4. The trivial $SL(E)$-modules among the $S_\pi E$ are the $S_{\delta^n}E$. For $|\mu| = d$, the Schur module $S_\mu E$ can be characterized as $S_\mu E = \mathrm{Hom}_{\mathfrak{S}_d}([\mu], E^{\otimes d})$. Consider the vector space $K^\pi_{\mu\nu} := \mathrm{Hom}_{\mathfrak{S}_d}\big([\pi],[\mu]\otimes[\nu]\big)$ defined for partitions μ,ν,π of d. Its dimension is the Kronecker coefficient $k_{\pi\mu\nu}$. Thus

$$(S_\pi W)^{SL(E)\times SL(F)} = S_{\delta^n}E\otimes S_{\delta^n}F\otimes K^\pi_{\mu\nu}.$$

Next the $\mathbb{Z}_2$-action needs to be taken into account. Tensor product and composition give rise to a canonical $GL(E)\times GL(F)$-equivariant map,

$$\begin{aligned}\mathrm{Hom}_{\mathfrak{S}_d}([\mu], E^{\otimes d})\otimes \mathrm{Hom}_{\mathfrak{S}_d}([\nu], F^{\otimes d})\otimes \mathrm{Hom}_{\mathfrak{S}_d}\big([\pi],[\mu]\otimes[\nu]\big)&\\ \to \mathrm{Hom}_{\mathfrak{S}_d}\big([\pi],(E\otimes F)^d\big),&\end{aligned}$$

i.e., to

$$S_\mu E\otimes S_\nu F\otimes K^\pi_{\mu\nu} \to S_\pi(E\otimes F), \tag{13.6.5}$$

$$\alpha\otimes\beta\otimes\gamma \mapsto (\alpha\otimes\beta)\circ\gamma. \tag{13.6.6}$$

The action of $\tau\in\mathbb{Z}_2\subset GL(E\otimes E)$ determines an involution of $S_\pi(E\otimes E)$ (recall that $E = F$). The corresponding action on the left-hand side of (13.6.5) may be understood as follows: The isomorphism $[\mu]\otimes[\nu]\to[\nu]\otimes[\mu]$ defines a linear map $\sigma^\pi_{\mu\nu}\colon K^\pi_{\mu\nu}\to K^\pi_{\nu\mu}$ such that $\sigma^\pi_{\nu\mu}\sigma^\pi_{\mu\nu} = \mathrm{Id}$. Then

$$\tau\cdot((\alpha\otimes\beta)\circ\gamma) = (\beta\otimes\alpha)\circ\sigma^\pi_{\mu\nu}(\gamma). \tag{13.6.7}$$

In the case $\mu = \nu$, we obtain a linear involution $\sigma^\pi_{\mu\mu}$ of $K^\pi_{\mu\mu}$ with fixed set $\mathrm{Hom}_{\mathfrak{S}_d}([\pi],\mathrm{Sym}^2[\mu])$. Define the corresponding *symmetric Kronecker coefficient*:

$$sk^\pi_{\mu\mu} := \dim \mathrm{Hom}_{\mathfrak{S}_d}([\pi],\mathrm{Sym}^2[\mu]). \tag{13.6.8}$$

So $sk^\pi_{\mu\mu}$ equals the multiplicity of $[\pi]$ in $S^2[\mu]$. Note that $sk^\pi_{\mu\mu}\le k_{\pi\mu\mu}$ and the inequality may be strict. See [**226**] for some examples. The above discussion implies the following.

Proposition 13.6.2.1 ([**51**]).

$$\mathbb{C}[GL(W)\cdot \det_n]_{\text{poly}} = \bigoplus_{\delta\geq 0}\bigoplus_{\pi\,|\,|\pi|=n\delta}(S_\pi W^*)^{\oplus s k^\pi_{\delta^n\delta^n}}, \tag{13.6.9}$$

$$\mathbb{C}[\overline{GL(W)\cdot \det_n}]_\delta \subseteq \bigoplus_{\pi\,|\,|\pi|=n\delta}(S_\pi W^*)^{\oplus s k^\pi_{\delta^n\delta^n}}, \tag{13.6.10}$$

where the subscript "poly" denotes the subspace of modules occurring in the polynomial algebra.

Example: $\mathbb{C}[\ell^s p]$. Suppose $W = A\oplus A'\oplus B$ and $x=\ell^s p\in S^d W$, where $p\in S^{d-s}A$ is generic and $\dim A'=1$, $\ell\in A'\setminus\{0\}$. Assume that $d-s, \mathbf{a} > 3$. It is straightforward to show that, with respect to bases adapted to the splitting $W=A\oplus A'\oplus B$,

$$GL(W)(x) = \left\{ \begin{pmatrix}\psi\,\mathrm{Id} & 0 & *\\ 0 & \eta & *\\ 0 & 0 & *\end{pmatrix} \,\middle|\, \eta^s\psi^{d-s}=1\right\}.$$

First observe that the $GL(W)(x)$-invariants in $S_\pi W$ must be contained in $S_\pi(A\oplus A')$. By Proposition 6.7.2.2, this is the sum of the $S_{\pi'}A\otimes S^{|\pi|-|\pi'|}A'$, for $\pi\mapsto\pi'$. Here $\pi\mapsto\pi'$ means that $\pi_1\geq\pi_1'\geq\pi_2\geq\pi_2'\geq\cdots\geq 0$. The action of $GL(W)(x)$ on such a factor is by multiplication with $\psi^{|\pi'|}\eta^{|\pi|-|\pi'|}$; hence the conditions for invariance are $|\pi'|=\delta(d-s)$ and $|\pi|=\delta d$ for some δ. In summary:

$$\mathbb{C}[GL(W)\cdot x] = \bigoplus_{\delta\geq 0}\bigoplus_{\substack{|\pi|=\delta d,\ |\pi'|=\delta(d-s),\\ \pi\mapsto\pi'}}(S_\pi W^*)^{\oplus \dim S_{\pi'}A},$$

$$\mathbb{C}[\overline{GL(W)\cdot x}]_\delta \subseteq \bigoplus_{\substack{|\pi|=\delta d,\ |\pi'|=\delta(d-s),\\ \pi\mapsto\pi'}}(S_\pi W^*)^{\oplus \dim S_{\pi'}A}. \tag{13.6.11}$$

When we specialize to $p=\mathrm{perm}_m$, an explicit, but complicated description in terms of representation theory is given in [**51**]. Symmetric Kronecker coefficients, Kronecker coefficients, and $\dim(S_\mu E)_0^{\mathfrak{S}_m}$ need to be computed. As remarked in [**51**], by [**140**], if $|\mu|=n\delta$, then $\dim(S_\mu E)_0^{\mathfrak{S}_m} = \mathrm{mult}(S_\mu E, S^n(S^\delta E))$.

GL v. SL. As mentioned above, it is more desirable to work with GL than SL because the coordinate ring inherits a grading. However, each SL-module corresponds to an infinite number of GL-modules. Refer to Chapter 16 for explanations of the terminology in what follows.

The irreducible SL_M-modules are obtained by restricting the irreducible GL_M-modules. The weight lattice of Λ_{SL_M} of SL_M is $\mathbb{Z}^{M-1}$ and the dominant integral weights $\Lambda^+_{SL_M}$ are the nonnegative linear combinations of the fundamental weights $\omega_1, \dots, \omega_{M-1}$. A Schur module $S_\pi\mathbb{C}^M$ considered as an SL_M-module has highest weight

$$\lambda = \boldsymbol{\lambda}(\pi) = (\pi_1 - \pi_2)\omega_1 + (\pi_2 - \pi_3)\omega_2 + \cdots + (\pi_{M-1} - \pi_M)\omega_{M-1}.$$

Write $S_\pi\mathbb{C}^M = V_{\boldsymbol{\lambda}(\pi)}(SL_M)$.

Let $\boldsymbol{\pi}(\lambda)$ denote the smallest partition such that the GL_M-module $S_{\boldsymbol{\pi}(\lambda)}\mathbb{C}^M$, considered as an SL_M-module, is V_λ. That is, $\boldsymbol{\pi}$ is a map from $\Lambda^+_{SL_M}$ to $\Lambda^+_{GL_M}$, mapping $\lambda = \sum_{j=1}^{M-1} \lambda_j\omega_j$ to

$$\boldsymbol{\pi}(\lambda) = \Big(\sum_{j=1}^{M-1} \lambda_j, \sum_{j=2}^{M-1} \lambda_j, \dots, \lambda_{M-1}\Big).$$

With this notation,

$$\mathbb{C}[SL(W) \cdot \det_n] = \mathbb{C}[\overline{SL(W) \cdot \det_n}] = \bigoplus_{\lambda \in \Lambda^+_{SL(W)}} (V_\lambda^*)^{\oplus sk^{\boldsymbol{\pi}(\lambda)}_{\delta n \delta n}}, \quad \delta = |\boldsymbol{\pi}(\lambda)|/n.$$

As mentioned above, despite the failure of $SL(W) \cdot \ell^{n-m}\operatorname{perm}_m$ to be closed, it is still possible to study its coordinate ring via representation theory. The following is a special case of a result in [**245**]:

Theorem 13.6.2.2 ([**51, 245**]). *Let $W = A \oplus A' \oplus B$, $\dim A = \mathbf{a}$, $\dim A' = 1$, $z \in S^{d-s}A$, $\ell \in A' \setminus \{0\}$. Assume that $SL(A) \cdot z$ is closed. Write $v = \ell^s z$. Set $R = SL(A)$, and take P to be the parabolic subgroup of $GL(W)$ preserving $A \oplus A'$; write its Levi decomposition as $P = KU$, where U is unipotent, so $K = GL(A \oplus A') \times GL(B)$. Assume further that $U \subset GL(W)([z]) \subset P$.*

(1) *A module $S_\nu W^*$ occurs in $\mathbb{C}[\overline{GL(W) \cdot v}]_\delta$ if and only if $S_\nu(A \oplus A')^*$ occurs in $\mathbb{C}[\overline{GL(A \oplus A') \cdot v}]_\delta$. There is then a partition ν' such that $\nu \mapsto \nu'$ and $V_{\boldsymbol{\lambda}(\nu')}(SL(A)) \subset \mathbb{C}[\overline{SL(A) \cdot [v]}]_\delta$.*

(2) *Conversely, if $V_\lambda(SL(A)) \subset \mathbb{C}[\overline{SL(A) \cdot [v]}]_\delta$, then there exist partitions π, π' such that $S_\pi W^* \subset \mathbb{C}[\overline{GL(W) \cdot [v]}]_\delta$, $\pi \mapsto \pi'$ and $\boldsymbol{\lambda}(\pi') = \lambda$.*

(3) *A module $V_\lambda(SL(A))$ occurs in $\mathbb{C}[\overline{SL(A) \cdot [v]}]$ if and only if it occurs in $\mathbb{C}[SL(A) \cdot v]$.*

Theorem 13.6.2.2 establishes a connection between $\mathbb{C}[\overline{GL(W) \cdot v}]$, which we are primarily interested in but cannot compute, and $\mathbb{C}[SL(A) \cdot v]$, which in principle can be described using (13.6.4).

While there is an explicit description of the coordinate rings in terms of representation-theoretic data, obtaining enough information to make any progress on Conjecture 13.6.1.2 seems beyond the reach of current technology.

13.6.3. The extension problem. The GL-orbit closures come with a grading, which, if they could be exploited, would vastly simplify the problem. To pass from the coordinate ring of the GL-orbit to the coordinate ring of the closure, one must determine which modules of functions on the orbit extend to give modules of functions on the boundary. For example, $\mathbb{C}[\mathbb{C}^*] = \mathbb{C}[z, z^{-1}]$, and the functions that extend are $\mathbb{C}[z]$, as z^{-k} has a pole at the origin. Recently S. Kumar [**196**] proved that neither $\mathcal{P}erm_n^m$ nor $\mathcal{D}et_n$ is *normal* (i.e., the rings $\mathbb{C}[\mathcal{P}erm_n^m]$ and $\mathbb{C}[\mathcal{D}et_n]$ are not integrally closed; see e.g., [**289**, p. 128]), which makes the extension problem considerably more difficult. (In the first case, $n > 2m$ is required in Kumar's proof, but that is the situation of interest.)

A first step to solving the extension problem is thus to determine the components of the boundary. The action of $GL(W)$ on S^nW extends to $\mathrm{End}(W)$. Using this action, define a rational map

$$\psi_n : \mathbb{P}(\mathrm{End}(W)) \dashrightarrow \mathbb{P}(S^nW^*),$$
$$[u] \mapsto [\det_n \circ u].$$

Its indeterminacy locus $Ind(\psi_n)$ is, set-theoretically, given by the set of $[u]$ such that $\det(u.X) = 0$ for all $X \in W = Mat_{n\times n}$. Thus

$$Ind(\psi_n) = \{u \in \mathrm{End}(W) \mid \mathrm{image}(u) \subset \hat{\sigma}_{n-1}\},$$

where $\hat{\sigma}_{n-1} \subset W$ denotes the hypersurface of noninvertible matrices. Since $\mathrm{image}(u)$ is a vector space, this relates to the classical problem of determining linear subspaces on $\sigma_{n-1} \subset \mathbb{P}(\mathrm{End}(W))$; see, e.g., [**120**].

By Hironaka's theorems [**163**] one can in principle resolve the indeterminacy locus of ψ_n by a sequence of smooth blowups (see §15.2.1 for the definition of a blowup), and $\mathcal{D}et_n$ can then be obtained as the image of the resolved map.

A component of the boundary of $\mathcal{D}et_n$. Recall that when n is odd, $\mathbb{P}\Lambda^2 E \subset \sigma_{n-1}(Seg(\mathbb{P}E \times \mathbb{P}E))$. Decompose a matrix X into its symmetric and skew-symmetric parts X_S and X_Λ. Define a polynomial $P_\Lambda \in S^n(E\otimes E)^*$ by letting

$$P_\Lambda(X) = \det_n(X_\Lambda, \ldots, X_\Lambda, X_S).$$

This is zero for n even so assume that n is odd. $P_\Lambda = P_{\Lambda,n}$ can be expressed as follows: Let $Pf_i(X_\Lambda)$ denote the Pfaffian of the skew-symmetric matrix, of even size, obtained from X_Λ by suppressing its i-th row and column and write $X_S = (s_{ij})$. Then

$$P_\Lambda(X) = \sum_{i,j} s_{ij} Pf_i(X_\Lambda) Pf_j(X_\Lambda).$$

Proposition 13.6.3.1 ([**213**]). $[P_{\Lambda,n}] \in \mathcal{D}et_n$. *Moreover,* $\overline{GL(W)\cdot P_\Lambda}$ *is an irreducible codimension one component of the boundary of* $\mathcal{D}et_n$, *not contained in* $\mathrm{End}(W)\cdot[\det_n]$. *In particular,* $\overline{dc}(P_{\Lambda,n}) = n < dc(P_{\Lambda,n})$.

Proof. For $t \neq 0$ let $u_t \in GL(W)$ be defined by $u_t(X_\Lambda + X_S) = X_\Lambda + tX_S$. Since the determinant of a skew-symmetric matrix of odd size vanishes,

$$(u_t\cdot \det_n)(X) = \det_n(X_\Lambda + tX_S) = nt\det_n(X_\Lambda,\dots,X_\Lambda,X_S) + O(t^2),$$

and therefore $u_t\cdot[\det_n]$ converges to $[P_\Lambda]$ as t goes to zero, proving the first assertion.

To prove the second assertion, compute the stabilizer of P_Λ inside $GL(E{\otimes}E)$. The action of $GL(E)$ on $E{\otimes}E$ by $X \mapsto gXg^T$ preserves P_Λ up to scale, and the Lie algebra of the stabilizer of $[P_\Lambda]$ is thus a $GL(E)$-submodule of $\mathrm{End}(E{\otimes}E)$. We have the decomposition into $GL(E)$-modules:

$$\begin{aligned}\mathrm{End}(E{\otimes}E) &= \mathrm{End}(\Lambda^2E\oplus S^2E)\\ &= \mathrm{End}(\Lambda^2E)\oplus\mathrm{End}(S^2E)\oplus\mathrm{Hom}(\Lambda^2E,S^2E)\oplus\mathrm{Hom}(S^2E,\Lambda^2E).\end{aligned}$$

Moreover, $\mathrm{End}(\Lambda^2E) = \mathfrak{gl}_n\oplus E\Lambda$ and $\mathrm{End}(S^2) = \mathfrak{gl}_n\oplus ES$, where $E\Lambda$ and ES are distinct irreducible $GL(E)$-modules. Similarly, $\mathrm{Hom}(\Lambda^2E,S^2E) = \mathfrak{sl}_n\oplus E\Lambda S$ and $\mathrm{Hom}(S^2E,\Lambda^2E) = \mathfrak{sl}_n\oplus ES\Lambda$, where $E\Lambda S$ and $ES\Lambda$ are irreducible, pairwise distinct and different from $E\Lambda$ and ES. Then one can check that the modules $E\Lambda, ES, E\Lambda S, ES\Lambda$ are not contained in the stabilizer, and that the contribution of the remaining terms is isomorphic with $\mathfrak{gl}_n\oplus\mathfrak{gl}_n$. In particular, it has dimension $2n^2$, which is one more than the dimension of the stabilizer of $[\det_n]$. This implies that $\overline{GL(W)\cdot P_\Lambda}$ has codimension one in $\mathcal{D}et_n$. Since it is not contained in the orbit of the determinant, it must be an irreducible component of its boundary. Since the zero set is not a cone (i.e., the equation involves all the variables), P_Λ cannot be in $\mathrm{End}(W)\cdot\det_n$, which consists of $GL(W)\cdot\det_n$ plus cones. $\square$

The hypersurface defined by P_Λ has interesting properties.

Proposition 13.6.3.2 ([**213**]). *The dual variety of the hypersurface* $Z(P_\Lambda)$ *is isomorphic to the Zariski closure of*

$$\mathbb{P}\{v^2\oplus v\wedge w\in S^2\mathbb{C}^n\oplus\Lambda^2\mathbb{C}^n,\ v,w\in\mathbb{C}^n\}\subset\mathbb{P}(\mathbb{C}^n{\otimes}\mathbb{C}^n).$$

As expected, $Z(P_\Lambda)^\vee$ is similar to $Seg(\mathbb{P}^{n-1}\times\mathbb{P}^{n-1})$.

13.6.4. Direct geometric methods. In §13.6.2 the focus was on $\mathbb{C}[\mathcal{D}et_n] = S^\bullet W^*/I(\mathcal{D}et_n)$ and $\mathbb{C}[\mathcal{P}erm_n^m]$. In this subsection I switch perspective to focus on the ideals, looking for explicit equations that $\mathcal{D}et_n$ satisfies but $\mathcal{P}erm_n^m$ fails to satisfy when m is large.

To obtain such equations, it is useful to ask: *How is the hypersurface* $\mathrm{Zeros}(\det_n) \subset \mathbb{P}^{n^2-1}$ *special?* Say it satisfies some property (P) that is preserved under degenerations; more precisely, (P) holds for all hypersurfaces in $\mathcal{D}et_n$. Then one can study the variety $\Sigma_P \subset \mathbb{P}(S^n\mathbb{C}^{n^2})$ of all hypersurfaces satisfying this property and find equations for it, which will also be equations for $\mathcal{D}et_n$.

In [**213**] this strategy was carried out for the property of having a degenerate dual variety: Recall from §8.2 that $\mathrm{Zeros}(\det_n)^\vee = Seg(\mathbb{P}^{n-1} \times \mathbb{P}^{n-1})$, whereas for a general polynomial, the dual variety of its zero set is a hypersurface.

Let $\mathcal{D}ual_{q,d,N} \subset \mathbb{P}S^d\mathbb{C}^N$ denote the Zariski closure of the set of irreducible hypersurfaces with degenerate dual varieties of dimension q. In [**213**] equations for $\mathcal{D}ual_{q,d,N}$ were found. A large component of $\mathcal{D}ual_{q,d,N}$ consists of the cones, $Sub_q(S^d\mathbb{C}^N)$. Note that $Sub_{2n-2}(S^n\mathbb{C}^{n^2})$ has dimension much greater than $\mathcal{D}et_n$. Nevertheless:

Theorem 13.6.4.1 ([**213**])**.** *The scheme* $\mathcal{D}ual_{2n-2,n,n^2}$ *is smooth at* $[\det_n]$, *and* $\mathcal{D}et_n$ *is an irreducible component of* $\mathcal{D}ual_{2n-2,n,n^2}$.

The equations for $\mathcal{D}ual_{q,d,N}$ were found by specializing Segre's dimension formula of §8.2.4 in the form that if $\mathrm{Zeros}(P) \in \mathcal{D}ual_{q,d,N}$, then P divides the determinant of its Hessian restricted to any linear space of dimension $q+3$. One then restricts this condition to an affine line, where one can perform Euclidean division of univariate polynomials. One then extracts the part of the remainder under the division that is invariant under the choice of affine line in the projective line. The result is explicit equations, including the following:

Theorem 13.6.4.2 ([**213**])**.** *The irreducible* SL_{n^2}*-module with highest weight*

$$2n(n-1)(n-2)\omega_1 + (2n^2 - 4n - 1)\omega_2 + 2\omega_{2n+1}$$

is in $I_{n^2-n}(\mathcal{D}ual_{2n-2,n,n^2})$.

From these considerations, one also obtains the bound:

Theorem 13.6.4.3 ([**213**])**.** $\overline{dc}(\mathrm{perm}_m) \geq \frac{m^2}{2}$.

13.7. Other complexity classes via polynomials

In [**244**] the authors apply GCT to other sequences of polynomials beyond the permanent and determinant to capture other complexity classes. Here are some additional sequences of polynomials.

13.7.1. Circuits. The class **VP** has a restriction, considered undesirable by some in computer science, that the degrees of polynomials are bounded.

Instead, one might define a complexity class consisting of sequences of polynomials (f_n) subject to the requirement that the sequences are computable by arithmetic circuits of size polynomial in n. Such a class includes **VP** and perhaps more closely models **P**. A sequence of polynomials complete for this class was defined in [**244**, §6] as follows: Define a generic circuit of width n and depth ℓ as follows: on the first row put n variables $x_1 =: v_1^0, \dots, x_n =: v_n^0$. The second row consists of, e.g., n addition gates, each with a new variable. The sums of all the x-variables are computed n times and multiplied by the new variable. Then the third row should be n multiplication gates, each computes the product of each of the terms in the row above multiplied by a new variable. The fourth row is again addition gates, etc., up to the ℓ-th row. The result is a polynomial in ℓn^2+n variables of degree approximately $n^{\frac{\ell}{2}}$.

This polynomial, called $H_{n,\ell}$, has a symmetry group including many copies of $\mathfrak{S}_n$ as well as a torus $(\mathbb{C}^*)^{(n-1)}$ for each multiplication gate.

The formula size is superexponential.

13.7.2. A polynomial modeling NP. Let X_0, X_1 be $n\times n$ matrices with variable entries. For $\sigma : [n] \to \{0,1\}$ let X_σ denote the matrix whose i-th column is the i-th column of $X_{\sigma(i)}$. Write $X = (X_0, X_1)$, which we may consider as linear coordinates on $\mathbb{C}^{2n^2}$. Define

$$E(X) = \prod_{\sigma:[n]\to\{0,1\}} \det(X_\sigma).$$

Note that $\deg(E(X)) = n^{2^n}$.

If $E(X)$ is zero, there must exist σ such that $\det(X_\sigma) = 0$. Thus, if one is handed σ, it is very fast to verify whether $E(X)$ is zero. We conclude:

Proposition 13.7.2.1. *Deciding if $E(X) = 0$ belongs to* **NP**.

Gurvits [**153**] showed moreover that deciding whether $E(X) = 0$ is **NP**-complete.

13.7.3. Expression size is a familiar polynomial. In [**22**] the authors show that the sequence (M_3^n) of the matrix multiplication operator for n square matrices of size 3 is $\mathbf{VP}_e$-complete. This is the operator $(X_1, \dots, X_n) \mapsto \operatorname{tr}(X_1 \cdots X_n)$. If $A_j = \mathbb{C}^9$, $1 \le j \le n$, then we view this operator as a polynomial by the inclusion $A_1 \otimes \cdots \otimes A_n \subset S^n(A_1 \oplus \cdots \oplus A_n)$.

13.8. Vectors of minors and homogeneous varieties

In this section I describe another way to recover defining equations for Grassmannians that works for homogeneous varieties that are *cominuscule*—the two other cases of relevance for tensors will be the *Lagrangian Grassmannian* and the *Spinor varieties*, which arise naturally in statistics and complexity theory. See [**203**] for the general cominuscule case.

13.8.1. The Grassmannian and minors of matrices. Consider the Grassmannian $G(k,W)$ where $\dim W = k+p$. It may be locally parametrized as follows: Let $E = \langle e_1, \dots, e_k\rangle$, $F = \langle f_1, \dots, f_p\rangle$ be complementary subspaces in W, so $W = E \oplus F$. Parametrize the k-planes near E as follows: Let $x = (x^u_i)$, $1 \le i, j \le k$, $1 \le s, u \le p$, and write

$$E(x) = \langle e_1 + x^s_1 f_s, \dots, e_k + x^s_k f_s\rangle. \tag{13.8.1}$$

Exercises 13.8.1.1:

(1) Verify that this indeed gives a local parametrization. Which k-planes are not included?

(2) Show that this parametrization allows one to recover the fact that the tangent space at E is isomorphic to $E^*{\otimes}V/E$.

The open subset of $G(k, k+p)$ obtained by the image of $\mathbb{C}^p \otimes \mathbb{C}^k$ via (13.8.1) has thus been parametrized by the set of $k \times p$ matrices.

Now consider the image of $E(x)$ under the Plücker embedding. The local parametrization about $E = [v(0)]$ is

$$[v(x^s_i)] = [(e_1 + x^s_1 f_s) \wedge \cdots \wedge (e_k + x^s_k f_s)].$$

In what follows I will also need to work with $G(k, W^*)$. In dual bases, a local parametrization about $E^* = \langle e^1, \dots, e^k\rangle = [\alpha(0)]$ is

$$[\alpha(y^s_j)] = [(e^1 + y^1_s f^s) \wedge \cdots \wedge (e^k + y^k_s f^s)].$$

I next explain how to interpret the image in the Plücker embedding of open subset of $G(k,W)$ described above as the vector of minors for $E^*{\otimes}F$.

$\hat{G}(k,W)$ *as vectors of minors.* Recall the decomposition of $\Lambda^k(E \oplus F)$ as a $GL(E) \times GL(F)$ module:

$$\begin{aligned}\Lambda^k(E \oplus F) =&(\Lambda^k E{\otimes}\Lambda^0 F) \oplus (\Lambda^{k-1}E{\otimes}\Lambda^1 F) \oplus (\Lambda^{k-2}E{\otimes}\Lambda^2 F)\\ &\oplus \cdots \oplus (\Lambda^1 E{\otimes}\Lambda^{k-1}F) \oplus (\Lambda^0 E{\otimes}\Lambda^k F).\end{aligned}$$

Assume that E has been equipped with a volume form in order to identify $\Lambda^s E \simeq \Lambda^{k-s}E^*$, so one has the $SL(E) \times GL(F)$ decomposition:

$$\begin{aligned}\Lambda^k(E \oplus F) =&(\Lambda^0 E^*{\otimes}\Lambda^0 F) \oplus (\Lambda^1 E^*{\otimes}\Lambda^1 F) \oplus (\Lambda^2 E^*{\otimes}\Lambda^2 F)\\ &\oplus \cdots \oplus (\Lambda^{k-1}E^*{\otimes}\Lambda^{k-1}F) \oplus (\Lambda^k E^*{\otimes}\Lambda^k F).\end{aligned}$$

Recall from §2.7.3 that $\Lambda^s E^*{\otimes}\Lambda^s F \subset S^s(E^*{\otimes}F)$ has the geometric interpretation as the space of $s \times s$ minors on $E{\otimes}F^*$. That is, choosing bases, write an element f of $E{\otimes}F^*$ as a matrix; then a basis of $\Lambda^s E^*{\otimes}\Lambda^s F$ evaluated on f will give the set of $s \times s$ minors of f.

To see these minors explicitly, note that the bases of E^*, F induce bases of the exterior powers. Expanding out v above in such bases (recall that the summation convention is in use), we obtain

$$\begin{aligned} v(x_i^s) = e_1 \wedge \cdots \wedge e_k & \\ &+ x_i^s e_1 \wedge \cdots \wedge e_{i-1} \wedge e_s \wedge e_{i+1} \wedge \cdots \wedge e_k \\ &+ (x_i^s x_j^t - x_j^s s_i^t) e_1 \wedge \cdots \wedge e_{i-1} \wedge e_s \wedge e_{i+1} \wedge \cdots \\ &\quad \wedge e_{j-1} \wedge e_t \wedge e_{j+1} \cdots \wedge e_k \\ &+ \cdots , \end{aligned}$$

i.e., writing v as a row vector in the induced basis, we have

$$v = (1, x_i^s, x_i^s x_j^t - x_i^t x_j^s, \ldots) = (1, \Delta_{i,s}(x), \ldots, \Delta_{I,S}(x), \ldots),$$

where $I = (i_1, \ldots, i_p)$, $S = (s_1, \ldots, s_p)$, and $\Delta_{I,S}(x)$ denotes the corresponding $p \times p$ minor of x. Similarly $\alpha = (1, y_s^j, y_s^j y_t^i - y_s^i y_t^j, \ldots)$.

Remark 13.8.1.2. There is a natural map $G(k, V) \times G(l, W) \to G(k+l, V \oplus W)$ given by $[v_1 \wedge \cdots \wedge v_k] \times [w_1 \wedge \cdots \wedge w_l] \mapsto [v_1 \wedge \cdots \wedge v_k \wedge w_1 \wedge \cdots \wedge w_l]$. In terms of vectors of minors, if $v = v(x_i^s)$ is given as the minors of a matrix x and $w = w(y_u^g)$, then the image $k+l$ plane is given as the minors of the block matrix

$$\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}.$$

This elementary observation is important for holographic algorithms.

Equations for Grassmannians. The minors of a matrix satisfy relations. Choose $q \leq k$ and I, J of cardinality q; the minor $\Delta_J^I(x)$ has various Laplace expansions which are equal. Label the coordinates on $\Lambda^k V = \bigoplus_{s=0}^k \Lambda^s E^* \otimes \Lambda^s F$ by $z_{I,J}$, $0 \leq |I| = |J| \leq k$, $I \subset \{1, \ldots, k\}$, $J \subset 1, \ldots, p\}$. The expansions give quadratic relations among the $z_{I,J}$ which must be satisfied if the vector $(z_{I,J})$ is the set of minors of a $k \times p$ matrix. For a complete list of these relations in coordinates, see e.g. [**268**, p. 503, eq. (2.1.4)] or [**303**, §3.1]. These equations span the $GL(V)$-modules

$$\sum_{s>0} S_{2^{k-2s}1^{4s}} V^* = I_2(G(k, V)) \subset S^2(\Lambda^2 V^*),$$

which generate the ideal.

Fast Pairings.

Proposition 13.8.1.3. *The characteristic polynomial of a product of a $k \times \ell$ matrix x with an $\ell \times k$ matrix y is:*

$$\text{charpoly}(xy)(t) = \det(\text{Id}_E + txy) = \sum_{\substack{I \subset [k], S \subset [\ell] \\ |I| = |S|}} \Delta_{I,S}(x) \Delta_{S,I}(y) t^{|I|}. \tag{13.8.2}$$

Proof. For a linear map $f : A \to A$, recall the induced linear maps $f^{\wedge k} : \Lambda^k A \to \Lambda^k A$, where, if one chooses a basis of A and represents f by a matrix, then the entries of the matrix representing $f^{\wedge k}$ in the induced basis on $\Lambda^k A$ will be the $k \times k$ minors of the matrix of f. In particular, if $\dim A = \mathbf{a}$, then $f^{\wedge \mathbf{a}}$ is multiplication by a scalar which is $\det(f)$.

Recall the decomposition:

$$\mathrm{End}(E^* \oplus F) = (E^* {\otimes} F) \oplus (E^* {\otimes} E) \oplus (F {\otimes} F^*) \oplus (F^* {\otimes} E).$$

To each $x \in E^*{\otimes}F$, $y \in E{\otimes}F^*$, associate the element

$$-x + \mathrm{Id}_E + \mathrm{Id}_F + y \in \mathrm{End}(E^* \oplus F). \tag{13.8.3}$$

Note that

$$\det \begin{pmatrix} \mathrm{Id}_E & -x \\ y & \mathrm{Id}_F \end{pmatrix} = \det(\mathrm{Id}_E + xy).$$

Consider

$$\begin{aligned}
&(-x + \mathrm{Id}_E + \mathrm{Id}_F + ty)^{\wedge n} \\
&\quad = (\mathrm{Id}_E)^{\wedge k} \wedge (\mathrm{Id}_F)^{\wedge(n-k)} + t(\mathrm{Id}_E)^{\wedge k-1} \wedge (\mathrm{Id}_F)^{\wedge(n-k-1)} \wedge (-x) \wedge y \\
&\qquad + t^2(\mathrm{Id}_E)^{\wedge(k-2)} \wedge (\mathrm{Id}_F)^{\wedge(n-k-2)} \wedge (-x)^{\wedge 2} \wedge y^{\wedge 2} + \cdots \\
&\qquad + t^k(\mathrm{Id}_F)^{\wedge(n-2k)} \wedge (-x)^{\wedge k} \wedge y^{\wedge k}.
\end{aligned}$$

Let

$$e^1 \wedge \cdots \wedge e^k \wedge f_1 \wedge \cdots \wedge f_{n-k} \in \Lambda^n(E^*{\otimes}F)$$

be a volume form. All that remains to be checked is that, when the terms are reordered, the signs work out correctly, which is left to the reader. $\square$

Remark 13.8.1.4. The decomposition $\Lambda^k(E \oplus F) = \sum_{j=0}^k \Lambda^j E {\otimes} \Lambda^{k-j} F$ gives a geometric proof of the combinatorial identity

$$\binom{k+p}{k} = \sum_j \binom{k}{j}\binom{p}{k-j}.$$

13.8.2. The Lagrangian Grassmannian and minors of symmetric matrices. Now assume that $k = l$ and restrict the map from the space of $k \times k$ matrices to the symmetric matrices. One obtains the vectors of minors of symmetric matrices. The resulting variety lives in a linear subspace of $\mathbb{P}(\Lambda^k(S^2E))$ because there are redundancies among the minors.

Exercise 13.8.2.1: Show that the number of irredundant minors is $\binom{2k}{k} - \binom{2k-2}{k}$.

The closure of the image of this map is again homogeneous. To see this let $\omega \in \Lambda^2 V^*$ be nondegenerate and such that E, F are isotropic for ω, i.e., $\omega|_E = 0, \omega|_F = 0$.

Exercise 13.8.2.2: Show that ω establishes an identification $E^* \simeq F$.

Choose bases such that $\omega = e_1 \wedge f_1 + \cdots + e_k \wedge f_k$. Let $x = (x_i^s)$ and write

$$E(x) = \langle e_1 + x_1^s f_s, \ldots, e_k + x_k^s f_s \rangle$$

for a nearby k-plane as above.

Exercise 13.8.2.3: Show that $E(x)$ is isotropic if and only if x is a symmetric matrix.

Exercise 13.8.2.3 shows that the vectors of minors of symmetric matrices form an open subset of the *Lagrangian Grassmannian* $G_{Lag}(k, 2k) = Sp(V, \omega)/P$, where $P \subset Sp(V, \omega)$ is the subgroup preserving E. Since being isotropic is a closed property and using the (not proven here) fact that $G_{Lag}(k, 2k)$ is connected, taking the Zariski closure of this open set gives $G_{Lag}(k, 2k)$. As before, the relations among the minors furnish quadratic equations.

For those familiar with representation theory, as $C_k = SP_{2k}(\mathbb{C})$-modules, these equations are

$$\sum_{s>0} V_{2\omega_{k-2s}} = I_2(G_{Lag}(k, V)) \subset S^2(\Lambda^k V/\omega \wedge \Lambda^{k-2} V)^*,$$

which follows from Proposition 6.10.6.1.

Exercise 13.8.2.4: What combinatorial identity is implied by the $GL(E)$-module decomposition of $\Lambda^k(S^2 E)$?

13.8.3. Varieties of principal minors. For some applications, one is interested only in the principal minors of square matrices and symmetric matrices. These *varieties of principal minors* have been studied recently [**166, 253, 221**]. They are projections of homogeneous varieties. Although they are not homogeneous, they are G-varieties. In the case of symmetric matrices the group G is $SL_2^{\times k} \rtimes \mathfrak{S}_k$. For regular matrices, the group is $GL_2^{\times n} \rtimes \mathfrak{S}_n \subset GL_{2n}$. Here the reductive group G can be thought of as an enlargement of the normalizer of the torus in $SP(V)$. The point is that if one decomposes V_{ω_k} as a G-module, the center of projection is a G-submodule and thus the quotient is as well.

While in principle one could recover the equations using elimination theory, in practice this calculation is not feasible. Using geometric methods, L. Oeding [**253**] shows that the variety of vectors of principal minors of symmetric matrices is cut out set-theoretically by an irreducible G-module in degree four. Namely, writing $(\mathbb{C}^2)^{\otimes k} = V_1 \otimes \cdots \otimes V_k$, the module is the sum of the various tensor products of three $S_{22}V_j$'s with $(k-3)$ of the $S_4 V_j$'s.

In the case $k = 3$, this is Cayley's hyperdeterminant and the variety is the dual variety of $Seg(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)$. The $k = 3$ case was observed in [**166**].

13.8.4. The spinor variety and Pfaffians. The set of minors of skew-symmetric $k \times k$ matrices does furnish a homogeneous variety to which I return below. With skew-symmetric matrices, it is more natural to deal with sub-Pfaffians than minors. The space of sub-Pfaffians of size k is parametrized by $\Lambda^k E$. (It is indexed by subsets $I \subset \{1, \dots, k\}$.) However, only subsets with $|I|$ even are relevant, so assume from now on that $k = 2q$ is even and throw away the trivial sub-Pfaffians.

Thus consider the map

$$\begin{aligned} \Lambda^2 E &\to \bigoplus_{j=0}^{q} \Lambda^{2j} E =: \mathcal{S}_+, \\ x &\mapsto \mathrm{sPfaff}(x), \end{aligned}$$

where $\mathrm{sPfaff}(x)$ is the set of sub-Pfaffians of x. One also has a map $\Lambda^2 E^* \to \bigoplus_{j=0}^{q} \Lambda^{2j+1} E^* =: \mathcal{S}_+^*$.

Taking the closure of the image in projective space yields a $\binom{k}{2}$-dimensional subvariety of $\mathbb{P}^{2^{n-1}-1}$. Endow $V = E \oplus F$ with a symmetric quadratic form such that E, F are isotropic. Arguing as above, one shows that the skew-symmetric matrices give a local parametrization of the isotropic q-planes near E, and the closure of this set is a component of the set of isotropic q-planes, a homogeneous variety called the *spinor variety* $\mathbb{S}_k$. It is homogeneous for the group $Spin_{2k}$ whose Lie algebra is $\mathfrak{so}(2k)$. The group is the simply connected double cover of SO_{2k}.

The first few spinor varieties are classical varieties in disguise (corresponding to coincidences of Lie groups in the first two cases and triality in the third):

$$\begin{aligned} \mathbb{S}_2 &= \mathbb{P}^1 \sqcup \mathbb{P}^1 \subset \mathbb{P}^2, \\ \mathbb{S}_3 &= \mathbb{P}^3 \subset \mathbb{P}^3, \\ \mathbb{S}_4 &= Q^6 \subset \mathbb{P}^7. \end{aligned}$$

Although the codimension grows very quickly, the spinor varieties are small in these cases. The next case $\mathbb{S}_5 \subset \mathbb{P}^{15}$ is of codimension 5 and is not isomorphic to any classical homogeneous variety.

For the traditional interpretation of spinor varieties as maximal isotropic subspaces on a quadric, see any of [**85, 162, 204**]. See [**203**] for the relationship between the description above and the description as parametrizing isotropic subspaces.

The following elementary observation is an analog of Remark 13.8.1.2, since it plays a crucial role in holographic algorithms, I record it as a proposition:

Proposition 13.8.4.1. *Let $E = E_1 \oplus \cdots \oplus E_k$, consider the spinor varieties $\hat{\mathbb{S}}_{E_j} \in \Lambda^{even} E_j$ and $\hat{\mathbb{S}}_E \in \Lambda^{even} E$. Then there is a natural inclusion $f : \hat{\mathbb{S}}_{E_1} \times \cdots \times \hat{\mathbb{S}}_{E_k} \to \hat{\mathbb{S}}_E$. If $v_j \in \hat{\mathbb{S}}_{E_j}$ is given as $v_j = \mathrm{sPfaff}(x_j)$, where x_j is a matrix, then $f(v_1, \ldots, v_k) = \mathrm{sPfaff}(x)$, where x is a block diagonal matrix with blocks the x_j.*

Equations for spinor varieties. There are classical identities among Pfaffians that arise from a variant of equating various Laplace expansions of the determinant:

Theorem 13.8.4.2. *Let $x = (x^i_j)$ be an $n \times n$ skew-symmetric matrix. Let $(j, I) = (i_1, \ldots, i_p, j, i_{p+1}, \ldots, i_K)$, where $i_p < j < i_{p+1}$. Then for each $I = (i_1, \ldots, i_K) \subset \{1, \ldots, n\}$, $J = (j_1, \ldots, j_L) \subset \{1, \ldots, n\}$,*

$$\sum_{s=1}^{L} (-1)^s Pf_{(j_s,I)}(x) Pf_{J\backslash j_s}(x) + \sum_{t=1}^{K} (-1)^t Pf_{I\backslash i_t}(x) Pf_{(i_t,J)}(x) = 0.$$

Corollary 13.8.4.3. *Give $\mathbb{C}^{2^{n-1}}$ linear coordinates z_I, where $I \subset \{1, \ldots, n\}$. Let $(j, I) = (i_1, \ldots, i_p, j, i_{p+1}, \ldots, i_K)$, where $i_p < j < i_{p+1}$. Equations for the spinor variety $\mathbb{S}_n \subset \mathbb{P}^{2^{n-1}-1}$ are*

$$\sum_{s=1}^{L} (-1)^s z_{(j_s,I)} z_{J\backslash j_s} + \sum_{t=1}^{K} (-1)^t z_{I\backslash i_t} z_{(i_t,J)} = 0.$$

For those familiar with representation theory, as $D_n = Spin_{2n}$-modules these equations are

$$\sum_{s>0} V_{\omega_{n-4s}} = I_2(\mathbb{S}_+) \subset S^2(V^*_{\omega_k}),$$

which follows from Proposition 6.10.6.1. For more on these equations, see [**228**, §2.4] or [**85**].

Using the set of minors instead of Pfaffians yields two copies of the quadratic Veronese reembedding of the spinor variety, or more precisely the quadratic Veronese reembedding of the spinor variety and the isomorphic variety in the dual space $v_2(\mathbb{S}) \sqcup v_2(\mathbb{S}_*) \subset \mathbb{P}(S^2\mathbb{C}^{2^{n-1}} \oplus S^2\mathbb{C}^{2^{n-1}*})$.

Fast pairings. Recall the decomposition

$$\Lambda^2(E \oplus E^*) = \Lambda^2 E \oplus E \otimes E^* \oplus \Lambda^2 E^*.$$

Consider $x + \mathrm{Id}_E + y \in \Lambda^2(E \oplus E^*)$. Observe that

$$(x + \mathrm{Id}_E + y)^{\wedge n} = \sum_{j=0}^{n} (\mathrm{Id}_E)^{\wedge(n-j)} \wedge x^{\wedge j} \wedge y^{\wedge j} \in \Lambda^{2n}(E \oplus E^*). \tag{13.8.4}$$

Let $\Omega = e_1 \wedge e^1 \wedge e_2 \wedge e^2 \wedge \cdots \wedge e_n \wedge e^n \in \Lambda^{2n}(E \oplus E^*)$ be a volume form. The coefficient of the j-th term in (13.8.4) is the sum

$$\sum_{|I|=2j} \mathrm{sgn}(I)\, \mathrm{Pf}_I(x)\, \mathrm{Pf}_I(y),$$

where for an even set $I \subseteq [n]$, define $\sigma(I) = \sum_{i \in I} i$ and define $\mathrm{sgn}(I) = (-1)^{\sigma(I)+|I|/2}$. Put more invariantly, the j-th term is the pairing

$$\langle y^{\wedge j}, x^{\wedge j} \rangle.$$

For a matrix z define a matrix $\tilde{z}$ by setting $\tilde{z}^i_j = (-1)^{i+j+1} z^i_j$. Let z be an $n \times n$ skew-symmetric matrix. Then for every even $I \subseteq [n]$,

$$\mathrm{Pf}_I(\tilde{z}) = \mathrm{sgn}(I)\, \mathrm{Pf}_I(z).$$

For $|I| = 2p$, $p = 1, \dots, \lfloor \frac{n}{2} \rfloor$,

$$\mathrm{Pf}_I(\tilde{z}) = (-1)^{i_1+i_2+1} \cdots (-1)^{i_{2p-1}+i_{2p}+1}\, \mathrm{Pf}_I(z) = \mathrm{sgn}(I)\, \mathrm{Pf}_I(z).$$

The following theorem is the key to a geometric interpretation of holographic algorithms:

Theorem 13.8.4.4 ([**214**]). *Let z, y be skew-symmetric $n \times n$ matrices. Then*

$$\langle \mathrm{subPf}(z), \mathrm{subPf}^{\vee}(y) \rangle = \mathrm{Pf}(\tilde{z} + y).$$

In particular, the pairing $\mathcal{S}_+ \times \mathcal{S}_+^* \to \mathbb{C}$ restricted to $\hat{\mathbb{S}}_n \times \hat{\mathbb{S}}_{n*} \to \mathbb{C}$ can be computed in polynomial time.

13.9. Holographic algorithms and spinors

I follow [**212**] in this section. In [**317**], L. Valiant stunned the complexity community by showing a counting problem where counting the number of solutions mod 2 was **NP**-hard and counting the number of solutions mod 7 was in **P**. This was done, in Valiant's words, by "imitating fragments of quantum computation on a classical computer". In this section I explain these algorithms and their relations to spinors.

13.9.1. Counting problems as vector space pairings $A^* \times A \to \mathbb{C}$. For simplicity of exposition, I restrict our consideration to the complexity problem of counting the number of solutions to equations c_s over $\mathbb{F}_2$ with variables x_i. This problem is called SAT in the complexity literature. (In complexity theory one usually deals with Boolean variables and clauses, which is essentially equivalent to equations over $\mathbb{F}_2$ but some care must be taken in the translation.)

To convert a counting problem to a vector space pairing, proceed as follows:

Step 1. For an instance of a problem construct a bipartite graph $\Gamma = (V_x, V_c, E)$ that encodes the instance. Here V_x, V_c are the two sets of vertices and E is the set of edges. V_x corresponds to the set of variables, V_c to the set of equations (or "clauses"), and there is an edge e_{is} joining the vertex of the variable x_i to the vertex of the equation c_s if and only if x_i appears in c_s.

For example, if the instance has two equations and four variables, with equations $c_1 = x_1x_2 + x_1x_3 + x_2x_3 + x_1 + x_2 + x_3 + 1$, $c_2 = x_1x_2 + x_1x_4 + x_2x_4 + x_1 + x_2 + x_4 + 1$, one obtains the graph of Figure 13.9.1.

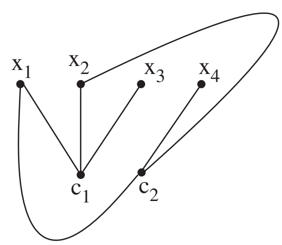


Figure 13.9.1. Graph for 3NAE.

Step 2. Construct "local" tensors that encode the information at each vertex. To do this first associate to each edge e_{is} a vector space $A_{is} = \mathbb{C}^2$ with basis $a_{is|0}, a_{is|1}$ and dual basis $\alpha_{is|0}, \alpha_{is|1}$ of A^*_{is}. Next, to each variable x_i associate the vector space

$$A_i := \bigotimes_{\{s|e_{is}\in E\}} A_{is}$$

and the tensor

$$g_i := \bigotimes_{\{s|e_{is}\in E\}} a_{is|0} + \bigotimes_{\{s|e_{is}\in E\}} a_{is|1} \in A_i, \tag{13.9.1}$$

which will encode that x_i should be consistently assigned either 0 or 1 each time it appears. Now to each equation c_s associate a tensor in $A^*_s := \bigotimes_{\{i|e_{is}\in E\}} A^*_{is}$ that encodes that c_s is satisfied.

To continue the example of Figure 13.9.1,

$$A_1 = A_{11}\otimes A_{12},\ A_2 = A_{21}\otimes A_{22},\ A_3 = A_{31},\ A_4 = A_{42},$$
$$g_1 = a_{11|0}\otimes a_{12|0} + a_{11|1}\otimes a_{12|1},\ g_3 = a_{31|0} + a_{31|1},$$
$$\begin{aligned} r_1 = {} & \alpha_{11|0}\otimes\alpha_{21|0}\otimes\alpha_{31|1} + \alpha_{11|0}\otimes\alpha_{21|1}\otimes\alpha_{31|0} + \alpha_{11|1}\otimes\alpha_{21|0}\otimes\alpha_{31|0} \\ & + \alpha_{11|1}\otimes\alpha_{21|1}\otimes\alpha_{31|0} + \alpha_{11|1}\otimes\alpha_{12|0}\otimes\alpha_{31|1} + \alpha_{11|0}\otimes\alpha_{21|1}\otimes\alpha_{31|1}. \end{aligned}$$

When x_i, x_j, x_k appear in c_s and

$$c_s(x_i, x_j, x_k) = x_ix_j + x_ix_k + x_jx_k + x_i + x_j + x_k + 1, \tag{13.9.2}$$

this equation is called 3NAE in the computer science literature, as it is satisfied over $\mathbb{F}_2$ as long as the variables x_i, x_j, x_k are not all 0 or all 1.

More generally, if c_s has $x_{i_1}, \dots, x_{i_{d_s}}$ appearing and c_s is d_sNAE, then one associates the tensor

$$r_s := \sum_{(\epsilon_1,\dots,\epsilon_{d_s})\neq(0,\dots,0),(1,\dots,1)} \alpha_{i_1,s|\epsilon_1}\otimes\cdots\otimes\alpha_{i_{d_s},s|\epsilon_{d_s}}. \tag{13.9.3}$$

Step 3. Tensor all the local tensors from V_x (resp. V_c) together to get two tensors in dual vector spaces with the property that their pairing counts the number of solutions. That is, consider $G := \bigotimes_i g_i$ and $R := \bigotimes_s r_s$, which are respectively elements of the vector spaces $A := \bigotimes_e A_e$ and $A^* := \bigotimes_e A_e^*$. Then the pairing $\langle G, R\rangle$ counts the number of solutions.

The original counting problem has now been replaced by the problem of computing a pairing $A \times A^* \to \mathbb{C}$, where the dimension of A is exponential in the size of the input data. Were the vectors to be paired arbitrary, there would be no way to perform this pairing in a number of steps that is polynomial in the size of the original data. Since $\dim A$ is a power of two, it is more natural to try to use spinor varieties than Grassmannians to get fast algorithms.

Exercise 13.9.1.1: Verify that in the example above, $\langle G, R\rangle$ indeed counts the number of solutions.

Remark 13.9.1.2. Up until now one could have just taken each $A_{is} = \mathbb{Z}_2$. Using complex numbers, or numbers from any larger field, will allow a group action. This group action will destroy the local structure but leave the global structure unchanged. Valiant's inspiration for doing this was quantum mechanics, where particles are replaced by wave functions.

13.9.2. Computing the vector space pairing in polynomial time. To try to move both G and R to a special position so that the pairing can be evaluated quickly, identify all the A_e with a single $\mathbb{C}^2$, and allow $SL_2\mathbb{C}$ to act. This action is very cheap. If it acts simultaneously on A and A^*, the pairing $\langle G, R\rangle$ will be unchanged.

To illustrate, we now restrict ourselves to 3NAE-SAT, which is still **NP**-hard.

The advantage of “3” is that the vectors r_s lie in $\mathbb{C}^8$, and the spinor variety in $\mathbb{C}^8$ is a hypersurface. Since we will give $\mathbb{C}^8 = (\mathbb{C}^2)^{\otimes 3}$ the structure of an SL_2-module in such a way that $SL_2 \not\subset Spin_8$, *any* vector in $\mathbb{C}^8$ can be moved into the spinor variety by the SL_2-action.

Explicitly, the tensor corresponding to a NAE clause r_s is (13.9.3), and $d_s = 3$ for all s. Let

$$T = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

be the basis change, the same in each A_e, sending $a_{e|0} \mapsto a_{e|0} + a_{e|1}$ and $a_{e|1} \mapsto a_{e|0} - a_{e|1}$, which induces the basis change $\alpha_{e|0} \mapsto \frac{1}{2}(\alpha_{e|0} + \alpha_{e|1})$ and $\alpha_{e|1} \mapsto \frac{1}{2}(\alpha_{e|0} - \alpha_{e|1})$ in A_e^*.

Then

$$\begin{aligned} T\left(\sum_{(\epsilon_1,\epsilon_2,\epsilon_3)\neq(0,0,0),(1,1,1)} \alpha_{i_1,s|\epsilon_1} \otimes \alpha_{i_2,s|\epsilon_2} \otimes \alpha_{i_3,s|\epsilon_3} \right) \\ = 6\alpha_{i_1,s|0} \otimes \alpha_{i_2,s|0} \otimes \alpha_{i_3,s|0} - 2(\alpha_{i_1,s|0} \otimes \alpha_{i_2,s|1} \otimes \alpha_{i_3,s|1} \\ + \alpha_{i_1,s|1} \otimes \alpha_{i_2,s|0} \otimes \alpha_{i_3,s|1} + \alpha_{i_1,s|1} \otimes \alpha_{i_2,s|1} \otimes \alpha_{i_3,s|0}). \end{aligned}$$

After this change of basis, $r_s \in \hat{\mathbb{S}}_4$ for all s.

The tensor g_i corresponding to a variable vertex x_i is (13.9.1). Applying T gives

$$\begin{aligned} &T(a_{i,s_{i_1}|0} \otimes \cdots \otimes a_{i,s_{i_{d_i}}|0} + a_{i,s_{i_1}|1} \otimes \cdots \otimes a_{i,s_{i_{d_i}}|1}) \\ &\qquad = 2 \sum_{\{(\epsilon_1,\ldots,\epsilon_{d_i}) | \sum \epsilon_\ell = 0 \,(\mathrm{mod}\ 2)\}} a_{i,s_{i_1}|\epsilon_1} \otimes \cdots \otimes a_{i,s_{i_{d_i}}|\epsilon_{d_i}}. \end{aligned}$$

Somewhat miraculously, after this change $g_i \in \hat{\mathbb{S}}_{\#\{s | e_{is} \in E\}}$ for all i.

What is even more tantalizing, by Proposition 13.8.4.1, it is possible to glue together all the g_i's in such a way that $G \in \hat{\mathbb{S}}_{|E|}$. Similarly, it is also possible to glue together all the r_s such that $R \in \hat{\mathbb{S}}^*_{|E|}$.

13.9.3. NP, in fact #P, is pre-holographic.

Definition 13.9.3.1. Let P be a counting problem. Define P to be *pre-holographic* if it admits a formulation such that the vectors g_i, r_s are all simultaneously representable as vectors of sub-Pfaffians.

Remark 13.9.3.2. If P is pre-holographic, then there exist an order such that $G \in \hat{\mathbb{S}}$ and an order such that $R \in \hat{\mathbb{S}}_*$.

The following was proved (although not stated) in [**214**].

Theorem 13.9.3.3 ([**212**]). *Any problem in* **NP**, *in fact* #**P**, *is pre-holographic.*

Proof. To prove the theorem it suffices to exhibit one #**P** complete problem that is pre-holographic. Counting the number of solutions to 3NAE-SAT is one such. □

13.9.4. What goes wrong. While for 3NAE-SAT it is always possible to give V and V^* structures of the spin representations $\mathcal{S}_+$ and $\mathcal{S}_+^*$, so that $[G] \in \mathbb{P}V$ and $[R] \in \mathbb{P}V^*$ both lie in spinor varieties, these structures *may not be compatible*! What goes wrong is that the ordering of pairs of indices (i, s) that is good for V may not be good for V^*. The "only" thing that can go wrong is the signs of the sub-Pfaffians as I explain below. Thus the gap is reminiscent of the difference between permanent and determinant. (Had the signs worked out in either case, we would have had $\mathbf{P} = \mathbf{NP}$.)

Sufficient conditions for there to be a good ordering of indices were determined in [**214**]. In particular, if the bipartite graph Γ is planar, then these sufficient conditions hold. Using this result, one recovers Valiant's algorithms from a geometric perspective.

13.9.5. Sufficient conditions for a good ordering.

Definition 13.9.5.1. Call an edge order such that edges incident on each $x_i \in V_x$ (resp. $c_s \in V_c$) are adjacent a *generator order* (resp. *recognizer order*) and denote such by $\bar{E}_G$ (resp. $\bar{E}_R$).

A generator order exactly corresponds to the inclusion of spinor varieties in Proposition 13.8.4.1; similarly for a recognizer order. The difficulty is finding an order that works simultaneously. Any order is obtained by permuting either a generator order or a recognizer order. When one performs the permutation, by the interpretation of the Pfaffian given in Exercises 13.2.1, one keeps the same terms, only the signs change.

Definition 13.9.5.2. An order $\bar{E}$ is *valid* if it simultaneously renders $G \in \hat{\mathbb{S}}_{|E|}$ and $R \in \hat{\mathbb{S}}_{|E|*}$.

Thus if an order is valid, then

$$\langle G, R\rangle = \sum_I \mathrm{Pf}_I(z)\,\mathrm{Pf}_{I^c}(y) \tag{13.9.4}$$

for appropriate skew-symmetric matrices z, y. By Theorem 13.8.4.4 one can evaluate the right hand side of (13.9.4) in polynomial time.

In [**214**] a class of valid orderings is found for planar graphs. The key point is the interpretation of the Pfaffian in Exercise 13.2.1(3). The upshot is as follows:

Theorem 13.9.5.3 ([**214**])**.** *Let $\mathcal{P}$ be pre-holographic. If there exists a valid edge order (e.g., if the graph Γ_P is any planar bipartite graph), then $\mathcal{P}$ can be computed in polynomial time.*

To obtain the polynomial time computation of an instance P of $\mathcal{P}$, normalize $\pi(G)$ (resp. $\tau(R)$) so that the first (resp. last) entry becomes one. Say we need to divide by α, β respectively (i.e., $\alpha = \prod_i \alpha_i$ where $G_i = \alpha_i \operatorname{subPf}(x_i)$ and similarly for β). Consider skew-symmetric matrices x, y where x^i_j is the entry of (the normalized) $\pi(G)$ corresponding to $I = (i,j)$ and y^i_j is the entry of (the normalized) $\tau(R)$ corresponding to $I^c = (i,j)$. Then the number of satisfying assignments to P is given by $\alpha\beta \operatorname{Pf}(\tilde{x} + y)$.

Example 13.9.5.4. In the complexity literature the problem #Pl-Mon-3NAE-SAT refers to equations where the corresponding bipartite graph of connection variables and equations is planar (Pl), and each equation involves exactly three variables and is the equation 3NAE (cf. (13.9.2)).

Figure 13.9.2 shows an instance of #Pl-Mon-3NAE-SAT, with an edge order given by a path through the graph. The corresponding matrix, $\tilde{z} + y$,

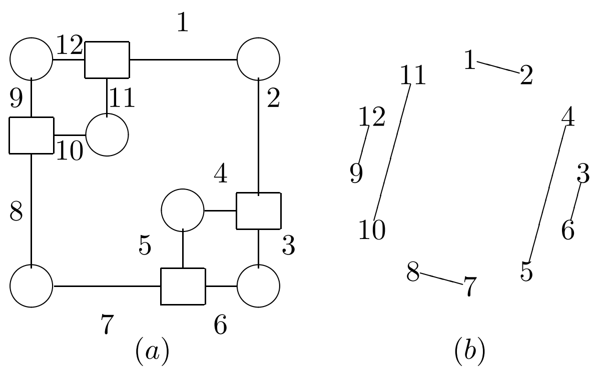


Figure 13.9.2. Example 13.9.5.4, and the term

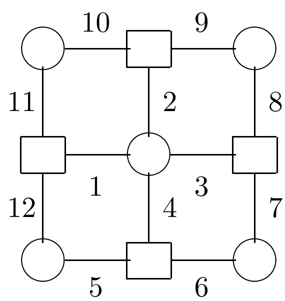
$$S = (1,2)(3,6)(4,5)(7,8)(9,12)(10,11)$$

in the Pfaffian (which has no crossings). Circles are variable vertices and rectangles are clause vertices.

is given below. In a generator order, each variable corresponds to a $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ block. In a recognizer order, each clause corresponds to a 3×3 block with $-1/3$ above the diagonal. Sign flips $z \mapsto \tilde{z}$ occur in a checkerboard pattern with the diagonal flipped; here no flips occur. We pick up a factor of $\frac{6}{2^3}$ for each clause and 2 for each variable, so $\alpha = 2^6$, $\beta = (\frac{6}{2^3})^4$, and $\alpha\beta\,\mathrm{Pf}(\tilde{z}+y) =$ 26 satisfying assignments. Here is the corresponding matrix:

$$\tilde{z}+y = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1/3 & -1/3 \\ -1 & 0 & -1/3 & -1/3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/3 & 0 & -1/3 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/3 & 1/3 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1/3 & -1/3 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1/3 & 0 & -1/3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/3 & 1/3 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1/3 & -1/3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/3 & 0 & -1/3 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/3 & 1/3 & 0 & 1 & 0 \\ 1/3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1/3 \\ 1/3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1/3 & 0 \end{pmatrix}.$$

Another instance of #Pl-Mon3NAE-SAT which is not read-twice (i.e., variables may appear in more than two clauses) and its $\tilde{z}+y$ matrix are shown in Figure 13.9.3. The central variable has a submatrix which again has ones above the diagonal and also contributes 2 to α, so $\alpha = 2^5$, $\beta = (\frac{6}{2^3})^4$. Four sign changes are necessary in $\tilde{z}$. The result is $\alpha\beta\,\mathrm{Pf}(\tilde{z}+y) = 14$ satisfying assignments.



$$\begin{pmatrix} 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1/3 & -1/3 \\ -1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & -1/3 & -1/3 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 & -1/3 & -1/3 & 0 & 0 & 0 & 0 \\ -1 & 1 & -1 & 0 & -1/3 & -1/3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/3 & 0 & -1/3 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1/3 & 1/3 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/3 & 0 & 0 & -1 & 0 & -1/3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/3 & 0 & 0 & 0 & 1/3 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1/3 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1/3 & 0 & 0 \\ 0 & 1/3 & 0 & 0 & 0 & 0 & 0 & 0 & 1/3 & 0 & 1 & 0 \\ 1/3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1/3 \\ 1/3 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1/3 & 0 \end{pmatrix}$$

Figure 13.9.3. Another #Pl-Mon-3NAE-SAT example and its $\tilde{z}+y$ matrix.

13.9.6. History and discussion. In Valiant's original formulation of holographic algorithms (see [**314**]–[**319**]), the step of forming Γ is the same, but then Valiant replaced the vertices of Γ with weighted graph fragments to get a new weighted graph Γ' in such a way that the number of (weighted) perfect matchings of Γ' equals the answer to the counting problem. Then, if

Γ' is planar, one can appeal to the FKT algorithm discussed below to compute the number of weighted perfect matchings in polynomial time. Valiant also found certain algebraic identities that were necessary conditions for the existence of such graph fragments. Pfaffian sums were an ingredient in his formulation from the beginning.

Cai [**58**]–[**64**] recognized that Valiant's procedure could be reformulated as a pairing of tensors as in Steps 2 and 3, and that the condition on the vertices was that the local tensors g_i, r_s could, possibly after a change of basis, be realized as a vector of sub-Pfaffians. In Cai's formulation one still appeals to the existence of Γ' and the FKT algorithm in the last step.

The holographic algorithm that shocked the complexity theory community was for $\#_7$Pl-Rtw-MON-3CNF, that is, equations in which the associated bipartite graph is planar (Pl), each variable appears in exactly two equations (Rtw = read twice), and each equation has exactly three variables. The problem is to count the number of solutions mod 7, and Valiant proved that this problem is in **P**. The reason this is shocking is that the problem $\#_2$Pl-Rtw-MON-3CNF, i.e., with the same type of equations but to count the number of solutions mod 2, is **NP**-hard. From the $\#_7$Pl-Rtw-MON-3CNF example, it seems that the difficulty of a problem could be very sensitive to the field, and it would be interesting to have more investigation into the arithmetic aspects of holographic algorithms and complexity classes in general.

Cai and his collaborators have explored the limits of holographic algorithms; see, e.g., [**65**]. They proved several dichotomy theorems. While it originally appeared that holographic algorithms might be used to prove **P** = **NP**, for all problems known to be **NP**-hard, the problem of signs appears to be an obstruction to a fast algorithm. On the other hand, for problems that are not hard, the holographic approach and its cousins using, e.g., Grassmannians instead of spinor varieties (see, e.g., [**214**, §5]), provide an explicit method for constructing fast algorithms that might be implemented in practice.

To aid the reader in comparing the formulation of holographic algorithms presented here, where Pfaffian sums arise directly from evaluating the vector space pairing, and the traditional formulation, where they arise in implementing the FKT algorithm, I include a description and proof of the algorithm in the appendix below.

13.9.7. Appendix: The FKT algorithm. (This subsection is not used elsewhere in the text.) The FKT (Fisher-Kasteleyn-Temperley) algorithm [**185**] computes the number of perfect matchings of a planar graph using the Pfaffian. The key to the FKT algorithm is that for certain sparse skew-symmetric matrices x, one has all terms in the Pfaffian positive. Such classes of matrices have been studied extensively; see [**309**].

Let Γ be a weighted graph with skew-adjacency matrix $y = (y^i_j)$. The goal is to change the weightings to get a new matrix $\tilde{y}$ such that the perfect matching polynomial (see §13.2.3) is converted to the Pfaffian:

$$\mathrm{PerfMat}(y) = \mathrm{Pf}(\tilde{y}).$$

This will hold if any odd permutation contains an odd number of variables whose sign has been changed, and any even permutation contains an even number of variables whose sign has been changed.

Another way to say this, is as follows: Given two perfect matchings, M, M', form a union of cycles in Γ, denoted $M\Delta M'$, and call this union the *superposition* of M on M'. The union $M\Delta M'$ is obtained by starting at a vertex, then traveling an edge of M to the next vertex, then a vertex of M' to the next, and so on, alternating. If M, M' share common edges, one must start at new vertices again in the middle of the procedure, but with whatever choices one makes, one obtains the same set of cycles.

Exercise 13.9.7.1: Show that all cycles appearing in $M\Delta M'$ are of even length.

Let D be an orientation of the edges of Γ. Define

$$\mathrm{sgn}_D(M) := \mathrm{sgn}\begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & 2k-1 & 2k \\ i_1 & j_j & i_2 & j_2 & \cdots & i_k & j_k \end{pmatrix}, \tag{13.9.5}$$

where the edges of M are listed as $(i_1 j_1), \dots, (i_k j_k)$. Note that $\mathrm{sgn}_D(M)$ is the sign of the monomial appearing in $\mathrm{Pf}(y)$ corresponding to M. Thus one needs an orientation of the edges of Γ such that $\mathrm{sgn}_D(M) = \mathrm{sgn}_D(M')$ for all perfect matchings M, M'.

Definition 13.9.7.2. Give an orientation to the plane and define a notion of "clockwise" for traversing a cycle. Say the edges of Γ are also oriented. Call a cycle in Γ *clockwise odd* if it contains an odd number of edges with the same orientation as that induced by traversing the cycle clockwise.

Lemma 13.9.7.3. *In the notation as above, given perfect matchings M, M' on an oriented plane graph, $\mathrm{sgn}_D(M) = \mathrm{sgn}_D(M')$ if and only if each cycle in $M\Delta M'$ is clockwise odd.*

Exercise 13.9.7.4: Prove Lemma 13.9.7.3.

Theorem 13.9.7.5 (FKT [**185, 305**])**.** *If Γ is a planar graph, it admits an orientation such that all superposition cycles are clockwise odd. Thus with this orientation the number of perfect matchings of Γ can be computed by evaluating a Pfaffian.*

Note that computing the orientation can be done very quickly.

The proof of the theorem will be a consequence of the following two lemmas:

Lemma 13.9.7.6. *A finite planar graph can be oriented so that all elementary cycles bounding a (finite) face will be clockwise odd.*

Proof. Pick any face F_1, with any orientation on the edges of ∂F_1 except one. Give that one edge the orientation such that ∂F_1 is clockwise odd. Take F_2 such that $\partial F_1 \cap \partial F_2$ contains an edge, and do the same for F_2. Continue—the only potential problem is that one arrives at an edge that completes two ∂F_j's simultaneously, but this can be avoided by insisting at each step k that $\bigcup_{i=1}^{k} F_i$ is simply connected. $\square$

Lemma 13.9.7.7. *If all elementary cycles bounding a (finite) face are clockwise odd, then every cycle enclosing an even number of vertices of* Γ *is clockwise odd.*

Proof. Proceed by induction on the number of faces an elementary cycle encloses. If there is one face, we are done by hypothesis. Say the lemma has been proved up to n faces. Consider a cycle C with $n+1$ faces. Select a face F with $\partial F \cap C \neq \emptyset$ and such that the union of the other enclosed faces is bounded by a cycle we label C'. Write $\partial F = C''$.

Let v_C denote the number of vertices enclosed by C, e_C the number of edges clockwise oriented with C, and similarly for C', C''. It remains to show that $v_C \not\equiv e_C \bmod 2$. We have

$$\begin{aligned} v_C &= v_{C'} + v_{C''} = v_{C'}, \\ e_C &= e_{C'} + e_{C''} - \text{overlaps}, \\ v_{C'} &\not\equiv e_{C'} \bmod 2 \text{ by induction.} \end{aligned}$$

Exercise 13.9.7.8: Finish the proof.

Chapter 14

Varieties of tensors in phylogenetics and quantum mechanics

This chapter discusses two constructions of varieties of tensors from graph-theoretic data. The first, tensor network states, in §14.1, comes from physics, more precisely, quantum information theory. The second, Bayesean networks in algebraic statistics, in §14.2, as the name suggests, comes from statistics. Algebraic statistics has already been written about considerably by geometers, so just one case is presented in detail: models arising in the study of phylogenetics. This work of Allmann and Rhodes [**9, 8**] has motivated considerable work by geometers already described extensively in this book. I present these two constructions together because, despite their very different origins, they are very similar.

14.1. Tensor network states

Recall from §1.5.2 that tensor network states were defined in physics as subsets of spaces of tensors that are state spaces for quantum-mechanical systems that are physically feasible. A practical motivation for them was to reduce the number of variables one needed to deal with in computations by restricting one's study to a small subset of the state space. This section follows [**216**].

14.1.1. Definitions and notation. For a graph Γ with edges e_s and vertices v_i, $s \in e(j)$ means e_s is incident to v_j. If Γ is directed, $s \in in(j)$ are the incoming edges and $s \in out(j)$ the outgoing edges.

Let $V_1, \dots, V_n$ be vector spaces and let $\mathbf{v}_i = \dim V_i$. Let Γ be a graph with n vertices v_j, $1 \leq j \leq n$, and m edges e_s, $1 \leq s \leq m$, and let $\vec{\mathbf{e}} = (\mathbf{e}_1, \dots, \mathbf{e}_m) \in \mathbb{N}^m$. Associate V_j to the vertex v_j and an auxiliary vector space E_s of dimension $\mathbf{e}_s$ to the edge e_s. Make Γ into a directed graph. (The choice of directions will not effect the end result.) Let $\mathbf{V} = V_1 \otimes \cdots \otimes V_n$.

Let

(14.1.1)
$$TNS(\Gamma, \vec{\mathbf{e}}, \mathbf{V}) := \{T \in V_1 \otimes \cdots \otimes V_n \mid \exists T_j \in V_j \otimes (\otimes_{s \in in(j)} E_s) \otimes (\otimes_{t \in out(j)} E_t^*) \text{ such that } T = \mathrm{Con}(T_1 \otimes \cdots \otimes T_n)\},$$

where Con is the contraction of all the E_s's with all the E_s^*'s.

Example 14.1.1.1. Let Γ be a graph with two vertices and one edge connecting them; then $TNS(\Gamma, \mathbf{e}_1, V_1 \otimes V_2)$ is just the set of elements of $V_1 \otimes V_2$ of rank at most $\mathbf{e}_1$, denoted $\hat{\sigma}_{\mathbf{e}_1}(Seg(\mathbb{P}V_1 \times \mathbb{P}V_2))$, the (cone over the) $\mathbf{e}_1$-st secant variety of the Segre variety. To see this, let $\epsilon_1, \dots, \epsilon_{\mathbf{e}_1}$ be a basis of E_1 and $\epsilon^1, \dots, \epsilon^{\mathbf{e}_1}$ the dual basis of E^*. Assume, to avoid trivialities, that $\mathbf{v}_1, \mathbf{v}_2 \geq \mathbf{e}_1$. Given $T_1 \in V_1 \otimes E_1$, we may write $T_1 = u_1 \otimes \epsilon_1 + \cdots + u_{\mathbf{e}_1} \otimes \epsilon_{\mathbf{e}_1}$ for some $u_\alpha \in V_1$. Similarly, given $T_2 \in V_2 \otimes E_1^*$ we may write $T_1 = w_1 \otimes \epsilon^1 + \cdots + w_{\mathbf{e}_1} \otimes \epsilon^{\mathbf{e}_1}$ for some $w_\alpha \in V_2$. Then $\mathrm{Con}(T_1 \otimes T_2) = u_1 \otimes w_1 + \cdots + u_{\mathbf{e}_1} \otimes w_{\mathbf{e}_1}$.

The graph used to define a set of tensor network states is often modeled to mimic the physical arrangement of the particles, with edges connecting nearby particles, as nearby particles are the ones likely to be entangled.

Remark 14.1.1.2. The construction of tensor network states in the physics literature does not use a directed graph, because all vector spaces are Hilbert spaces, and thus self-dual. However, the sets of tensors themselves do not depend on the Hilbert space structure of the vector space, which is why we omit this structure. The small price to pay is that the edges of the graph must be oriented, but all orientations lead to the same set of tensor network states.

14.1.2. Grasedyck's question.

Lars Grasedyck asked:

Is $TNS(\Gamma, \vec{\mathbf{e}}, \mathbf{V})$ Zariski closed? That is, given a sequence of tensors $T_\epsilon \in \mathbf{V}$ that converges to a tensor T_0, if $T_\epsilon \in TNS(\Gamma, \vec{\mathbf{e}}, \mathbf{V})$ for all $\epsilon \neq 0$, can we conclude that $T_0 \in TNS(\Gamma, \vec{\mathbf{e}}, \mathbf{V})$?

He mentioned that he could show this to be true when Γ was a tree, but did not know the answer when Γ is a triangle (see Figure 14.1.1).

Definition 14.1.2.1. A dimension $\mathbf{v}_i$ is *critical*, resp. *subcritical*, resp. *supercritical*, if $\mathbf{v}_i = \prod_{s \in e(j)} \mathbf{e}_s$, resp. $\mathbf{v}_i \leq \prod_{s \in e(j)} \mathbf{e}_s$, resp. $\mathbf{v}_i \geq \prod_{s \in e(j)} \mathbf{e}_s$.

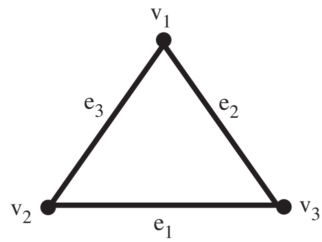


Figure 14.1.1

If $TNS(\Gamma, \vec{\mathbf{e}}, \mathbf{V})$ is critical for all i, we say that $TNS(\Gamma, \vec{\mathbf{e}}, \mathbf{V})$ is critical, and similarly for sub- and supercritical.

Theorem 14.1.2.2. *$TNS(\Gamma, \vec{\mathbf{e}}, \mathbf{V})$ is not Zariski closed for any Γ containing a cycle whose vertices have nonsubcritical dimensions. In particular, if Γ is a triangle and $\mathbf{v}_i \geq \mathbf{e}_j \mathbf{e}_k$ for all $\{i, j, k\} = \{1, 2, 3\}$, then $TNS(\Gamma, \vec{\mathbf{e}}, \mathbf{V})$ is not Zariski closed.*

Nonsubcritical reductions are explained below.

14.1.3. Connection to the GCT program. The triangle case is especially interesting because, as explained below, in the critical dimension case it corresponds to

$$\mathrm{End}(V_1) \times \mathrm{End}(V_2) \times \mathrm{End}(V_3) \cdot \mathrm{MMult}_{\mathbf{e}_3, \mathbf{e}_2, \mathbf{e}_1},$$

where, setting $V_1 = E_2^* \otimes E_3$, $V_2 = E_3^* \otimes E_1$, and $V_3 = E_2 \otimes E_1^*$, $\mathrm{MMult}_{\mathbf{e}_3, \mathbf{e}_2, \mathbf{e}_1} \in V_1 \otimes V_2 \otimes V_3$ is the matrix multiplication operator, that is, as a tensor, $\mathrm{MMult}_{\mathbf{e}_3, \mathbf{e}_2, \mathbf{e}_1} = \mathrm{Id}_{E_3} \otimes \mathrm{Id}_{E_2} \otimes \mathrm{Id}_{E_1}$. In [**55**] a *geometric complexity theory* (GCT) study of MMult and its $GL(V_1) \times GL(V_2) \times GL(V_3)$ orbit closure is considered. One sets $\mathbf{e}_1 = \mathbf{e}_2 = \mathbf{e}_3 = n$ and studies the geometry as $n \to \infty$. It is a toy case of the varieties introduced by Mulmuley and Sohoni discussed in §13.6.

The critical loop case with $\mathbf{e}_s = 3$ for all s is also related to the GCT program, as it corresponds to the multiplication of n matrices of size three. As a tensor, it may be thought of as a map $(X_1, \dots, X_n) \mapsto \mathrm{tr}(X_1 \cdots X_n)$. This sequence of functions indexed by n, considered as a sequence of homogeneous polynomials of degree n on $V_1 \oplus \cdots \oplus V_n$, is complete for the class $\mathbf{VP}_e$ of sequences of polynomials of small formula size; see §13.7.3.

14.1.4. Critical loops.

Proposition 14.1.4.1. *Let $\mathbf{v}_1 = \mathbf{e}_2 \mathbf{e}_3, \mathbf{v}_2 = \mathbf{e}_3 \mathbf{e}_1, \mathbf{v}_3 = \mathbf{e}_2 \mathbf{e}_1$. Then*

$$TNS(\triangle, (\mathbf{e}_2 \mathbf{e}_3, \mathbf{e}_3 \mathbf{e}_1, \mathbf{e}_2 \mathbf{e}_1), V_1 \otimes V_2 \otimes V_3)$$

consists of matrix multiplication (up to relabeling) and its degenerations, i.e.,

$$\begin{aligned}&TNS(\triangle,(\mathbf{e}_2\mathbf{e}_3,\mathbf{e}_3\mathbf{e}_1,\mathbf{e}_2\mathbf{e}_1),V_1\otimes V_2\otimes V_3)\\&\qquad=\mathrm{End}(V_1)\times\mathrm{End}(V_2)\times\mathrm{End}(V_3)\cdot M_{\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_1}.\end{aligned}$$

It has dimension $\mathbf{e}_2^2\mathbf{e}_3^2+\mathbf{e}_2^2\mathbf{e}_1^2+\mathbf{e}_3^2\mathbf{e}_1^2-(\mathbf{e}_2^2+\mathbf{e}_3^2+\mathbf{e}_1^2-1)$.

More generally, if Γ *is a critical loop,* $TNS(\Gamma,\vec{\mathbf{e}},\mathbf{V})$ *is* $\mathrm{End}(V_1)\times\cdots\times\mathrm{End}(V_n)\cdot M_{\vec{\mathbf{e}}}$*, where* $M_{\vec{\mathbf{e}}}:V_1\times\cdots\times V_n\to\mathbb{C}$ *is the matrix multiplication operator* $(X_1,\dots,X_n)\mapsto\mathrm{tr}(X_1\cdots X_n)$.

Proof. For the triangle case, a generic element $T_1\in E_2\otimes E_3^*\otimes V_1$ may be thought of as a linear isomorphism $E_2^*\otimes E_3\to V_1$, identifying V_1 as a space of $\mathbf{e}_2\times\mathbf{e}_3$-matrices, and similarly for V_2,V_3. Choosing bases $e_s^{u_s}$ for E_s^*, with dual basis $e_{u_s,s}$ for E_s, one obtains induced bases $x_{u_3}^{u_2}$ for V_1, etc. Let $1\le i\le\mathbf{e}_2$, $1\le\alpha\le\mathbf{e}_3$, $1\le u\le\mathbf{e}_1$. Then

$$\mathrm{Con}(T_1\otimes T_2\otimes T_3)=\sum x_\alpha^i\otimes y_u^\alpha\otimes z_i^u,$$

which is the matrix multiplication operator. The general case is similar. $\square$

Let $S(GL(E_1)\times\cdots\times GL(E_n))$ denote the subgroup of $GL(E_1)\times\cdots\times GL(E_n)$ such that the product of the determinants is 1, and let $\mathfrak{s}(\mathfrak{gl}(E_1)\oplus\cdots\oplus\mathfrak{gl}(E_n))$ denote its Lie algebra.

Proposition 14.1.4.2. *The Lie algebra of the stabilizer of* $M_{\mathbf{e}_n\mathbf{e}_1,\mathbf{e}_1\mathbf{e}_2,\dots,\mathbf{e}_{n-1}\mathbf{e}_n}$ *in* $GL(V_1)\times\cdots\times GL(V_n)$ *is* $\mathfrak{s}(\mathfrak{gl}(E_1)\oplus\cdots\oplus\mathfrak{gl}(E_n))$ *embedded by*

$$\begin{aligned}\alpha_1\oplus\cdots\oplus\alpha_n\mapsto&(\mathrm{Id}_{E_n}\otimes\alpha_1,\alpha_1^T\otimes\mathrm{Id}_{E_2},0,\dots,0)]]\\&+(0,\mathrm{Id}_{E_1}\otimes\alpha_2,\alpha_2^T\otimes\mathrm{Id}_{E_3},0,\dots,0)\\&+\cdots+(\alpha_n^T\otimes\mathrm{Id}_{E_1},0,\dots,0,\mathrm{Id}_{E_{n-1}}\otimes\alpha_n).\end{aligned}$$

The proof is safely left to the reader.

Large loops are referred to as "1-D systems with periodic boundary conditions" in the physics literature and are the prime objects people use in practical simulations today. By Proposition 14.1.4.2, for a critical loop, $\dim(TNS(\Gamma,\vec{\mathbf{e}},\mathbf{V}))=\mathbf{e}_1^2\mathbf{e}_2^2+\cdots+\mathbf{e}_{n-1}^2\mathbf{e}_n^2+\mathbf{e}_n^2\mathbf{e}_1^2-(\mathbf{e}_1^2+\cdots+\mathbf{e}_n^2-1)$, compared with the ambient space which has dimension $\mathbf{e}_1^2\cdots\mathbf{e}_n^2$. For example, when $\mathbf{e}_j=2$, $\dim(TNS(\Gamma,\vec{\mathbf{e}},\mathbf{V}))=12n+1$, while $\dim\mathbf{V}=4^n$.

14.1.5. Zariski closure.

Theorem 14.1.5.1. *Let* $\mathbf{v}_1=\mathbf{e}_2\mathbf{e}_3,\mathbf{v}_2=\mathbf{e}_3\mathbf{e}_1,\mathbf{v}_3=\mathbf{e}_2\mathbf{e}_1$*. Then*

$$TNS(\triangle,(\mathbf{e}_2\mathbf{e}_3,\mathbf{e}_3\mathbf{e}_1,\mathbf{e}_2\mathbf{e}_1),V_1\otimes V_2\otimes V_3)$$

is not Zariski closed. More generally, any $TNS(\Gamma,\mathbf{e},\mathbf{V})$*, where* Γ *contains a cycle with no subcritical vertex, is not Zariski closed.*

Proof. Were $T(\triangle) := TNS(\triangle, (\mathbf{e}_2\mathbf{e}_3, \mathbf{e}_3\mathbf{e}_1, \mathbf{e}_2\mathbf{e}_1), V_1 \otimes V_2 \otimes V_3)$ Zariski closed, it would be

$$\overline{GL(V_1) \times GL(V_2) \times GL(V_3) \cdot M_{\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_1}}.$$

To see this, note that the $G = GL(V_1) \times GL(V_2) \times GL(V_3)$-orbit of matrix multiplication is a Zariski open subset of $V_1 \otimes V_2 \otimes V_3$ of the same dimension as $T(\triangle)$.

Although it is not necessary for the proof, to make the parallel with the GCT program clearer, this Zariski closure may be described as the cone over (the closure of) the image of the rational map

$$\begin{aligned} &\mathbb{P}\operatorname{End}(V_1) \times \mathbb{P}\operatorname{End}(V_2) \times \mathbb{P}\operatorname{End}(V_3) \dashrightarrow \mathbb{P}(V_1 \otimes V_2 \otimes V_3), \\ &\qquad\qquad ([X],[Y],[Z]) \mapsto (X,Y,Z) \cdot [M_{\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_1}]. \end{aligned} \tag{14.1.2}$$

(Compare with the map ψ_n in §13.6.3.)

Note that $(X,Y,Z) \cdot M_{\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_1}(P,Q,R) = \operatorname{tr}(XP, YQ, ZR)$, so the indeterminacy locus consists of $([X],[Y],[Z])$ such that for all triples of matrices P, Q, R, $\operatorname{tr}((X \cdot P)(Y \cdot Q)(Z \cdot R)) = 0$. In principle, one can obtain $T(\triangle)$ as the image of a map from a succession of blowups of $\mathbb{P}\operatorname{End}(V_1) \times \mathbb{P}\operatorname{End}(V_2) \times \mathbb{P}\operatorname{End}(V_3)$.

One way to attain a point in the indeterminacy locus is as follows: We have $X : E_2^* \otimes E_3 \to E_2^* \otimes E_3$, $Y : E_3^* \otimes E_1 \to E_3^* \otimes E_1$, $Z : E_1^* \otimes E_2 \to E_1^* \otimes E_2$. Take subspaces $U_{E_2E_3} \subset E_2^* \otimes E_3$, $U_{E_3E_1} \subset E_3^* \otimes E_1$. Let $U_{E_1E_2} := \operatorname{Con}(U_{E_2E_3}, U_{E_3E_1}) \subset E_2^* \otimes E_1$ be the images of all the $pq \in E_2^* \otimes E_1$ where $p \in U_{E_2E_3}$ and $q \in U_{E_3E_1}$ (i.e., the matrix multiplication of all pairs of elements). Take X_0, Y_0, Z_0 respectively to be the projections to $U_{E_2E_3}$, $U_{E_3E_1}$, and ${U_{E_1E_2}}^{\perp}$. Then $([X_0],[Y_0],[Z_0])$ is in the indeterminacy locus. Taking a curve in G that limits to this point may or may not give something new.

The simplest type of curve is to let X_1, Y_1, Z_1 be the projections to the complementary spaces (so $X_0 + X_1 = \operatorname{Id}$) and to take the curve (X_t, Y_t, Z_t) with $X_t = X_0 + tX_1$, $Y_t = Y_0 + tY_1$, $Z_t = Z_0 + tZ_1$. Then the limiting tensor (if it is not zero), as a map $V_1^* \times V_2^* \times V_3^* \to \mathbb{C}$, is the map

$$(P,Q,R) \mapsto \operatorname{tr}(P_0Q_0R_1) + \operatorname{tr}(P_0Q_1R_0) + \operatorname{tr}(P_1Q_0R_0).$$

Call this tensor $\tilde{M}$. We first observe that

$$\tilde{M} \notin Sub_{\mathbf{e}_2\mathbf{e}_3-1,\mathbf{e}_2\mathbf{e}_1-1,\mathbf{e}_3\mathbf{e}_1-1}(V_1 \otimes V_2 \otimes V_3),$$

so it is either a relabeling of matrix multiplication or a point in the boundary that is not in $\operatorname{End}(V_1) \times \operatorname{End}(V_2) \times \operatorname{End}(V_3) \cdot M_{\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_1}$.

It remains to show that there exist $\tilde{M}$ such that $\tilde{M} \notin G \cdot M_{\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_1}$. To prove an $\tilde{M}$ is a point in the boundary, we compute the Lie algebra of its stabilizer and show that it has dimension greater than the dimension of the stabilizer of matrix multiplication. One may take block matrices, e.g.,

$$X_0 = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}, \ X_1 = \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix},$$

and take Y_0, Y_1 of similar shape, but take Z_0, Z_1 with the shapes reversed.

For another example, if one takes $\mathbf{e}_j = \mathbf{e}$ for all j, takes X_0, Y_0, Z_1 to be the diagonal matrices and X_1, Y_1, Z_0 to be the matrices with zero on the diagonal, then one obtains a stabilizer of dimension $4\mathbf{e}^2 - 2\mathbf{e} > 3\mathbf{e}^2 - 1$.

The same construction works for larger loops and cycles in larger graphs as it is essentially local—one just leaves all other curves stationary at the identity. $\square$

Remark 14.1.5.2. When $\mathbf{e} = 2$, we obtain a codimension one component of the boundary. In general, the stabilizer is much larger than the stabilizer of M, so the orbit closures of these points do not give rise to codimension one components of the boundaries. It remains an interesting problem to find the codimension one components of the boundary.

14.1.6. Reduction from the supercritical case to the critical case with the same graph. The supercritical cases may be realized, in the language of Kempf, as a "collapsing of a bundle" (see Chapter 17) over the critical cases as follows.

For a vector space W, let $G(k, W)$ denote the Grassmannian of k-planes through the origin in W. Let $\mathcal{S} \to G(k, W)$ denote the tautological rank k vector bundle whose fiber over $E \in G(k, W)$ is the k-plane E. Assume that $\mathbf{f}_j \leq \mathbf{v}_j$ for all j with at least one inequality strict. Form the vector bundle $\mathcal{S}_1 \otimes \cdots \otimes \mathcal{S}_n$ over $G(\mathbf{f}_1, V_1) \times \cdots \times G(\mathbf{f}_n, V_n)$, where $\mathcal{S}_j \to G(\mathbf{f}_j, V_j)$ are the tautological subspace bundles. Note that the total space of $\mathcal{S}_1 \otimes \cdots \otimes \mathcal{S}_n$ maps to $\mathbf{V}$ with image $Sub_{\vec{\mathbf{f}}}(\mathbf{V})$. Define a fiber subbundle, whose fiber over $(U_1 \times \cdots \times U_n) \in G(\mathbf{f}_1, V_1) \times \cdots \times G(\mathbf{f}_n, V_n)$ is $TNS(\Gamma, \vec{\mathbf{e}}, U_1 \otimes \cdots \otimes U_n)$. Denote this bundle by $TNS(\Gamma, \vec{\mathbf{e}}, \mathcal{S}_1 \otimes \cdots \otimes \mathcal{S}_n)$.

Proposition 14.1.6.1. *Suppose that* $\mathbf{f}_j := \prod_{s \in e(j)} \mathbf{e}_s \leq \mathbf{v}_j$. *Then* $TNS(\Gamma, \vec{\mathbf{e}}, \mathbf{V})$ *is the image of the bundle* $TNS(\Gamma, \vec{\mathbf{e}}, \mathcal{S}_1 \otimes \cdots \otimes \mathcal{S}_n)$ *under the map to* $\mathbf{V}$. *In particular*

$$\dim(TNS(\Gamma, \vec{\mathbf{e}}, \mathbf{V})) = \dim(TNS(\Gamma, \vec{\mathbf{e}}, \mathbb{C}^{\mathbf{f}_1} \otimes \cdots \otimes \mathbb{C}^{\mathbf{f}_n})) + \sum_{j=1}^{n} \mathbf{f}_j(\mathbf{v}_j - \mathbf{f}_j).$$

Proof. If $\prod_{s \in e(j)} \mathbf{e}_s \leq \mathbf{v}_j$, then any tensor

$$T \in V_j \otimes \Big(\bigotimes_{s \in in(j)} E_s \Big) \otimes \Big(\bigotimes_{t \in out(j)} E_t^* \Big),$$

must lie in some $V_j'\otimes(\bigotimes_{s\in in(j)} E_s)\otimes(\bigotimes_{t\in out(j)} E_t^*)$ with $\dim V_j' = \mathbf{f}_j$. The space $TNS(\Gamma, \vec{\mathbf{e}}, \mathbf{V})$ is the image of this subbundle under the map to $\mathbf{V}$. $\square$

Using the techniques in [**333**], one may reduce questions about a supercritical case to the corresponding critical case.

The special case where $TNS(\Gamma, \vec{\mathbf{e}}, U_1\otimes\cdots\otimes U_n) = U_1\otimes\cdots\otimes U_n$ gives rise to the *subspace variety* $\hat{S}ub_{\mathbf{f}_1,\dots,\mathbf{f}_n}(\mathbf{V})$, which is well understood.

14.1.7. Reduction of cases with subcritical vertices of valence 1. The subcritical case in general can be understood in terms of projections of critical cases, but this is not useful for extracting information. However, if a subcritical vertex has valence 1, one may simply reduce to a smaller graph as follows.

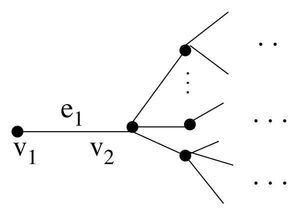


Figure 14.1.2. Vertices of valence 1, such as v_1 in the picture, can be eliminated when $\mathbf{v}_1 \le \mathbf{e}_1$.

Proposition 14.1.7.1. *Let $TNS(\Gamma, \overrightarrow{\mathbf{e}}, \mathbf{V})$ be a tensor network state and let v be a vertex of Γ with valence 1. Relabel the vertices so that $v = v_1$ and v_1 is attached by e_1 to v_2. If $\mathbf{v}_1 \le \mathbf{e}_1$, then $TNS(\Gamma, \overrightarrow{\mathbf{e}}, V_1\otimes\cdots\otimes V_n) = TNS(\tilde{\Gamma}, \overrightarrow{\tilde{\mathbf{e}}}, \tilde{V}_1\otimes V_3\otimes\cdots\otimes V_n)$, where $\tilde{\Gamma}$ is Γ with v_1 and e_1 removed, $\overrightarrow{\tilde{\mathbf{e}}}$ is the vector $(\mathbf{e}_2,\dots,\mathbf{e}_n)$ and $\tilde{V}_1 = V_1\otimes V_2$.*

Proof. A general element in $TNS(\Gamma, \overrightarrow{\mathbf{e}}, V_1\otimes\cdots\otimes V_n)$ is of the form $\sum_i^{\mathbf{e}_1}\sum_{z=1}^{\mathbf{e}_2} u_i\otimes v_{iz}\otimes w_z$, where $w_z\in V_3\otimes\cdots\otimes V_n$. Obviously,

$$TNS(\Gamma, \overrightarrow{\mathbf{e}}, V_1\otimes\cdots\otimes V_n) \subseteq TNS(\tilde{\Gamma}, \overrightarrow{\tilde{\mathbf{e}}}, \tilde{V}_1\otimes V_3\otimes\cdots\otimes V_n) =: TNS(\tilde{\Gamma}, \overrightarrow{\tilde{\mathbf{e}}}, \tilde{\mathbf{V}}).$$

Conversely, a general element in $TNS(\tilde{\Gamma}, \overrightarrow{\tilde{\mathbf{e}}}, \tilde{\mathbf{V}})$ is of the form $\sum_z X_z\otimes w_z$, $X_z\in V_1\otimes V_2$. Since $\mathbf{v}_1\le\mathbf{e}_1$, we may express X_z in the form $\sum_{i=1}^{e_1} u_i\otimes v_{iz}$, where $u_1,\dots,u_{v_1}$ is a basis of V_1. Therefore, $TNS(\Gamma, \overrightarrow{\mathbf{e}}, \mathbf{V}) \supseteq TNS(\tilde{\Gamma}, \overrightarrow{\tilde{\mathbf{e}}}, \tilde{\mathbf{V}})$. $\square$

14.1.8. Trees. With trees one can apply the two reductions successively to reduce to a tower of bundles where the fiber in the last bundle is a linear space. The point is that a critical vertex is both sub- and supercritical, so one can reduce at valence one vertices iteratively. Here are a few examples in the special case of chains. The result is similar to the Allman-Rhodes reduction theorem for phylogenetic trees, Theorem 14.2.3.1.

Example 14.1.8.1. Let Γ be a chain with three vertices. If it is supercritical, $TNS(\Gamma, \overrightarrow{\mathbf{e}}, \mathbf{V}) = V_1 \otimes V_2 \otimes V_3$. Otherwise

$$TNS(\Gamma, \overrightarrow{\mathbf{e}}, \mathbf{V}) = Sub_{\mathbf{e}_1, \mathbf{e}_1\mathbf{e}_2, \mathbf{e}_2}(V_1 \otimes V_2 \otimes V_3).$$

Example 14.1.8.2. Let Γ be a chain with four vertices. If $\mathbf{v}_1 \leq \mathbf{e}_1$ and $\mathbf{v}_4 \leq \mathbf{e}_3$, then, writing $W = V_1 \otimes V_2$ and $U = V_3 \otimes V_4$, by Proposition 14.1.7.1, $TNS(\Gamma, \overrightarrow{\mathbf{e}}, \mathbf{V}) = \hat{\sigma}_{\mathbf{e}_2}(Seg(\mathbb{P}W \times \mathbb{P}U))$. Other chains of length 4 have similar complete descriptions.

Example 14.1.8.3. Let Γ be a chain with five vertices. Assume that $\mathbf{v}_1 \leq \mathbf{e}_1$, $\mathbf{v}_5 \leq \mathbf{e}_4$, $\mathbf{v}_1\mathbf{v}_2 \geq \mathbf{e}_2$, and $\mathbf{v}_4\mathbf{v}_5 \geq \mathbf{e}_3$. Then $TNS(\Gamma, \overrightarrow{\mathbf{e}}, \mathbf{V})$ is the image of a bundle over $G(\mathbf{e}_2, V_1 \otimes V_2) \times G(\mathbf{e}_3, V_4 \otimes V_5)$ whose fiber is the set of tensor network states associated to a chain of length 3.

14.2. Algebraic statistics and phylogenetics

Recall the discussion of algebraic statistics from §1.5.1.

Consider making statistical observations of repeated outcomes of n events, where the outcome of the p-th event has $\mathbf{a}_p$ different possibilities. (For example, throws of n dice where the i-th die has $\mathbf{a}_i$ faces.) The set of discrete probability distributions for this scenario is

$$PD_{\vec{\mathbf{a}}} := \Big\{T \in \mathbb{R}^{\mathbf{a}_1} \otimes \cdots \otimes \mathbb{R}^{\mathbf{a}_n} \mid T_{i_1,\dots,i_n} \geq 0, \sum_{i_1,\dots,i_n} T_{i_1,\dots,i_n} = 1\Big\}.$$

Given a particular $T \in PD_{\vec{\mathbf{a}}}$ representing some probability distribution, the events are independent if and only if $\operatorname{rank}(T) = 1$.

Thus there is a simple test to see if the distribution represented by T represents independent events: the 2×2 minors $\Lambda^2 \mathbb{R}^{\mathbf{a}_i *} \otimes \Lambda^2(\mathbb{R}^{\mathbf{a}_1} \otimes \cdots \otimes \hat{\mathbb{R}}^{\mathbf{a}_i} \otimes \cdots \otimes \mathbb{R}^{\mathbf{a}_n})^*$ must be zero.

A *discrete probability distribution* is a point in $PD_{\vec{\mathbf{a}}}$. For example, say we have two biased coins. Then $V = \mathbb{R}^2 \otimes \mathbb{R}^2$ and a point corresponds to a matrix

$$\begin{pmatrix} p_{h,h} & p_{h,t} \\ p_{t,h} & p_{t,t} \end{pmatrix},$$

where $p_{h,h}$ is the probability that both coins, when tossed, come up heads, etc.

A *statistical model* is a family of probability distributions given by a set of constraints that these distributions must satisfy, i.e., a subset of $PD_{\vec{\mathbf{a}}}$. An *algebraic statistical model* consists of all joint probability distributions that are the common zeros of a set of polynomials on V. For example, the model for independent events is $PD_{\vec{\mathbf{a}}}$ intersected with the Segre variety.

14.2.1. Graphical models. Certain algebraic statistical models can be represented pictorially by graphs, where each observable event is represented by a vertex of valence 1 (an external vertex), and the events that cannot be observed that influence event j are given vertices attached to j by an edge.

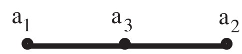


Figure 14.2.1. The numbers over nodes are the number of outcomes associated to node's event.

As discussed in §1.5.1, this picture reprents a situation where we can only observe two of the outcomes and the third event that cannot be measured influences the other two. Sum over all possibilities for the third factor to get a 2×2 matrix whose entries are

$$p_{i,j} = p_{i,j,1} + \cdots + p_{i,j,\mathbf{a}_3}, \quad 1 \leq i \leq \mathbf{a}_1, \ 1 \leq j \leq \mathbf{a}_2. \tag{14.2.1}$$

The algebraic statistical model here is the set of rank at most $\mathbf{a}_3$ matrices in the space of $\mathbf{a}_1 \times \mathbf{a}_2$ matrices, i.e., $\hat{\sigma}_{\mathbf{a}_3}(Seg(\mathbb{RP}^{\mathbf{a}_1-1} \times \mathbb{RP}^{\mathbf{a}_2-1}))$. Thus, given a particular model, e.g., a fixed value of $\mathbf{a}_3$, to test if the data (as points of $\mathbb{R}^{\mathbf{a}_1} \otimes \mathbb{R}^{\mathbf{a}_2}$) fit the model, one can check if they (mostly) lie inside $\hat{\sigma}_{\mathbf{a}_3}(Seg(\mathbb{RP}^{\mathbf{a}_1-1} \times \mathbb{RP}^{\mathbf{a}_2-1}))$.

Similarly, if we had n events where the i-th event had $\mathbf{a}_i$ possible outcomes, and we suspected there was a single hidden event influencing the n, we would test the data sets with equations for $\sigma_r(Seg(\mathbb{P}^{\mathbf{a}_1-1} \times \cdots \times \mathbb{P}^{\mathbf{a}_n-1}))$ for various r. With more hidden variables, one gets more complicated models.

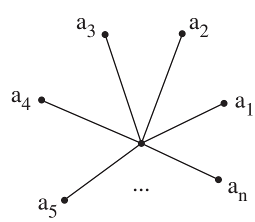


Figure 14.2.2. n events are influenced by a single hidden event.

Because complex polynomials always have roots, it is easier to solve this problem first over the complex numbers and then return to the real situation later. Thus to test models of the type discussed above, one needs defining equations for the relevant algebraic varieties.

For more on algebraic statistics, see [**168, 259**].

14.2.2. Phylogenetic invariants. In order to determine a tree that describes the evolutionary descent of a family of extant species, J. Lake [**200**], and J. Cavender and J. Felsenstein [**83**] proposed the use of what is now called *algebraic statistics* by viewing the four bases composing DNA as the possible outcomes of a random variable.

Evolution is assumed to progress by a bifurcating tree, beginning with a common ancestor at the root, whose DNA sequence one cannot observe. The common ancestor has two descendents. Each descendent in the tree either (1) survived to the present day, in which case one knows its sequence, or (2) did not, in which case one cannot observe its sequence, and we assume it splits into two descendents (and each of its descendents may itself have survived, or have two descendents). Thus we assume the tree is binary, but we do not know its combinatorial structure. (We will see below that if there are intermediate species in the mutations that do not bifurcate, one gets the same algebraic statistical model.)

One also assumes that evolution follows an identical process that depends only on the edge. This is called *independent and identically distributed random variables* and abbreviated (i.i.d.)—each random variable has the same probability distribution as the others and all are mutually independent. The probabilities along various edges of the tree depend only on the immediate ancestor. This is called a *Markov* assumption:

$$p_A = M_{AF} p_F,$$

where M_{AF} is a *Markov matrix*; that is, the entries are numbers between 0 and 1 and the entries of the rows each sum to 1. $(M_{AF})_{ij}$ represents the conditional probability that if base i appears at vertex A at a particular site, then base j appears at F at that site.

Allmann and Rhodes [**9, 8**] have observed that for such trees, one can work *locally* on the tree and glue together the results to get a *phylogenetic variety* that includes the possible probability distributions associated to a given evolutionary tree. I illustrate their algorithm below.

14.2.3. Reconstructing an evolutionary tree. Given a collection of extant species, one would like to assess the likelihood of each of the possible evolutionary trees that could have led to them. To do this, one can test the various DNA sequences that arise to see which algebraic statistical model for evolution fits best. The invariants discussed below identify the trees up to some unavoidable ambiguities.

The simplest situation is where one species gives rise to two new species; see Figure 14.2.3.

There are three species involved, the parent F and the two offspring A_1, A_2, so the DNA occupies a point of the positive coordinate simplex in

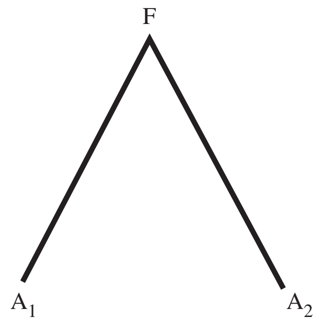


Figure 14.2.3. Ancestor F mutates to two new species, A_1 and A_2.

$\mathbb{R}^4 \otimes \mathbb{R}^4 \otimes \mathbb{R}^4$. To simplify the study, I work with $\mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4$. In terms of Markov matrices, the point is $(p_F, M_{A_1F}p_F, M_{A_2F}p_F)$. One can measure the DNA of the two new species but not the ancestor, so the relevant algebraic statistical model is just the set of 4×4 matrices $\mathbb{C}^4 \otimes \mathbb{C}^4 = \hat{\sigma}_4(Seg(\mathbb{P}^3 \times \mathbb{P}^3))$, which is well understood. Here $\mathbf{a}_1 = \mathbf{a}_2 = \mathbf{a}_3 = 4$ in the analogue of equation (14.2.1), and one sums over the third factor. In this case there is nothing new to be learned from the model.

Now consider the situation of an initial species, mutating to just one new species, which then bifurcates to two new ones.

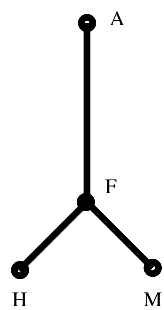


Figure 14.2.4. Ancestor mutates to a single new species, which is the father of both humans and monkeys.

Were we able to measure the probabilities of the ancestors in the tree, we would have a tensor

$$\begin{aligned}(p_A, p_F, p_H, p_M) &= (p_A, M_{FA}p_A, M_{HF}M_{FA}p_A, M_{MF}M_{FA}p_A) \\ &\in \mathbb{R}^\kappa_A \otimes \mathbb{R}^\kappa_F \otimes \mathbb{R}^\kappa_H \otimes \mathbb{R}^\kappa_M;\end{aligned}$$

note that this tensor is indeed rank one. Now we cannot measure either A or F. First note that if we take F as unknown, then A becomes irrelevant, so we might as well begin with

$$(p_F, p_H, p_M) = (p_F, M_{HF}p_F, M_{MF}p_F) \in \mathbb{R}^\kappa_F \otimes \mathbb{R}^\kappa_H \otimes \mathbb{R}^\kappa_M$$

and note this is still rank one. Finally, we sum over all possibilities for p_F to obtain a tensor of rank at most κ in $\mathbb{R}^\kappa_H \otimes \mathbb{R}^\kappa_M$.

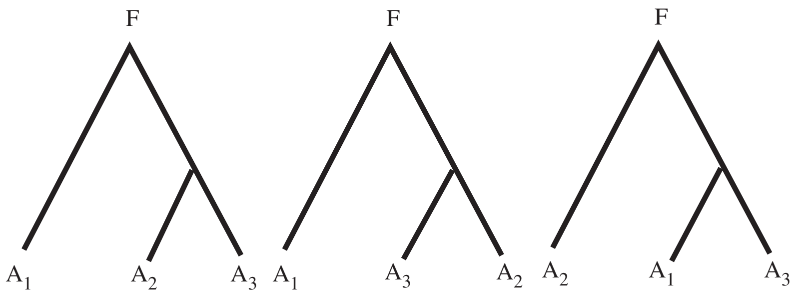


Figure 14.2.5. These three trees all give rise to the same statistical model.

Moving the case of three extant species, we assume a parent F gives rise to three extant species A_1, A_2, A_3. One might think that this gives rise to three distinct algebraic statistical models, as one could have F giving rise to A_1 and G, then G splitting to A_2 and A_3 or two other possibilities. However, all three scenarios give rise to the same algebraic statistical model: $\hat{\sigma}_4(Seg(\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3))$—the $(4, 4, 4)$-arrays of border rank at most four. See [**8**]. In other words, the pictures in Figure 14.2.3 all give rise to the same algebraic statistical models.

Now consider the case where there are four new species A_1, A_2, A_3, A_4 all from a common ancestor F. Here finally there are three different scenarios that give rise to distinct algebraic statistical models.

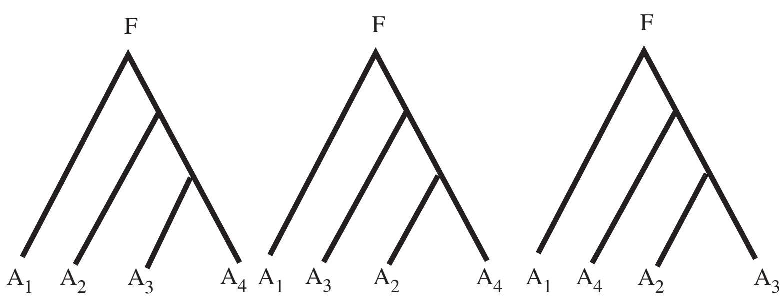


Figure 14.2.6. These three trees all give rise to diferent statistical models.

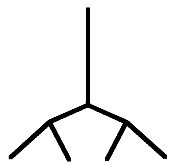


Figure 14.2.7. Trees such as this are statistically equivalent to the ones in Figure 14.2.3

Note that there are no pictures such as the one shown in Figure 14.2.7 because such pictures give rise to equivalent algebraic statistical models to the exhibited trees.

Say the parent F first gives rise to A_1 and E, and then E gives rise to A_2 and G, and G gives rise to A_3 and A_4, as well as the equivalent (by the discussion above) scenarios. The resulting (complexified, projectivized) algebraic statistical model is

$$\begin{aligned}\Sigma_{12,34} := \sigma_4(&Seg(\mathbb{P}A_1 \times \mathbb{P}A_2 \times \mathbb{P}(A_3 \otimes A_4)) \\ &\cap \sigma_4(Seg(\mathbb{P}(A_1 \otimes A_2) \times \mathbb{P}A_3 \times \mathbb{P}A_4)).\end{aligned}$$

Similarly one gets the other two possibilities:

$$\begin{aligned}\Sigma_{13,24} := \sigma_4(&Seg(\mathbb{P}A_1 \times \mathbb{P}A_3 \times \mathbb{P}(A_2 \otimes A_4)) \\ &\cap \sigma_4(Seg(\mathbb{P}(A_1 \otimes A_3) \times \mathbb{P}A_2 \times \mathbb{P}A_4))\end{aligned}$$

and

$$\begin{aligned}\Sigma_{14,23} := \sigma_4(&Seg(\mathbb{P}A_1 \times \mathbb{P}A_4 \times \mathbb{P}(A_2 \otimes A_3)) \\ &\cap \sigma_4(Seg(\mathbb{P}(A_1 \otimes A_4) \times \mathbb{P}A_2 \times \mathbb{P}A_3)).\end{aligned}$$

These three are isomorphic as projective varieties; that is, there are linear maps of the ambient spaces that take each to each of the others, but the varieties are situated differently in the ambient space $A_1 \otimes A_2 \otimes A_3 \otimes A_4$, so they have different defining equations. An essential result of [**8**] is:

Theorem 14.2.3.1 ([**8**]). *Once one has defining equations for $\sigma_4(Seg(\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3))$, one has defining equations for all algebraic statistical models corresponding to bifurcating phylogenetic trees.*

Theorem 14.2.3.1 is clear for the examples above as equations for $\sigma_4(Seg(\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^{15})$ are inherited from those of $\sigma_4(Seg(\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3))$; see §3.7.1. The actual variety is an intersection of secant varieties of Segre varieties, so its ideal is the sum of the two ideals. The general case is similar.

After proving Theorem 14.2.3.1, E. Allmann, who resides in Alaska, offered to hand-catch, smoke, and send an Alaskan salmon to anyone who could determine the equations to $\sigma_4(Seg(\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3))$, which were unknown at the time. This spurred considerable research, and such equations were found by S. Friedland, as explained in §7.7.3, which earned him the prize. However, the actual prize was to find generators of the ideal of the variety, and this question (which would merit another smoked salmon) is still open at this writing.

Part 4

Advanced topics

Chapter 15

Overview of the proof of the Alexander-Hirschowitz theorem

This chapter contains a brief overview of the ideas that go into the proof of the Alexander-Hirshowitz theorem on the nondegeneracy of secant varieties of Veronese varieties. It is intended as an introduction to the paper [**36**].

I assume the reader is familiar with the discussion of the Alexander-Hirshowitz theorem in Chapter 5.

The main tool for proving nondefectivity of secant varieties is Terracini's lemma, §5.3. To prove nondefectivity of $\sigma_r(v_d(\mathbb{P}V))$, one needs to pick r points on $v_d(\mathbb{P}V)$ and show that the span of their affine tangent spaces has dimension $\min(r\mathbf{v}, \binom{\mathbf{v}+d-1}{d})$. If one picks completely general points, there is the best chance of success, but the calculation is impossible. On the other hand, if one picks points that are special and easy to write down, they may be too special and their tangent spaces may fail to span. To prove the theorem, one would like to use an induction argument based on $\dim V$ and/or d. Since the case $d = 2$ is very defective, attempting to use induction on d starting with $d = 2$ would be doomed to failure, so the discussion begins with the case $d = 3$ in §15.1.1. Already back in 1916, Terracini had ideas for performing the induction which are explained in §15.1.2. Things did not advance much after that until Alexander and Hirschowitz had the idea to work not just with sets of points, but with limits of such sets as well.

This idea is discussed in §15.2, where a brief overview of the rest of their argument is given.

I thank G. Ottaviani for considerable help in writing this chapter. It is based on his lectures at MSRI, July 2008.

Throughout this chapter I adopt the notation $\dim V = \mathbf{v} = n+1$.

15.1. The semiclassical cases

15.1.1. The case $d=3$. Consider first the case of $d=3$. Note that

$$\frac{\dim S^3V}{\dim V} = \frac{\binom{n+3}{3}}{n+1} = \frac{(n+3)(n+2)}{6}$$

is an integer if and only if $n \not\equiv 2 \bmod 3$. This indicates that an induction from $n-1$ to n probably would not work, and that it is better to use $n-3$ to n instead.

Thus we will need three initial cases. Suppose we have these (e.g., by checking random points in small dimensions using a computer, or using a large sheet of paper), and now we want to handle the case $\dim V = n+1$, assuming that we have done all smaller cases. For simplicity assume that $n \not\equiv 2 \bmod 3$ so $\frac{(n+3)(n+2)}{6}$ is an integer.

Let $L \subset V$ be a codimension 3 subspace. We need to pick $\frac{(n+3)(n+2)}{6}$ points on $X = v_d(\mathbb{P}V)$, i.e., on $\mathbb{P}V$. Take $\frac{n(n-1)}{6}$ of them on $\mathbb{P}L$ and the remaining $n+1$ points outside. Consider the exact sequence of vector spaces

$$0 \to S^3L^\perp \to S^3V^* \to S^3L^* \to 0.$$

Note that $(\hat{T}_pX)^\perp \subset S^3V^*$ has codimension $n+1$. If $p \notin L$, then it maps to zero in S^3L^*. If $p \in L$, then $(\hat{T}_pX)^\perp \subset S^3V^*$ maps to an $(n-2)$-dimensional subspace of S^3L^*, and thus is the image of a codimension 3 subspace in $(S^3L)^\perp = (S^3V/S^3L)^*$.

The cubic case thus reduces to the following:

Proposition 15.1.1.1. *Let $L \subset V$ have codimension 3, let $\Sigma \subset \mathbb{P}V$ be a collection of $\frac{(n+3)(n+2)}{6}$ points, of which exactly $\frac{n(n-1)}{6}$ lie on $\mathbb{P}L$, but are otherwise in general position. Then there does not exist a cubic hypersurface in $\mathbb{P}V$ that contains $\mathbb{P}L$ and is singular at the points of Σ.*

In Proposition 15.1.1.1, a cubic hypersurface is to be thought of as determining a hyperplane in $\mathbb{P}S^3V$ that would contain all the tangent spaces of the $\frac{n(n-1)}{6} + n + 1$ points.

Proof. Choose a second linear subspace $\mathbb{P}M \subset \mathbb{P}V$ of codimension 3. Place

- $n-2$ points on $\mathbb{P}M \backslash (\mathbb{P}M \cap \mathbb{P}L)$;
- $n-2$ points on $\mathbb{P}L \backslash (\mathbb{P}M \cap \mathbb{P}L)$;

- three points on $\mathbb{P}V\backslash(\mathbb{P}M \cup \mathbb{P}L)$;
- and the remaining $\frac{n(n-1)}{6} - (n-2) = \frac{(n+3)(n+2)}{6} - 2(n-2) - 3$ points on $\mathbb{P}M \cap \mathbb{P}L$.

The space of cubics on $\mathbb{P}V$ which vanish on $\mathbb{P}L \cup \mathbb{P}M$ is $9(n-1)$-dimensional (note a linear function of n), the space of cubics on $\mathbb{P}V$ which vanish on $\mathbb{P}L$ is $(\frac{3}{2}n^2 + \frac{3}{2}n + 1)$-dimensional, and the space of cubics on $\mathbb{P}M$ which vanish on $\mathbb{P}(L \cap M)$ is $(\frac{3}{2}(n-3)^2 + \frac{3}{2}(n-3) + 1)$-dimensional, and there is an exact sequence between these spaces.

We are reduced to the following proposition:

Proposition 15.1.1.2. *There does not exist a cubic hypersurface $Z \subset \mathbb{P}V$ containing $\mathbb{P}L \cup \mathbb{P}M$ that is singular at $n-2$ points of $\mathbb{P}L$, $n-2$ points of $\mathbb{P}M$, and three points outside $\mathbb{P}L \cup \mathbb{P}M$.*

To prove Proposition 15.1.1.2 we choose yet a third codimension 3 subspace $\mathbb{P}N \subset \mathbb{P}V$ such that our last three points are on $\mathbb{P}N$ and there are $n-5$ points on $\mathbb{P}L \cap \mathbb{P}M$ among the $n-2$ on $\mathbb{P}L$, and similarly for $\mathbb{P}M$. We have an exact sequence of vector spaces of the following dimensions:

$$0 \to 27 \to 9(n-1) \to 9(n-4) \to 0.$$

Since 27 is independent of n, we may apply induction starting, e.g., in $\mathbb{P}^8$ and taking $\mathbb{C}^9 = L \oplus M \oplus N$. The cubics vanishing on $\mathbb{P}L \cup \mathbb{P}M \cup \mathbb{P}N$ are all of the form $\alpha\beta\gamma$ with $\alpha \in L^{\perp}, \beta \in M^{\perp}, \gamma \in N^{\perp}$, i.e., the union of three disjoint hyperplanes, and these fail to have the stated singularities. $\square$

15.1.2. An easy proof of most cases. The idea will be to use double induction, specializing to hyperplanes.

Let

$$s_1(d,n) = \lfloor \frac{\binom{n+d}{d}}{n+1} \rfloor - (n-2),$$
$$s_2(d,n) = \lceil \frac{\binom{n+d}{d}}{n+1} \rceil + (n-2).$$

Theorem 15.1.2.1 (essentially Terracini, 1916; see [**36**])**.** *For all $k \leq s_1(d,n)$ and all $k \geq s_2(d,n)$, the variety $\sigma_k(v_d(\mathbb{P}^n))$ is of the expected dimension $\min\{\binom{n+d}{d} - 1, kn + (k-1)\}$.*

"Essentially" because Terracini did not make such a statement, but, according to [**36**], everything in the proof is in Terracini's article; see [**36**] for details.

Proof. Use induction on n and d; the case $d = 3$ is done in §15.1.1, and assume the case $n = 2$. (This was done in 1892 by Campbell [**67**] with a better proof by Terracini in 1915; see [**308**]).

Here is the case $k \leq s_1(d,n)$; the other case is similar. Assume that $k = s_1(d,n)$, and specialize $s_1(d,n-1)$ points to H, so there are $s_1(d,n) - s_1(d,n-1)$ points off H. We consider the exact sequence of {hypersurfaces of degree $d-1$ on $\mathbb{P}V$ } to {hypersurfaces of degree d on $\mathbb{P}V$ } to {hypersurfaces of degree d on H }, where if $h \in V^*$ is an equation of H, the sequence is

$$\begin{gathered} 0 \to S^{d-1}V^* \to S^d V \to S^d(V^*/\langle h\rangle) \to 0, \\ P \mapsto Ph. \end{gathered}$$

We get to lower degree in the term on the left hand side and lower dimension on the right hand side. I now explain how to use this.

Let $V' \subset V$ be a hyperplane so $X = v_d(\mathbb{P}V) \supset X' = v_d(\mathbb{P}V')$. Consider the projection $\pi : \mathbb{P}(S^dV) \dashrightarrow \mathbb{P}(S^dV/S^dV')$ and let X'' denote the image of X.

We need to apply Terracini's lemma to a collection of points on X. The idea is to take two types of points: points whose tangent spaces are independent on X' and points whose tangent spaces are independent off X'.

Claim: X'' is isomorphic to $v_{d-1}(\mathbb{P}V)$. First note that $\dim(S^dV/S^dV') = \dim(S^{d-1}V)$. In fact $S^dV/S^dV' \simeq (V/V') \circ S^{d-1}V$.

Consider $x \in X\backslash X'$; then $\mathbb{P}\hat{T}_xX \cap \mathbb{P}(S^dV') = \emptyset$, so $d\pi_x$ is an isomorphism. If $y \in X'$, then $\mathbb{P}\hat{T}_yX \cap \mathbb{P}(S^dV') = \mathbb{P}\hat{T}_yX'$, so while π is not defined at y, it is defined at any pair $(y,[v])$, and the image of $\mathbb{P}\hat{T}_yX = Bl_xX$ is a point. (See §15.2.1 below for an explanation of blowups.)

Consider points $[z] \in \sigma_r(v_d(\mathbb{P}V))$ such that $z = x_1 + \cdots + x_p + y_1 + \cdots + y_{r-p}$ with $[x_j] \in X\backslash X'$ and $[y_s] \in X'$. We can use Terracini's lemma essentially separately for the two sets of points. More precisely, consider $S^dV = S^dV' \oplus S^{d-1}V \circ (V/V')$ and use Terracini's lemma separately for each factor. The first factor picks up the span of the $\hat{T}_{[y_s]}X'$ and the second factor picks up the span of the $\hat{T}_{[x_j]}X''$ plus the "image" of each $\hat{T}_{[y_s]}X$, which is a line. In summary:

Lemma 15.1.2.2 ([**36**, Theorem 4.1]). *If $\sigma_{k-u}(v_d(\mathbb{P}^{n-1})) \neq \mathbb{P}(S^d\mathbb{C}^n)$ and* codim $(\sigma_u(v_{d-1}(\mathbb{P}^n))) \geq k-u$, *then $\sigma_k(v_d(\mathbb{P}^n))$ is not defective.*

The problem has been reduced by one degree for one set of points (those on X'') and by one dimension for the other set (those on X').

Recall from Exercise 5.3.2.3 that $\dim \sigma_2(C) = 3$ for any curve not contained in a plane, so in particular $\sigma_2(v_3(\mathbb{P}^1)) = \mathbb{P}^3$. Note also that $\dim(S^d\mathbb{C}) = 1$, so $\sigma_1(v_d(\mathbb{P}^0))$ is its entire ambient space.

Recall from Exercise 5.3.2.4 that it is sufficient to verify the nondegeneracy of the last secant variety that is not the entire ambient space.

Exercises.

(1) Apply Lemma 15.1.2.2 to show that $\sigma_k(v_d(\mathbb{P}^1))$ is nondegenerate for all d, k.

(2) Show that the induction applies to show that $v_5(\mathbb{P}^3)$ has nondegenerate secant varieties, and that $\sigma_8(v_4(\mathbb{P}^3))$ is nondegenerate, but that the lemma does not apply to $\sigma_9(v_4(\mathbb{P}^3))$.

(3) (If you have some time on your hands.) Show that Lemma 15.1.2.2 applies to all $\sigma_r(v_d(\mathbb{P}^2))$ with $30 \geq d \geq 6$ except for $(d, r) = (6, 21), (9, 55), (12, 114), (15, 204), (21, 506), (27, 1015), (30, 1364)$.

This induction procedure gives a proof of almost all cases.

Note that $v_3(\mathbb{P}^n)$ requires special treatment because $v_2(\mathbb{P}^n)$ has degenerate secant varieties. □

15.2. The Alexander-Hirschowitz idea for dealing with the remaining cases

15.2.1. Blowups. To discuss the remaining cases we will need the concept of a blowup in what follows. Given a variety or a complex manifold X and a subvariety $Y \subset X$, let $U \subset X$ be a coordinate patch with coordinates $(z_1, \dots, z_n)$ such that $Y \cap U$ has local defining equations $f_j(z) = 0$, $1 \leq j \leq m = \operatorname{codim}(Y)$. Consider

$$\tilde{U}_Y := \{([Z_1, \dots, Z_m], (z_1, \dots, z_n)) \in \mathbb{P}^{m-1} \times U \mid Z_i f_j(z) = Z_j f_i(z)\ \forall i, j, \leq m\}.$$

Then $\tilde{U}_Y$ is a smooth subvariety of $\mathbb{P}^{m-1} \times U$. There is a map $\tau : \tilde{U}_Y \to Y$ that is an isomorphism off $U \backslash (Y \cap U)$, and over $Y \cap U$ the fiber can be identified with the projectivized conormal bundle $\mathbb{P}N^*Y|_U$. Performing the same procedure over a covering of X by coordinate patches, the resulting maps can be glued together to obtain a global map $\tau : \tilde{X}_Y \to X$, which is called the *blowup* of X along Y. It consists of removing Y from X and replacing it by its projectivized conormal bundle. One use of blowups is desingularization: Say $C \subset \mathbb{P}^2$ is a curve having a double point at a point $y \in \mathbb{P}^2$. If we blow up $\mathbb{P}^2$ along y, the inverse image of C in $\tilde{\mathbb{P}}^2_y$ is smooth. Another use is to keep track of infinitesimal data in limits, which is how we will use them here. Blowups are also used to "resolve" rational maps $X \dashrightarrow Z$, i.e., blowing up X where the rational map $X \dashrightarrow Z$ is not defined, one obtains a new rational map $X' \dashrightarrow Z$, and one can blow up again; continuing, eventually one arrives at a variety $\tilde{X}$ and a regular map $\tilde{X} \to Z$.

15.2.2. Dancing points and the curvilinear lemma. We now consider $\sigma_{21}(v_6(\mathbb{P}^3))$, a typical case where the above induction does not apply. Counting dimensions, we expect that $\sigma_{21}(v_6(\mathbb{P}^3)) = \mathbb{P}(S^6\mathbb{C}^4) = \mathbb{P}^{83}$. Back with $\sigma_{20}(v_6(\mathbb{P}^3))$ everything is fine as we can put 11 points on X''

and 9 on X', and since $[11(4)-1]+9=52<55=\dim \mathbb{P}S^{d-1}V$ and $9(3)-1=26<27=\dim \mathbb{P}S^dV'$, Lemma 15.1.2.2 applies. But now there is no place to put the 21-st point and still apply the lemma.

The idea of Alexander and Hirschowitz is: instead of using four points, take three points and a tangent vector.

The following lemma simplifies checking the independence of triples of tangent spaces:

Lemma 15.2.2.1 (The curvilinear lemma; see [**84**])**.** *Given $p_1, p_2, p_3 \in X$, their embedded tangent spaces are linearly independent (i.e., $\hat{T}_{p_3}X \cap \langle \hat{T}_{p_1}X, \hat{T}_{p_2}X\rangle = 0$ and similarly for permutations) if and only if for all triples of lines $\ell_j \in \mathbb{P}\hat{T}_{p_j}X$, the ℓ_j are linearly independent (i.e., $\ell_3 \cap \langle \ell_1, \ell_2\rangle = 0$ and similarly for permutations).*

See, e.g., [**36**, §6.1] for a streamlined proof.

K. Chandler [**84**] simplified the original arguments by introduction of the *dancing point*. I illustrate how the remaining cases can be treated by proving a typical case: the nondegeneracy of $\sigma_{21}(v_6(\mathbb{P}^3))$. Say it was degenerate. Assume we have already chosen 20 points $x_1, \dots, x_{20}$ whose tangent planes do not intersect using the $11, 9$ split; then for a generic choice of x_{21} and for all $v \in \mathbb{P}\hat{T}_{x_{21}}X$, by the curvilinear lemma, we would get a line intersecting the other tangent spaces. Consider a curve of such points $(x_{21,t}, v_t)$ with $x_{21,0} \in X'$. Naïvely, we would need to obtain an $84 = \dim S^6\mathbb{C}^4$-dimensional span somehow, but in fact it is enough to have at least an 82-dimensional span using the curvilinear lemma. The key point is to consider $\sigma_{11}(v_4(\mathbb{P}^3))$, which fills its ambient space. Thanks to the curvilinear lemma, instead of having to prove that the affine tangent spaces of the 21 points span an 84-dimensional space, we can just use one tangent line to the 21-st point $x_{21,0}$ (and we will use the tangent line through v_0). Thus the strategy is to assume that the span of the original 20 points plus the one tangent line is at most an 82-dimensional space and derive a contradiction.

There are two cases:

Case 1: $[v_0] \notin \mathbb{P}\hat{T}_{x_{21,0}}X'$. This case is easy, as then $(x_{21,0}, v_0)$ projects to a single point of X'' and we have room to apply the induction step. In this case we have the span of 11 embedded tangent spaces on X'', (11×4) plus 9 points plus the point corresponding to the image of $(x_0, [v_0])$. Thus, $44 + 9 + 1 = 54 < \dim S^5\mathbb{C}^4 = 56$. Similarly, on X' we have the span of 9 embedded tangent spaces plus our last point, giving $9 \times 3 + 1 = 28 = \dim S^6(\mathbb{C}^3) = 28$. Finally $54 + 28 = 82 > 81$.

Case 2: $[v_0] \in \mathbb{P}\hat{T}_{x_{21,0}}X'$. In this case we need to work harder. I will be sketchy here and refer the reader to [**36**] for details.

The difficult point is to show that the span in $X'' = v_5(\mathbb{P}^3)$ of the 11 embedded tangent spaces plus the 9 points plus the new point $\pi(x_0, [v_0])$ is of the expected dimension.

Now $\sigma_9(v_5(\mathbb{P}^3))$ is easily seen to be nondegenerate, i.e., its affine tangent space at a smooth point is of dimension 40, but we want to take 9 general points on a $\mathbb{P}^2 \subset \mathbb{P}^3$ and still have all 9 tangent spaces independent. This can be done because there does not exist a quintic surface in $\mathbb{P}^3$ that is the union of a plane and a quartic surface containing the double points.

Now $\sigma_{11}(v_4(\mathbb{P}^2))$ equals its ambient space, thus the projection map from the span of the tangent spaces to the 11 points on $X \backslash X'$ to the span of the tangent spaces of their image in X'' has a $56 - 44 = 12$-dimensional image.

It follows that the span of these spaces and the line through v_0 plus the 9 points coming from the image of the tangent spaces to points on X' span $S^5(V/V')$.

Thus by semicontinuity, we also see that the span in X'' of the 11 embedded tangent spaces plus the 9 points plus $\tilde{\pi}(x_t, [v_t])$ is of the expected dimension.

Thus for $t \neq 0$ we have the correct $9 \times 3 = 27$-dimensional span on X', but then again by semicontinuity, this must still hold at $t = 0$.

It follows that v_0 and the 9 simple points give 12 independent conditions on the plane quintics. Hence there are no quintic surfaces containing $[v_0]$, the 9 simple points on the plane and the 11 double points, as we wanted.

Chapter 16

Representation theory

The purpose of this chapter is three-fold: to state results needed to apply Weyman's method in Chapter 17, to prove Kostant's theorem that the ideal of a rational homogeneous variety is generated in degree two, and to put concepts such as *inheritance* in a larger geometric context. It assumes that the reader is already familiar with basic representation theory. Definitions and a terse overview are presented primarily to establish notation. For proofs and more complete statements of what follows, see any of [**34, 33, 135, 170, 187, 143**].

In §16.1, basic definitions regarding root systems and weights are given. The Casimir operator is defined and Kostant's theorem is proved in §16.2. In §16.3 homogeneous vector bundles are defined and the Bott-Borel-Weil theorem is stated. The chapter concludes in §16.4 with a generalization of the *inheritance* technique discussed in Chapter 7. I continue to work exclusively over the complex numbers.

The reader should keep in mind the examples of groups and rational homogeneous varieties G/P that have already been discussed in Chapters 6 and 7 and §13.8: Grassmannians $G(k,V)$, Lagrangian Grassmannians $G_{Lag}(n,2n)$, spinor varieties $\mathbb{S}_k$, flag varieties, etc.

16.1. Basic definitions

16.1.1. Reduction to the simple case. A Lie algebra $\mathfrak{g}$ (resp. Lie group G) is called *simple* if it has no nontrivial ideals (resp. normal subgroups), and *semisimple* if it is the direct sum of simple Lie algebras (resp. product of simple groups). Semisimple Lie algebras are *reductive*, that is, all their finite-dimensional representations decompose into a direct sum of irreducible representations. Not all reductive Lie algebras are semisimple. For example, $\mathfrak{sl}(V)$ is simple, while $\mathfrak{gl}(V) = \mathfrak{sl}(V) \oplus \{\lambda \operatorname{Id}\}$ is reductive but not semisimple. However, all reductive Lie algebras are either semisimple or semisimple plus an abelian Lie algebra.

The irreducible representations of a semisimple Lie algebra are the tensor products of the irreducible representations of its simple components, so it will suffice to study irreducible representations of simple Lie algebras. Thanks to the adjoint action of $\mathfrak{g}$ on itself, without loss of generality, we may assume that $\mathfrak{g} \subset \mathfrak{gl}(V)$ is a matrix Lie algebra.

16.1.2. The maximal torus and weights. Simple Lie algebras may be studied via maximal diagonalizable subalgebras $\mathfrak{t} \subset \mathfrak{g}$ that generalize the set of diagonal matrices in $\mathfrak{sl}_n$. I will fix one such and decompose $\mathfrak{g}$-modules V with respect to $\mathfrak{t}$. This was done for $\mathfrak{gl}(V)$-modules in §6.8 using the diagonal matrices.

Exercise 16.1.2.1: Let $A_1, \dots, A_r$ be $n \times n$ matrices that commute. Show that if each A_j is diagonalizable, i.e., if there exists $g_j \in GL(n)$ such that $g_j A_j g_j{}^{-1}$ is diagonal, then $A_1, \dots, A_r$ are simultaneously diagonalizable. (That is, there exists $g \in GL(n)$ such that gA_jg^{-1} is diagonal for all j.) ⊚

If a matrix A is diagonalizable, then V may be decomposed into eigenspaces for A, and there is an eigenvalue associated to each eigenspace. Let $\mathfrak{t} = \langle A_1, \dots, A_r \rangle \subset \mathfrak{g}$ be the subspace spanned by diagonalizable commuting endomorphisms $A_1, \dots, A_r$. Assume that $\mathfrak{t}$ is maximal in the sense that no diagonalizable element of $\mathfrak{g}$ not in $\mathfrak{t}$ commutes with it. Then V may be decomposed into simultaneous eigenspaces for all $A \in \mathfrak{t}$. For each eigenspace V_j, define a function $\lambda_j : \mathfrak{t} \to \mathbb{R}$ by the condition that $\lambda_j(A)$ is the eigenvalue of A associated to the eigenspace V_j. Note that λ_j is a *linear* map, so $\lambda_j \in \mathfrak{t}^*$.

If there are p distinct $\mathfrak{t}$-eigenspaces of V, then these λ_j give p elements of $\mathfrak{t}^*$, called the *weights* of V. The dimension of V_j is called the *multiplicity* of λ_j in V. The decomposition $V = \bigoplus_j V_j$ is called the *weight space decomposition* of V.

Exercise 16.1.2.2: Show that all irreducible representations of an abelian Lie algebra $\mathfrak{t}$ are one-dimensional. ⊚

Now, back to our simple Lie algebra $\mathfrak{g}$. There always exists a maximal diagonalizable subalgebra $\mathfrak{t} \subset \mathfrak{g}$, unique up to conjugation by G, called a *maximal torus*. The *rank* of $\mathfrak{g}$ is defined to be the dimension of a maximal torus. Amazingly, irreducible (finite-dimensional) representations of $\mathfrak{g}$ are completely determined up to equivalence by the action of $\mathfrak{t}$.

This notion of weights agrees with that of §6.8. In that section T-weights of a $GL(V)$-action were also mentioned. They have a natural generalization to T-weights for arbitrary reductive G. When I refer to weights of a G-module in this chapter, I will mean weights of the corresponding $\mathfrak{g}$-module. (Not every $\mathfrak{g}$-module is a G-module if G is not simply connected. For example, the spin representations in §13.8.4 are not $SO(2n)$-modules.)

16.1.3. Roots. Let a simple Lie algebra $\mathfrak{g}$ act on itself by the adjoint action. Write its weight space decomposition as

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha,$$

where $R \subset \mathfrak{t}^* \backslash 0$ is some finite subset called the set of *roots* of $\mathfrak{g}$, and $\mathfrak{g}_\alpha$ is the eigenspace associated to the root α. Here $\mathfrak{g}_0 = \mathfrak{t}$ is the weight zero subspace. Fact: The eigenspaces $\mathfrak{g}_\alpha$ for $\alpha \neq 0$ are all one-dimensional.

Remark 16.1.3.1 (Why the word "root"?)**.** Consider the adjoint representation $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ restricted to $\mathfrak{t}$. The roots of the characteristic polynomial $p_\lambda(X) = \det(\mathrm{ad}(X) - \lambda \,\mathrm{Id}_{\mathfrak{g}})$ are the eigenvalues of X. By varying X one obtains linear functions on $\mathfrak{t}$ which are the roots of $\mathfrak{g}$. Weights were called "generalized roots" when the theory was being developed because the weights of a representation $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ are the roots of the characteristic polynomial $p_\lambda(X) = \det(\rho(X) - \lambda \,\mathrm{Id}_V)$.

Exercise 16.1.3.2: Let V and W be $\mathfrak{g}$-modules. If the weights of V are $\mu_1, \dots, \mu_k$ and the weights of W are $\nu_1, \dots, \nu_s$ (each counted with multiplicity), show that the weights of $V \otimes W$ are $\mu_i + \nu_l$. ⊚

Exercise 16.1.3.3: Let $\mathfrak{g} = \mathfrak{sl}_{\mathbf{v}}$. Here the rank of $\mathfrak{g}$ is $\mathbf{v} - 1$. Let $V = \mathbb{C}^{\mathbf{v}}$ have weight basis $v_1, \dots, v_{\mathbf{v}}$. In matrices with respect to this basis, if we write the weight of v_i as ϵ_i, then

$$\epsilon_i \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_{\mathbf{v}} \end{pmatrix} = \lambda_i.$$

Give V^* dual basis v^j. Let $e^i_j = v^i \otimes v_j \in V^* \otimes V$. Show that the roots are $\epsilon_i - \epsilon_j$, $i \neq j$, and $\mathfrak{g}_{\epsilon_i - \epsilon_j} = \langle e^i_j \rangle$. Show that $\Lambda^2 V$ has weights $\epsilon_i + \epsilon_j$ with $i < j$, also each with multiplicity one. What are the weights of V^*? The weights of $S^2 V$?

The *Killing form* $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ is given by $B(X, Y) = \mathrm{tr}(\mathrm{ad}(X) \circ \mathrm{ad}(Y))$, where $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ is the adjoint representation. The restriction of the Killing form to $\mathfrak{t}$ induces an inner product on $\mathfrak{t}^*$, which will be denoted $\langle \cdot, \cdot \rangle$.

The classification of complex simple Lie algebras $\mathfrak{g}$ (due to Killing, with some help from Cartan) is based on classifying the possible *root systems*, the collection of roots for $\mathfrak{g}$. The axioms for $R \subset \mathfrak{t}^*$ are:

(1) $\langle R \rangle = \mathfrak{t}^*$;

(2) for each $\alpha \in R$, reflection in the hyperplane perpendicular to α maps R to R;

(3) for all $\alpha, \beta \in R$, $2\langle \beta, \alpha \rangle / \langle \alpha, \alpha \rangle \in \mathbb{Z}$; and

(4) for all $\alpha \in R$, $2\alpha \notin R$;

(5) R cannot be split up into two separate root systems in complementary subspaces.

Exercise 16.1.3.4: Verify that the roots of $\mathfrak{sl}_n$ and $\mathfrak{so}_{2n}$ satisfy these axioms.

One may define an order on the roots by choosing a general hyperplane in $\mathfrak{t}^*$. Choosing a positive side to the hyperplane, R splits into two subsets $R^+ \cup R^-$, the *positive* and *negative* roots.

The Killing form restricted to the torus $\mathfrak{t}$ is $B|_{\mathfrak{t}} = \sum_{\alpha \in R^+} \alpha^2$.

Fixing a set of positive roots, a positive root is *simple* if it cannot be written as a sum of two other positive roots. For example, the simple roots of $\mathfrak{sl}_{\mathbf{v}}$ are the $\epsilon_i - \epsilon_{i+1}$, $1 \leq i \leq \mathbf{v} - 1$.

16.1.4. The weight lattice and dominant weights. Fix $\mathfrak{g}$ and a torus. Not every element of $\mathfrak{t}^*$ occurs as a weight. The elements that do occur as weights lie on a lattice, called the *weight lattice*, denoted $\Lambda_{\mathfrak{g}}$. The weight lattice is the set of $\ell \in \mathfrak{t}^*$ such that $\langle \ell, \alpha' \rangle \in \mathbb{Z}$ for all $\alpha' \in L_{R'}$, where $L_{R'}$ is the lattice generated by the *coroots* $\{\alpha' = \frac{2}{\langle \alpha, \alpha \rangle}\alpha \mid \alpha \in R\}$.

Place an order on the weight lattice by declaring $\lambda > \mu$ if $\lambda - \mu$ is a positive combination of positive roots. This order distinguishes a positive cone $\Lambda_{\mathfrak{g}}^+ \subset \Lambda_{\mathfrak{g}}$, called the set of *dominant weights*. When dealing with groups, I write Λ_G for the weight lattice of G, etc. We have the inclusion $\Lambda_G \subseteq \Lambda_{\mathfrak{g}}$ with equality if and only if G is simply connected. Otherwise it is a sublattice with index equal to the order of the fundamental group of G. A vector in V is called a *weight vector* if it is in an eigenspace for the torus, and a *highest weight vector* if the corresponding weight is a highest weight in the sense that a weight λ for an irreducible module V is a highest weight for V if $\lambda > \mu$ for all other weights μ of V.

Exercise 16.1.4.1: Show that if λ is a highest weight with highest weight vector v_λ, then $X_\alpha . v_\lambda = 0$ for all $\alpha \in R^-$ where $X_\alpha \in \mathfrak{g}_\alpha$.

The highest weight of an irreducible representation (which necessarily has multiplicity one) determines all other weights, along with their multiplicities. The elements of $\Lambda^+_{\mathfrak{g}}$ are the possible highest weights. An irreducible $\mathfrak{g}$-module V is generated by the spaces obtained by applying compositions of elements of $\mathfrak{g}$ to a highest weight line.

The weight lattice has $r = \dim \mathfrak{t}^*$ generators; denote them $\omega_1, \dots, \omega_r$. The irreducible representations correspond to r-tuples of nonnegative integers $(\ell_1, \dots, \ell_r)$ which determine a highest weight $\lambda = \ell_1\omega_1 + \cdots + \ell_r\omega_r$. Label the $\mathfrak{g}$-module with highest weight $\lambda \in \Lambda^+_{\mathfrak{g}}$ by V_λ.

Example 16.1.4.2. When $\mathfrak{g} = \mathfrak{sl}_{\mathbf{v}} = \mathfrak{sl}(V)$, the weight lattice is generated by $\omega_1 = \epsilon_1$, $\omega_2 = \epsilon_1 + \epsilon_2, \dots, \omega_{\mathbf{v}-1} = \epsilon_1 + \cdots + \epsilon_{\mathbf{v}-1}$. These are respectively the highest weights of the irreducible representations V, $\Lambda^2 V, \dots, \Lambda^{\mathbf{v}-1}V$.

Proposition 16.1.4.3. *The module $V_{\lambda+\mu}$ occurs in $V_\lambda \otimes V_\mu$ with multiplicity one. The submodule $V_{\lambda+\mu}$ is called the Cartan product of V_λ with V_μ.*

Proposition 16.1.4.3 elaborates on the discussion in §6.10.5.

The dimension of an irreducible module is given by the celebrated *Weyl dimension formula* . Let α_i denote the simple roots. Let

$$\rho = \frac{1}{2}(\alpha_1 + \cdots + \alpha_r) = \omega_1 + \cdots + \omega_r. \tag{16.1.1}$$

(While we have enough information to prove it, the equality in (16.1.1) is not obvious from our discussion.)

Remark 16.1.4.4. One geometric significance of ρ is that the minimal homogeneous embedding of the flag variety G/B is in $\mathbb{P}V_\rho$.

Theorem 16.1.4.5 (Weyl; see, e.g., [**187**, p. 267])**.** *In the above notation, the dimension of V_λ equals*

$$\frac{\prod_{\alpha\in R^+}\langle \lambda+\rho, \alpha\rangle}{\prod_{\alpha\in R^+}\langle \rho, \alpha\rangle}.$$

16.1.5. Exercises.

(1) Prove Proposition 16.1.4.3.

(2) Compute the dimension of the $SL(V)$-module $V_{d\omega_i}$.

16.2. Casimir eigenvalues and Kostant's theorem

In this section I define the Casimir operator and prove Kostant's theorem that the ideal of a rational homogeneous variety is generated in degree two.

16.2.1. The Casimir operator. For any Lie algebra $\mathfrak{g}$, a $\mathfrak{g}$-module V can be seen as a module over the *universal envelopping algebra*

$$U(\mathfrak{g}) := \mathfrak{g}^{\otimes}/\{X\otimes Y - Y\otimes X - [X,Y] \mid X, Y \in \mathfrak{g}\}.$$

The universal envelopping algebra $U(\mathfrak{g})$ acts on a $\mathfrak{g}$-module V by $X_{i_1}\otimes\cdots\otimes X_{i_q}.v = X_{i_1}(\cdots(X_{i_q}.v))$.

Exercise 16.2.1.1: Show that this action is well defined.

Let $\mathfrak{g}$ be semisimple, and let X_i and Y_i denote two bases of $\mathfrak{g}$ which are dual with respect to the Killing form (i.e., $B(X_i, Y_j) = \delta_{ij}$). Let

$$\mathrm{Cas} = \sum_i X_i \otimes Y_i \in U(\mathfrak{g})$$

denote the *Casimir operator*. That Cas does not depend on the choice of basis is a consequence of the following proposition:

Proposition 16.2.1.2. *The Casimir operator* Cas *commutes with the action of $\mathfrak{g}$ on any module V. In particular, if V is irreducible,* Cas *acts on V by a scalar.*

The scalar of Proposition 16.2.1.2 is called the *Casimir eigenvalue*.

Proof. If $Z \in \mathfrak{g}$, write $[X_i, Z] = \sum_j z_{ij} X_j$. By the invariance of the Killing form, $z_{ij} = B([X_i, Z], Y_j) = B(X_i, [Z, Y_j])$, hence $[Y_j, Z] = -\sum_i z_{ij} Y_i$. For any $v \in V$,

$$\begin{aligned}\mathrm{Cas}(Zv) = \textstyle\sum_i X_i Y_i Z v &= \textstyle\sum_i X_i([Y_i, Z] + Z Y_i)v \\ &= -\textstyle\sum_{i,j} z_{ji} X_i Y_j v + \sum_i ([X_i, Z] + Z X_i) Y_i v \\ &= -\textstyle\sum_{i,j} z_{ji} X_i Y_j v + \sum_{i,j} z_{ij} X_j Y_i v + \sum_i Z X_i Y_i v \\ &= Z(\mathrm{Cas}(v)).\end{aligned}$$

This proves the first claim. The second follows from Schur's lemma (in §6.1). □

Let us compute the Casimir eigenvalue of an irreducible $\mathfrak{g}$-module. We first need a lemma on preferred bases of a semisimple Lie algebra. Let $\mathfrak{t}$ be a maximal torus.

Lemma 16.2.1.3. *One may take a basis $(E_\alpha, E_{-\alpha}, H_j)$ of $\mathfrak{g}$ such that*

1. *The H_j form a Killing-orthonormal basis of $\mathfrak{t}$.*
2. *For $\alpha \in R^+$, $E_\alpha \in \mathfrak{g}_\alpha$ and $E_{-\alpha} \in \mathfrak{g}_{-\alpha}$.*
3. *$[E_\alpha, E_{-\alpha}] = h_\alpha$, where $h_\alpha \in \mathfrak{t}$ is defined by $B(h_\alpha, H) = \alpha(H)$ for all $H \in \mathfrak{t}$.*
4. *$B(E_\alpha, E_{-\alpha}) = 1$.*

Such a basis is called a Chevalley basis.

For example, the Chevalley basis for $\mathfrak{sl}_{\mathbf{V}}$ is just e^i_j as the E_α, and the H_j are the $e^i_i - e^j_j$, following the notation of Example 16.1.3.3.

Proof. The first two conditions are clear. The third and fourth are always true up to a scalar multiple (by the nondegeneracy of the Killing form on $\mathfrak{t}$ and its perfect pairing of $\mathfrak{g}_\alpha$ with $\mathfrak{g}_{-\alpha}$). Thus the only thing to prove is that the two scalars can be simultaneously adjusted to one. Fix $E_\alpha \in \mathfrak{g}_\alpha, E_{-\alpha} \in \mathfrak{g}_{-\alpha}$. Consider

$$\begin{aligned} B([E_\alpha, E_{-\alpha}], H) &= B(E_{-\alpha}, [H, E_\alpha]) \\ &= \alpha(H) B(E_{-\alpha}, E_\alpha). \end{aligned}$$

Thus requiring $B(E_{-\alpha}, E_\alpha) = 1$ exactly fixes $[E_\alpha, E_{-\alpha}] = h_\alpha$. □

Exercise 16.2.1.4: Find a Chevalley basis for $\mathfrak{so}_n$.

For any weight vector v_μ in any $\mathfrak{g}$-module,

$$h_\alpha(v_\mu) = \mu(h_\alpha) v_\mu = \langle \alpha, \mu \rangle v_\mu. \tag{16.2.1}$$

With respect to the Killing dual bases $(E_\alpha, H_i, E_{-\alpha})$ and $(E_{-\alpha}, H_i, E_\alpha)$:

$$\begin{aligned} \mathrm{Cas} &= \sum_{j=1}^r H_j^2 + \sum_{\alpha \in R^+} E_{-\alpha} E_\alpha + \sum_{\alpha \in R^+} E_\alpha E_{-\alpha} \\ &= \sum_{j=1}^r H_j^2 + \sum_{\alpha \in R^+} h_\alpha + 2 \sum_{\alpha \in R^+} E_{-\alpha} E_\alpha. \end{aligned}$$

Let H_ρ be defined by $B(H_\rho, H) = \rho(H)$, where ρ was defined by (16.1.1). With this notation,

$$\begin{aligned} \mathrm{Cas} &= \Big(\sum_{j=1}^r H_j^2 + 2H_\rho \Big) + 2 \sum_{\alpha \in R^+} E_{-\alpha} E_\alpha \\ &= E + N. \end{aligned}$$

Note that N will kill any highest weight vector in a representation, so if V_λ is an irreducible $\mathfrak{g}$-module of highest weight λ and v_λ is a highest weight vector, then

$$\mathrm{Cas}(v_\lambda) = \Big(\sum_{j=1}^r H_j^2 + 2H_\rho \Big)(v_\lambda) = (\langle \lambda, \lambda \rangle + 2\langle \rho, \lambda \rangle) v_\lambda.$$

In summary:

Proposition 16.2.1.5. *The Casimir operator acts on the irreducible $\mathfrak{g}$-module of highest weight λ as $c_\lambda \,\mathrm{Id}$, where*

$$c_\lambda = \langle \lambda + \rho, \lambda + \rho \rangle - \langle \rho, \rho \rangle.$$

Exercise 16.2.1.6: Using the labeling of the fundamental weights as in [**34**], show that the modules $V_{\omega_2+\omega_6}$ and V_{ω_5} for $\mathfrak{e}_6$ have the same Casimir eigenvalue.

16.2.2. Kostant's theorem on $I(G/P)$. For each $\lambda \in \Lambda_G^+$, consider the irreducible representation V_λ. The orbit of a highest weight line $[v_\lambda]$ in $\mathbb{P}V_\lambda$ is a rational homogeneous variety G/P, where P is the stabilizer of $[v_\lambda]$.

Exercise 16.2.2.1: Show that $I_d(G/P) = V_{d\lambda}{}^\perp \subset S^d V_\lambda^*$. ⊚

Thus the Hilbert function (Definition 4.9.4.5) of $G/P \subset \mathbb{P}V_\lambda$ is given by

$$Hilb_{G/P}(m) = \dim V_{m\lambda}.$$

Exercise 16.2.2.2: Compute the degree of the Grassmannian $G(k, V)$.

Exercise 16.2.2.3: Compute the degree of the Veronese variety $v_d(\mathbb{P}V)$.

I now prove Kostant's theorem that $I(G/P)$ is generated in degree two. By Exercise 16.2.2.1, we need to show that the multiplication map

$$V_{2\lambda}{}^\perp \otimes S^{d-2}V^* \to V_{d\lambda}{}^\perp \tag{16.2.2}$$

is surjective.

Lemma 16.2.2.4 (Kostant)**.** *Let $\nu, \nu' \in \Lambda_{\mathfrak{g}}^+$ be nonzero and such that $\nu - \nu' \in \Lambda_{\mathfrak{g}}^+$. Then $\langle \nu, \nu \rangle \geq \langle \nu', \nu' \rangle$ with equality holding if and only if $\nu = \nu'$.*

Proof. We have

$$\langle \nu, \nu \rangle - \langle \nu, \nu' \rangle = \langle \nu, \nu - \nu' \rangle \geq 0,$$
$$\langle \nu, \nu' \rangle - \langle \nu', \nu' \rangle = \langle \nu', \nu - \nu' \rangle \geq 0.$$

Thus $\langle \nu, \nu \rangle - \langle \nu', \nu' \rangle \geq 0$, and if equality holds, then $\langle \nu, \nu \rangle = \langle \nu, \nu' \rangle$. □

Rather than just study (16.2.2), consider the more general situation of the projection map to the Cartan product

$$V_{\mu_1} \otimes \cdots \otimes V_{\mu_k} \to V_{\mu_1 + \cdots + \mu_k},$$

where each V_{μ_i} is an irreducible $\mathfrak{g}$-module of highest weight μ_i. Write $V = V_{\mu_1} \otimes \cdots \otimes V_{\mu_k}$ and $\mu = \mu_1 + \cdots + \mu_k$.

Proposition 16.2.2.5 (Kostant)**.** *The $\mathfrak{g}$-module $V_\mu \subset V = V_{\mu_1} \otimes \cdots \otimes V_{\mu_k}$ is the unique submodule with Casimir eigenvalue c_μ.*

Proof. Say $V_\nu \subset V$ is such that $c_\nu = c_\mu$. Then $\nu = \nu_1 + \cdots + \nu_k$ with each ν_j a weight of V_{μ_j}, so $\nu_j \leq \mu_j$. Now apply Lemma 16.2.2.4 to $\mu + \rho$ and $\nu + \rho$. □

We are now ready to prove Kostant's theorem.

Theorem 16.2.2.6 (Kostant)**.** *The ideal of the closed orbit $X = G.[v_\lambda] \subset \mathbb{P}V_\lambda = \mathbb{P}V$ is generated in degree two.*

Proof. It will be easier to show the slightly stronger statement that for any tensor product as above, $V_\mu \subseteq (\bigcap_{i<j} V_{ij})$, where $V_{ij} \subset V$ is the submodule in which the $V_{\mu_i} \otimes V_{\mu_j}$-factor is replaced with the submodule $V_{\mu_i+\mu_j}$. For example, $V_{12} = V_{\mu_1+\mu_2} \otimes V_{\mu_3} \otimes \cdots \otimes V_{\mu_k}$.

The stronger statement implies the result by setting all $\mu_i = \lambda$, symmetrizing, and taking the annihilator to show that ${V_{k\lambda}}^\perp = {V_{2\lambda}}^\perp \circ S^{k-2}V^*$.

Since it is clear that $V_\mu \subseteq (\bigcap_{i<j} V_{ij})$, it remains to show that $V_\mu \supseteq (\bigcap_{i<j} V_{ij})$. Since V_μ is determined by its Casimir eigenvalue c_μ, thanks to Proposition 16.2.2.5, it will be sufficient to show that $(\bigcap_{i<j} V_{ij})$ is a Casimir eigenspace.

If $v = v_1 \otimes \cdots \otimes v_k \in V^{\otimes k}$ and X_s is an orthonormal basis of $\mathfrak{g}$, then

$$\begin{aligned}\mathrm{Cas}(v) &= v_1 \otimes \cdots v_{i-1} \otimes \sum_s X_s.(X_s.v_i) \otimes v_{i+1} \otimes \cdots \otimes v_k \\ &\quad + 2\sum_{i<j}\sum_s v_1 \otimes \cdots v_{i-1} \otimes X_s.v_i \otimes v_{i+1} \otimes \cdots \otimes v_{j-1} \otimes X_s.v_j \otimes v_{j+1} \otimes \cdots \otimes v_k.\end{aligned}$$

Let

(16.2.3)

$$A_{ij}(v) = \sum_s v_1 \otimes \cdots v_{i-1} \otimes X_s.v_i \otimes v_{i+1} \otimes \cdots \otimes v_{j-1} \otimes X_s.v_j \otimes v_{j+1} \otimes \cdots \otimes v_k.$$

On a module $W \subset V$,

$$\mathrm{Cas}|_W = (c_{\mu_1} + \cdots + c_{\mu_k})\,\mathrm{Id}_W + 2\sum_{i<j} A_{ij}|_W.$$

Exercises.

(1) Show that A_{ij} acts on V_{ij} as $\langle \mu_i, \mu_j \rangle\,\mathrm{Id}$.

(2) Show that thus, on $\bigcap V_{ij}$, the Casimir operator acts as the scalar

$$\sum_i c_{\mu_i} + 2\sum_{i<j} \langle \mu_i, \mu_j \rangle.$$

(3) Show that the above quantity is c_μ.

The theorem is proved. □

Remark 16.2.2.7. The above proof is Kostant's as it appeared in the unpublished work [**138**]. The proof appears in a more general setting in [**197**].

16.3. Cohomology of homogeneous vector bundles

16.3.1. Homogeneous vector bundles on G/P. Recall from §7.1 that natural desingularizations of subspace varieties are given by the total space of projective bundles over the Grassmannian. In this section I briefly review the homogeneous vector bundles on an arbitrary G/P.

Let G be a reductive algebraic (Lie) group and let G/P be a rational homogeneous variety. The homogeneous vector bundles on G/P are in one-to-one correspondence with integral P-modules. In turn, integral P-modules are in one-to-one correspondence with $\mathfrak{p}$-modules. Given a P-module W, define

$$E = G \times_P W,$$

where $(gp, w) \simeq (g, pw)$. The surjective map $E \to G/P$ has fiber isomorphic to W.

Write $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{n}$, where $\mathfrak{g}_0$ is a maximal reductive subalgebra (called a *Levi factor*) and $\mathfrak{n}$ is nilpotent. A $\mathfrak{p}$-module is completely reducible if and only if $\mathfrak{n}$ acts trivially, and it is irreducible if it is irreducible as a $\mathfrak{g}_0$-module. I focus almost exclusively on completely reducible $\mathfrak{p}$-modules as the more general case is significantly more difficult and not well understood. (See [**258**], where quiver methods are used to treat the general case.)

Example 16.3.1.1. Consider $G(r, W)$. The *tautological bundles* $\mathcal{S}$ and $\mathcal{Q}$ over $G(r, W)$ are the vector bundles whose fibers over $E \in G(r, W)$ are respectively E and W/E. (We already saw the subspace bundle in §7.1.2.) These are called the *subspace* and *quotient* bundles because $\mathcal{S}$ is a subbundle of the trivial bundle $\underline{W}$ with fiber W, and $\mathcal{Q}$ is the quotient $\underline{W}/\mathcal{S}$. Irreducible homogeneous bundles on $G(r, W)$ are all of the form $E_\lambda = S_\alpha Q^* \otimes S_\beta \mathcal{S}^*$, where $\alpha = (\alpha_1, \dots, \alpha_{\mathbf{w}-r})$ with $\alpha_1 \geq \cdots \geq \alpha_{\mathbf{w}-r} \in \mathbb{Z}$ and $\beta = (\beta_1, \dots, \beta_r)$ with $\beta_1 \geq \cdots \geq \beta_r \in \mathbb{Z}$, and S_α, S_β are the corresponding Schur functors. The $\mathfrak{g}_0$-module giving rise to E_λ is the irreducible module with highest weight $\lambda = (\beta_1 - \beta_2)\omega_1 + \cdots + (\beta_{r-1} - \beta_r)\omega_{r-1} + (\beta_r - \alpha_1)\omega_r + (\alpha_1 - \alpha_2)\omega_{r+1} + \cdots + (\alpha_{\mathbf{w}-r-1} - \alpha_{\mathbf{w}-r})\omega_{\mathbf{w}-1}$. Note that the coefficient of each ω_i except ω_r is indeed positive as required but that the coefficient of ω_r may be negative.

Example 16.3.1.2. Irreducible homogeneous bundles on $SL(W)/B = Flag_{1,2,\dots,(\mathbf{w}-1)}W$ are all of the form $F = L_1^{\otimes k_1} \otimes \cdots \otimes L_{\mathbf{w}-1}^{\otimes k_{\mathbf{w}-1}}$, where $k_j \in \mathbb{Z}$ and the fiber of L_j over a flag $(E_1, E_2, \dots, E_{\mathbf{w}-1}, W)$ is E_j/E_{j-1}.

Exercise 16.3.1.3: What is the weight of the irreducible $\mathfrak{g}_0 = \mathfrak{t}$-module giving rise to F in Example 16.3.1.2?

16.3.2. The Bott-Borel-Weil theorem. There are numerous excellent discussions of sheaf cohomology. I suggest [**146**] for the big picture and [**328**] for a concise discussion; either will suffice as a reference for this section.

Write $\mathfrak{g}_0 = \mathfrak{f}+\mathfrak{t}'$, where $\mathfrak{f} = [\mathfrak{g}_0, \mathfrak{g}_0]$ is semisimple and $\mathfrak{t}'$ acts on irreducible $\mathfrak{g}_0$-modules as a scalar times the identity.

The cohomology group $H^0(G/P, E)$ is, by definition, the (finite-dimensional vector) space of holomorphic sections of E. The other *sheaf cohomology groups* arise in a long exact sequence measuring the failure of a short exact sequence of vector bundles to give a short exact sequence of spaces of sections.

For example, $H^0(G(k, W), \mathcal{Q}) = W$, where the section at $E \in G(k, W)$ is given by $w \mapsto w \bmod E$. Similarly $H^0(G(k, W), \mathcal{S}^*) = W^*$, where the section is given by $\alpha \mapsto \alpha|_E$.

Exercise 16.3.2.1: Show that $H^0(G(k, W), \Lambda^j\mathcal{Q}) \supseteq \Lambda^j W$. (In fact, equality holds.)

Exercise 16.3.2.2: Show that $H^0(G/P, T(G/P)) \supseteq \mathfrak{g}$. (In fact, equality holds.) ⊚

An important fact is that the sheaf cohomology groups $H^k(G/P, E)$ are $\mathfrak{g}$-modules, just as in the examples above. The irreducible $\mathfrak{g}_0$-modules are indexed by lattice points in a cone $\Lambda^+_{\mathfrak{g}_0} \subseteq \Lambda_{\mathfrak{g}}$ that contains $\Lambda^+_{\mathfrak{g}}$. In the extreme case, $P = B$ is a Borel subgroup, $\Lambda^+_{\mathfrak{g}_0} = \Lambda_{\mathfrak{g}}$. We saw $\Lambda^+_{\mathfrak{g}_0}$ explicitly for $G(r, W)$ in Example 16.3.1.1.

The *Weyl group* $\mathcal{W} = \mathcal{W}_{\mathfrak{g}}$ is the finite reflection group generated by reflections along the hyperplanes in $\mathfrak{t}$ determined by the simple roots. These hyperplanes divide $\Lambda_{\mathfrak{g}}$ into chambers, called the *Weyl chambers.* The dominant weights $\Lambda^+_{\mathfrak{g}}$ form one such Weyl chamber, called the positive Weyl chamber. If $w \in \mathcal{W}$, define the *length* of w, $\ell(w)$, to be the minimal length of a word in the generators of $\mathcal{W}$ needed to express w.

Define the *affine action* of the Weyl group, for $w \in \mathcal{W}$ and $\lambda \in \Lambda_{\mathfrak{g}}$, by

$$w.\lambda = w(\lambda + \rho) - \rho.$$

Define the *Bott chambers* to be the shifts of the Weyl chambers by ρ; that is, if X is a Weyl chamber, the corresponding Bott chamber is $\{\rho + \lambda \mid \lambda \in X\}$.

Theorem 16.3.2.3 (Bott-Borel-Weil)**.** *With the above notation, let $E_\lambda \to G/P$ be an irreducible homogeneous vector bundle, where λ is a dominant weight for $\mathfrak{g}_0$. Then at most one cohomology group $H^k(G/P, E_\lambda)$ is nonzero. If λ is dominant for $\mathfrak{g}$, then $H^0(G/P, E_\lambda) = V^*_\lambda$, where V_λ is the irreducible $\mathfrak{g}$-module with highest weight λ. If λ is on a wall of the Bott chamber for*

$\mathfrak{g}$, *then* E_λ *has no cohomology, and otherwise there exists* $w \in \mathcal{W}$ *such that* $w.\lambda$ *is dominant for* $\mathfrak{g}$ *and* $H^{\ell(w)}(G/P, E_\lambda) = V^*_{w.\lambda}$.

There are three standard proofs of the Bott-Borel-Weil theorem. One is due to Demazure [**112**], and uses direct images of sheaves on G/B to reduce to the case of $G/P = \mathbb{P}^1$. Another is due to Kostant [**189**] and is via the use of Lie algebra cohomology and Dolbeault's theorem. A variant of Kostant's proof is due to Bernstein, Gelfand, and Gelfand [**25**], which also uses Lie algebra cohomology, but obtains the required results on Lie algebra cohomology via their famous BGG resolution rather than Hodge theory.

The part of the theorem due to Borel and Weil is the computation of H^0. The proofs for higher cohomology proceed by showing that H^k of the bundle in question is isomorphic to H^0 of another bundle.

What follows is an algorithm for implementing the BBW theorem, in the case $G = SL_{\mathbf{v}}$, following [**333**].

It is convenient to consider both the $GL(V)$ and $SL(V)$ structures of modules. Given a $GL(V)$-module $S_\pi V$, where $\pi = (p_1, \dots, p_{\mathbf{v}})$, the corresponding $SL(V)$-highest weight is $(p_1 - p_{\mathbf{v}})\omega_1 + \cdots + (p_{\mathbf{v}-1} - p_{\mathbf{v}})\omega_{\mathbf{v}-1}$. Note that ρ corresponds to the partition $(\mathbf{v}-1, \mathbf{v}-2, \dots, 1, 0)$, and is the first weight in the positive Bott chamber.

The weights of $\mathfrak{sl}_{\mathbf{v}}$ correspond to $\mathbf{v}$-tuples $(p_1, \dots, p_{\mathbf{v}})$ up to equivalence by shifting by the vectors $(c, \dots, c)$, and the highest weights correspond to partitions. The $\mathbf{v}$-tuple $(p_1, \dots, p_{\mathbf{v}})$ corresponds to the weight $(p_1 - p_2)\omega_1 + (p_2 - p_3)\omega_2 + \cdots + (p_{\mathbf{v}-1} - p_{\mathbf{v}})\omega_{\mathbf{v}-1}$. Note that $\rho = (\mathbf{v}, \mathbf{v}-1, \dots, 1)$. The Weyl group of $\mathfrak{sl}_{\mathbf{v}}$ is $\mathfrak{S}_{\mathbf{v}}$, and its action on the weights is to permute the elements of the $\mathbf{v}$-tuple. It is generated by the simple transpositions $\sigma_i = (i, i+1)$ associated to the simple roots $\epsilon_i - \epsilon_{i+1}$. Their affine action is as follows:

$$\sigma_i.(p_1, \dots, p_{\mathbf{v}}) = (p_1, \dots, p_{i-1}, p_{i+1} - 1, p_i + 1, p_{i+1}, \dots, p_{\mathbf{v}}).$$

To calculate cohomology for bundles on $SL(V)/P$, just apply successive reflections in simple roots that will reduce the number of adjacent terms that are increasing in $(p_1, \dots, p_{\mathbf{v}})$, which will have the effect of most efficiently moving into the positive Weyl chamber. At some point one will either have some $p_{j+1} = p_j + 1$, in which case there is no cohomology, as one has reflected onto a wall (and indeed, the corresponding reflection σ_j will leave the point stationary). Or, at some point, after say k such affine reflections, one arrives at a nonincreasing sequence representing an $\mathfrak{sl}(W)$ module M. The cohomology is then the dual module M^* in degree k and zero otherwise.

Example 16.3.2.4. Consider $\mathbb{P}^2 = \mathbb{P}W$ and $E_{d\omega_1} = \mathcal{O}(d)$. If $d \geq 0$, then $H^0(\mathbb{P}^2, \mathcal{O}(d)) = S^d W^*$. If $d = -1$, then $(p_1, p_2, p_3) = (-1, 0, 0)$ and there is no cohomology as $p_2 = p_1 + 1$. If $d = -2$, a first reflection gives $(-1, -1, 0)$ and there is no cohomology as $p_3 = p_2 + 1$. Now say $d = -k$, with $k > 2$.

A first reflection gives $(-1, 1-k, 0)$, and reflecting a second time gives $(-1, -1, 2-k) \equiv (k-3, k-3, 0) = (k-3)\omega_2$. We conclude that

$$H^i(\mathbb{P}^2, \mathcal{O}(d)) = \begin{cases} S^d W^*, & d \geq 0, \\ 0, & 0 > d \geq -2, \\ S^{-d+3} W, & d < -3. \end{cases}$$

Exercise 16.3.2.5: Use the BBW theorem to prove that more generally,

$$H^i(\mathbb{P}W, \mathcal{O}(d)) = \begin{cases} S^d W^*, & d \geq 0, \\ 0, & 0 > d \geq -\mathbf{w}+1, \\ S^{-d-\mathbf{w}} W, & d < -\mathbf{w}. \end{cases}$$

Remark 16.3.2.6. There is a pictorial recipe using marked Dynkin diagrams for determining the cohomology of irreducible homogeneous bundles applying the BBW theorem; see the wonderful book [**19**].

16.4. Equations and inheritance in a more general context

Let F, G be reductive groups with chosen simple root systems such that $\Lambda_F^+ \subset \Lambda_G^+$. Let $\lambda \in \Lambda_F^+$ and consider the corresponding F- and G-modules, which I will denote $W = W_\lambda$ and $V = V_\lambda$. There is a natural inclusion $W_\lambda \subset V_\lambda$ such that $V_\lambda = G \cdot W_\lambda$.

Let $Z_F \subset \mathbb{P}W$ be an F-variety and let $Z_G = G \cdot Z_F \subset \mathbb{P}V$ be the corresponding G-variety. If W_μ is an F-module, we say V_μ is the G-module *inherited* from W_μ.

Then $I(Z_G)$ contains the modules inherited from $I(Z_F)$. Often there is a subspace type variety such that $I(Z_G)$ is generated by the generators of $I(Z_F)$ and the generators of the ideal of the subspace variety.

Example. There are natural inclusions

$$\mathfrak{sl}_2 \subset (\mathfrak{sl}_2)^{\oplus 3} \subset \mathfrak{sp}_6 \subset \mathfrak{sl}_6 \subset \mathfrak{spin}_{12} \subset \mathfrak{e}_7,$$

giving rise to an inclusion of varieties

$$v_3(\mathbb{P}^1) \subset Seg(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1) \subset G_{Lag}(3,6) \subset G(3,6) \subset E_7/P_7.$$

If we set $a = -2/3, 0, 1, 2, 4, 8$ for the respective cases, then these varieties have dimension $3a+3$. If we call the ambient space V, then there are four G-orbits in $\mathbb{P}V$, corresponding to $G/P \subset \sigma_+(G/P) \subset \tau(G/P) \subset \mathbb{P}V$. Here $\sigma_+(G/P)$ denotes the points on a positive-dimensional family of secant lines. (This orbit does not occur in the case of $v_3(\mathbb{P}^1)$.) The other orbits have dimensions $5a+3$, $6a+6$, and $6a+7$. The equations may be understood uniformly via generalized inheritance. In particular, $\tau(G/P)$ is a hypersurface of degree 4 whose equation is a generalization of the classical discriminant, and $\sigma_+(G/P)$ is cut out in degree 3 by the derivatives of the quartic. See [**202**] for details and more examples.

Warning. Z_F may have different geometric interpretations, and Z_G might inherit only one of them. For example, if $W = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$, the invariant hypersurface in $\mathbb{P}W$ may be thought of as the dual of the Segre variety in the dual space, or the tangential variety of the Segre variety in the same space. For $V = A \otimes B \otimes C$ in general, only the tangential variety inheritance holds, and the inherited tangential variety will no longer be a hypersurface.

Chapter 17

Weyman's method

In our study of varieties in spaces of tensors we have tried to find equations for, and understand geometric properties of, G-varieties such as secant varieties of Segre and Veronese varieties, subspace varieties, etc. These varieties are often swept out by linear spaces where the sweeping is done by the action of G. In this chapter I explain the basics of a set of techniques, initiated by G. Kempf and vastly expanded, developed and utilized by J. Weyman, of studying G-varieties via a desingularization such that the desingularizing variety is the total space of a vector bundle over a homogeneous variety G/P. To fix ideas, recall the subspace variety

$$Sub_k(S^dW) = \{p \in S^dW \mid p \in S^dW' \text{ for some } W' \subset W,\ \dim W' = k\}.$$

It is singular at $p \in S^dW''$ with $\dim W'' < k$, so our desingularization must fiber over such points. Consider:

$$(17.0.1)\quad \{(p, W') \mid W' \subset W,\ \dim W' = k,\ p \in S^dW'\} \subset S^dW \times G(k, W).$$

This space is the total space of a homogeneous vector bundle over $G(k, W)$. Recall the subspace bundle $\mathcal{S} \to G(k, W)$ from §7.1.2 or Example 16.3.1.1. Then (17.0.1) is just the total space of $S^d\mathcal{S}$. The associated projective bundle desingularizes $Sub_k(S^dW)$. The idea of Weyman's method is to take the (sometimes) readily available information about the total space of a desingularizing vector bundle and to transport it to the original variety of interest. Another example is if $X \subset \mathbb{P}V$ is a smooth variety with strongly nondegenerate tangential variety (i.e., a general point on $\tau(X)$ is on a unique tangent plane), $\tau(X)$ is desingularized by the projectivization of the affine tangent bundle, $\mathbb{P}(\hat{T}X)$, where $\hat{T}X$ is a subbundle of the trivial bundle over $\mathbb{P}V$ with fiber V.

Regarding defining equations of G-varieties, a set of generators of the ideal of a variety $Z \subset \mathbb{P}V$ in degree d is given by a basis of a complement to the image of the symmetrization map

$$\pi_S : V^* \otimes I_{d-1}(Z) \to I_d(Z). \tag{17.0.2}$$

This may be extended to a *Koszul sequence*, which is discussed at the level of linear algebra in §17.2 and for vector bundles in §17.3.1. Weyman's "basic theorem" that expresses modules of generators of ideals in terms of cohomology groups of homogeneous vector bundles, is presented in §17.3.2. Then, in §17.3.3, the basic theorem is shown to apply in many cases of interest. The chapter concludes in §17.4 with a few examples. It begins, in §17.1, with basic definitions. The goal of this chapter is to outline Weyman's method, and the reader is directed to [**333**] for details.

17.1. Ideals and coordinate rings of projective varieties

For $Z \subset \mathbb{P}V$, its homogeneous coordinate ring equals the coordinate ring of $\hat{Z} \subset V$, $\mathbb{C}[Z] = \mathbb{C}[\hat{Z}] = S^\bullet V^*/I(Z)$. While the coordinate ring of an arbitrary affine variety is not graded, for cones over projective varieties one inherits a grading because a polynomial vanishes on a cone if and only if all its homogeneous components do. For a variety that is not embedded as a subvariety of projective space such as a desingularization $\tilde{Z}$ of a projective variety, one can only discuss functions locally and work with the *structure sheaf* $\mathcal{O}_{\tilde{Z}}$ (see, e.g., [**161**, Ch. II]). There is an identification $\mathcal{O}_{\hat{Z}}(\hat{Z}) = \mathbb{C}[\hat{Z}]$, where the left hand side is the ring associated to the sheaf $\mathcal{O}_{\hat{Z}}$ corresponding to the open subset $\hat{Z}$.

If an affine variety Z is *normal*, i.e., $\mathbb{C}[Z]$ is integrally closed (see, e.g., [**289**, §II.5]), then $\mathcal{O}_Z = q_*(\mathcal{O}_{\tilde{Z}})$, where $q : \tilde{Z} \to Z$ is a desingularization of Z.

Now assume that $Z \subset \mathbb{P}V$ is a singular projective variety with desingularization $\tilde{Z}$ that is the total space of a projective bundle $\mathbb{P}E$, and consider the maps

$$I_d(Z) \to S^d V^*, \tag{17.1.1}$$

$$S^d V^* \to H^0(X, S^d E^*), \tag{17.1.2}$$

where the first inclusion holds tautologically and the second map is defined because of the surjection $\underline{S^d V^*} \to S^d E^*$ and $H^0(X, \underline{S^d V^*}) = S^d V^*$. Here and in what follows, I use the convention that $\underline{W}$ denotes the trivial vector bundle with fiber W.

A projective variety Z is *projectively normal* if (i) Z is normal, i.e., its local rings are integrally closed, and (ii) the maps $S^d V^* \to H^0(Z, \mathcal{O}_Z(d))$ are surjective for all $d > 0$ (see, e.g., [**289**, II.5, Ex. 14(a)]). (Note that projective normality is an extrinsic property, while normality is an intrinsic property.)

Thus if Z is projectively normal, the map (17.1.2) is surjective with kernel the image of (17.1.1), and I claim that we obtain an exact sequence

$$0 \to I_d(Z) \to S^dV^* \to H^0(X, S^dE^*) \to 0. \tag{17.1.3}$$

To see this, let $\mathcal{E}$ denote the total space of E. Take an open subset $U \subset X$ over which E is trivial. If $r = \operatorname{rank} E$, then $E|_U \simeq U \times \mathbb{C}^r$, so

$$\mathcal{O}_{\mathcal{E}}(U) \simeq \mathcal{O}_U(U) \otimes S^\bullet \mathbb{C}^r = H^0(U, S^\bullet E^*).$$

In particular $\mathcal{O}_U(U) \otimes S^d(\mathbb{C}^r)^* = H^0(U, S^dE^*)$. Thus there is an injective map $H^0(X, S^dE^*) \to \mathcal{O}_{\mathcal{E}}(\mathcal{E})$, by restricting $f \in H^0(X, S^dE^*)$ to open subsets and using the above identification. On the other hand, the map is also surjective because any collection of local sections that patch together gives rise to a global section.

Now consider the push-forward of $\mathcal{O}_{\mathcal{E}}(\mathcal{E})$ to $\hat{Z}$, which is $\mathcal{O}_{\hat{Z}}(\hat{Z})$ by our normality assumption. But $\mathcal{O}_{\hat{Z}}(\hat{Z}) = \mathbb{C}[\hat{Z}] = S^\bullet V^*/I(Z)$.

In §17.3.2, I combine (17.0.2) and (17.1.3) to relate generators of the ideal of Z to cohomology groups of vector bundles related to E, to obtain Weyman's "basic theorem".

17.2. Koszul sequences

17.2.1. Koszul sequences of vector spaces. Let V^* be a complex vector space which we consider as a complex manifold and let $\Omega_{poly}(V^*)$ denote the space of polynomial differential forms on V^*. It is bigraded, $\Omega^{k,d}_{poly}(V^*) = S^dV \otimes \Lambda^k V$. (Here I use that the tangent space to a vector space at a point may be identified with the vector space itself.) The exterior derivative is a map $d : \Omega^{k,d}_{poly}(V^*) \to \Omega^{k+1,d-1}_{poly}(V^*)$. For example, if $P \in S^dV$, then $dP \in S^{d-1}V \otimes V$. The sacred equation $d^2 = 0$ gives rise to the exact sequence of $GL(V)$-modules (see Exercise 2.8.1(8)):

$$0 \to S^dV \to S^{d-1}V \otimes V \to S^{d-2}V \otimes \Lambda^2 V \to \cdots \to V \otimes \Lambda^{d-1}V \to \Lambda^d V \to 0. \tag{17.2.1}$$

In terms of modules, for the last map note that $V \otimes \Lambda^{d-1}V = S_{2,1^{d-1}}V \oplus \Lambda^d V$, the first factor is in the kernel of the last arrow, and the map is the identity on the second factor. A direct verification or Schur's lemma guarantees exactness. Note that the $GL(V)$-modules $S_\pi V$ that appear do so with specific realizations as submodules of $V^{\otimes}$.

Now consider the dual sequence:

$$0 \to \Lambda^d V^* \to \Lambda^{d-1}V^* \otimes V^* \to \cdots \to S^{d-1}V^* \otimes V^* \to S^dV^* \to 0. \tag{17.2.2}$$

Let $K_d \subset S^{d-1}V^* \otimes V^*$ denote the copy of $S_{d-1,1}V^*$. Explicitly,

$$K_d = \left\{ \sum \ell_i \otimes p_i \mid \ell_i \in V^*,\ p_i \in S^{d-1}V^*,\ \sum \ell_i p_i = 0 \right\}.$$

K_d is the *space of linear syzygies* among elements of $S^{d-1}V^*$.

Add up the sequences (17.2.2) for all d to get a sequence

$$(17.2.3) \qquad \begin{aligned} 0 \to \Lambda^{\mathbf{v}} V^* \otimes S^{\bullet} V^* \to \Lambda^{\mathbf{v}-1} V^* \otimes S^{\bullet} V^* \to \cdots \\ \to V^* \otimes S^{\bullet} V^* \to S^{\bullet} V^* \to 0, \end{aligned}$$

which is the prototype of a *Koszul sequence.* Explicitly, the maps are given by

$$v_1 \wedge \cdots \wedge v_k \otimes p \mapsto \sum_{i=1}^{k} (-1)^{i+1} v_1 \wedge \cdots \wedge v_{i-1} \wedge v_{i+1} \wedge \cdots \wedge v_k \otimes v_i p.$$

All terms in (17.2.3) are graded modules for the algebra $S^{\bullet}V^*$, and the maps are graded $S^{\bullet}V^*$-module maps. Since commutative algebraists like degree zero maps, they shift (i.e., they relabel) the grading of each space except for the last to have all maps of degree zero.

17.2.2. Minimal free resolutions of varieties. For the purposes of many topics in this book we will be content to have generators of the ideal of a variety X, but for a small amount of additional effort, Weyman's method can often give the entire minimal free resolution, so I discuss such resolutions briefly here.

Let $I \subset S^{\bullet}V^*$ be an ideal; a *free resolution* is obtained by adding maps on the left to

$$S^{\bullet}V^* \to S^{\bullet}V^*/I \to 0$$

to extend it to an exact (except at the last step) sequence of free $S^{\bullet}V^*$-modules. The resolution is *minimal* at each step if a basis of the j-th space can be mapped onto a minimal set of generators of the cokernel of the map to it.

If $X \subset \mathbb{P}V$ is a complete intersection, then the following Koszul sequence will be a minimal free resolution. Let $p_i \in S^{d_i}V^*$ be a set of generators of $I(X)$, which say are $\mathbf{a}$ in number, and let m_i be a basis of $M = \mathbb{C}(-d_1) \oplus \mathbb{C}(-d_2) \oplus \cdots \oplus \mathbb{C}(-d_n)$, where m_i has "degree" $-d_i$. Define $M \otimes S^{\bullet}V^* \to S^{\bullet}V^*$ by $m_i \otimes p \mapsto p_i p$. The kernel is generated by $m_i \otimes p_j - m_j \otimes p_i$, and continuing, we obtain a minimal free resolution of the $S^{\bullet}V^*$-module $\mathbb{C}[X] = S^{\bullet}V^*/I(X)$ by

$$\begin{aligned} 0 \to \Lambda^{\mathbf{a}} M \otimes S^{\bullet}V^* \to \cdots \to \Lambda^2 M \otimes S^{\bullet}V^* \\ \to M \otimes S^{\bullet}V^* \to S^{\bullet}V^* \to S^{\bullet}V^*/I(X) \to 0, \end{aligned}$$

where

$$m_{j_1} \wedge \cdots \wedge m_{j_k} \otimes p \mapsto \sum_{i=1}^{k} (-1)^{i+1} m_{j_1} \wedge \cdots \wedge m_{j_{i-1}} \wedge m_{j_{i+1}} \wedge \cdots \wedge m_{j_k} \otimes p_{j_i} p.$$

Many G-varieties are far from being complete intersections and there are nontrivial syzygies.

Exercise 17.2.2.1: Let $X = Seg(\mathbb{P}^1 \times \mathbb{P}^2) = Seg(\mathbb{P}A^* \times \mathbb{P}B^*) \subset \mathbb{P}(A^* \otimes B^*)$. Find a minimal free resolution of X.

If X is a G-variety, instead of taking a basis of generators, we can and will label M by a collection of modules.

Exercise 17.2.2.2: Let $X = \mathbb{P}E \subset \mathbb{P}V$ be a linear subspace. Write down the minimal free resolution of X invariantly (i.e., without choosing a basis of the generators of the ideal of X).

Exercise 17.2.2.3: Express Exercise 17.2.2.1 in terms of $GL(A) \times GL(B)$-modules.

Aside 17.2.2.4. When X is homogeneous, there are techniques available for writing down the minimal free resolution. For example, when X is subcominuscule, the minimal free resolution can be deduced pictorially from the restricted Hasse diagram; see [**121**]. However, the minimal free resolution is not even known for general triple Segre products or general Veronese varieties.

17.2.3. Koszul for linear subspaces. Now let $F \subset V$ be a linear subspace. Consider the restriction map $V^* \to F^*$ given by $\alpha \mapsto \alpha|_F$. For each j consider the complex

$$\begin{aligned} 0 \to &\Lambda^j (V/F)^* \to \Lambda^j V^* \to \Lambda^{j-1} V^* \otimes F^* \to \\ &\cdots \to \Lambda^2 V^* \otimes S^{j-2} F^* \to V^* \otimes S^{j-1} F^* \to S^j F^* \to 0. \end{aligned} \tag{17.2.4}$$

Here, the map $\Lambda^{j-s} V^* \otimes S^s F^* \to \Lambda^{j-s-1} V^* \otimes S^{s+1} F^*$ is given by the composition

$$\begin{aligned} \Lambda^{j-s} V^* \otimes S^s F^* &\to \Lambda^{j-s-1} V^* \otimes V^* \otimes S^s F^* \\ &\to \Lambda^{j-s-1} V^* \otimes F^* \otimes S^s F^* \to \Lambda^{j-s-1} V^* \otimes S^{s+1} F^*. \end{aligned}$$

Exercise 17.2.3.1: Show that the complex (17.2.4) is exact. ⊚

In fact, (17.2.4) is an exact sequence of P-modules, where $P \subset GL(V)$ is the subgroup preserving $F \subset V$. For example, $V^* \otimes S^{j-1} F^*$ contains $F^* \otimes S^{j-1} F^*$, which contains $S^j F^*$, the rest of the space maps to zero, and so on.

For what comes next, label the maps ϕ_j and split (17.2.4) into a collection of short exact sequences

(17.2.5)

$$0 \to \Lambda^j(V/F)^* \to \Lambda^j V^* \to \Lambda^j V^*/\ker(\phi_1) \to 0,$$
$$0 \to \Lambda^j V^*/\ker(\phi_1) \to \Lambda^{j-1}V^*\otimes F^* \to \Lambda^{j-2}V^*\otimes S^2F^*/\ker(\phi_2) \to 0,$$
$$0 \to \Lambda^{j-2}V^*\otimes S^2F^*/\ker(\phi_2) \to \Lambda^{j-2}V^*\otimes S^2F^*$$
$$\to \Lambda^{j-3}V^*\otimes S^3F^*/\ker(\phi_3) \to 0,$$
$$\vdots$$

(17.2.6)
$$0 \to \Lambda^2V^*\otimes S^{j-2}F^*/\ker(\phi_{j-2}) \to V^*\otimes S^{j-1}F^* \to S^jF^* \to 0.$$

17.3. The Kempf-Weyman method

17.3.1. Desingularizations via vector bundles. Now we go to the setting of the Kempf-Weyman method. Let $Z \subset \mathbb{P}V$ be a singular variety (in all our examples V will be a G-module and Z a G-variety), let X be a smooth projective variety (in all our applications we will take $X = G/P$), and let $E \to X$ be a vector bundle that is a subbundle of the trivial bundle with fiber V, which we denote $\underline{V}$ (in all our applications E will be homogeneous), such that $\mathbb{P}E \to Z$ is a desingularization.

The ideal of Z, as well as other spaces appearing in the minimal free resolution, will be obtained as vector spaces (in our case, as G-modules) that arise as the cohomology of the sheaf of sections of auxiliary vector bundles constructed from E.

Consider the exact sequence of vector bundles defined by (17.2.4), namely
(17.3.1)
$$0\to\Lambda^j(\underline{V}/E)^* \to \Lambda^j\underline{V}^* \to \Lambda^{j-1}\underline{V}^*\otimes E^* \to \cdots \to \underline{V}^*\otimes S^{j-1}E^* \to S^jE^* \to 0.$$

Label the maps $\phi_s : \Lambda^{j-s+1}\underline{V^*}\otimes S^{s-1}E^* \to \Lambda^{j-s}\underline{V^*}\otimes S^sE^*$. Each of the short exact sequences of vector bundles from (17.2.5) gives long exact sequences in sheaf cohomology. In what follows, write $H^k(U) := H^k(X,U)$. The first is

(17.3.2)
$$\cdots \to H^p(\Lambda^j\underline{V}^*) \to H^p(\Lambda^j\underline{V}^*/\ker(\phi_1))$$
$$\to H^{p+1}(\Lambda^j(\underline{V}^*/E)) \to H^{p+1}(\Lambda^j\underline{V}^*) \to \cdots,$$

and the others except the last are

$$(17.3.3)\qquad \begin{aligned} \cdots &\to H^p(\Lambda^{j-s}\underline{V^*}\otimes S^sE^*/\ker(\phi_s)) \to H^p(\Lambda^{j-s}\underline{V}^*\otimes S^sE^*) \\ &\to H^p(\Lambda^{j-(s+1)}\underline{V}^*\otimes S^{s+1}E^*/\ker(\phi_{s+1})) \\ &\to H^{p+1}(\Lambda^{j-s}\underline{V}^*\otimes S^sE^*/\ker(\phi_s)) \to \cdots . \end{aligned}$$

Since $\underline{V}^*$ is trivial, (17.3.2) gives $H^p(\Lambda^j\underline{V}^*/\ker(\phi_1)) \simeq H^{p+1}(\Lambda^j(\underline{V}/E))^*$ for $p > 0$ and (17.3.3) may be rewritten as

$$\begin{aligned} \cdots &\to \Lambda^{j-s}V^*\otimes H^p(S^sE^*/\ker(\phi_s)) \to \Lambda^{j-s}V^*\otimes H^p(S^sE^*) \\ &\to \Lambda^{j-s}V^*\otimes H^p(S^{s+1}E^*/\ker(\phi_{s+1})) \\ &\to \Lambda^{j-s}V^*\otimes H^{p+1}(S^sE^*/\ker(\phi_s)) \to \cdots . \end{aligned}$$

In the example of subspace varieties, $S^\bullet E^*$ was acyclic, i.e., $H^p(S^dE^*) = 0$ for all $p > 0$, $d \geq 0$. To apply Weyman's method, this will need to be the case (since E will always be a subbundle of a trivial bundle, this often holds—see §17.3.3 below). If $S^\bullet E^*$ is acyclic, we obtain the chain of equalities

$$(17.3.4)\qquad H^p(\Lambda^j(\underline{V}/E)^*) = H^{p-1}(\Lambda^{j-1}\underline{V}^*/\ker(\phi_1)),$$

$$(17.3.5)\qquad H^{p-1}(\Lambda^{j-1}\underline{V}^*/\ker(\phi_1)) = \Lambda^{j-1}V\otimes H^{p-2}(E^*/\ker(\phi_2)),$$

$$(17.3.6)\qquad \Lambda^{j-1}V^*\otimes H^{p-2}(E^*/\ker(\phi_2)) = \Lambda^{j-2}V^*\otimes H^{p-3}(S^2E^*/\ker(\phi_3)),$$

$$(17.3.7)\qquad \Lambda^{j-2}V^*\otimes H^{p-3}(S^2E^*/\ker(\phi_3)) = \Lambda^{j-3}V^*\otimes H^{p-4}(S^3E^*/\ker(\phi_4)),$$

$$\vdots$$

(17.3.8)

$$\Lambda^{j-(p-1)}V^*\otimes H^1(S^{p-2}E^*/\ker(\phi_{p-1})) = \Lambda^{j-p}V^*\otimes H^0(S^{p-1}E^*/\ker(\phi_p)).$$

Note that if we take $j = p + 1$, there is no kernel to quotient out by in the last term.

Finally (17.2.6) and (17.3.8) give the complex:

$$\cdots \to \Lambda^2V^*\otimes H^0(S^{d-2}E^*) \to V^*\otimes H^0(S^{d-1}E^*) \to H^0(S^dE^*) \to 0.$$

In summary:

Proposition 17.3.1.1. *Let $E \to X$ be a vector bundle that is a subbundle of a trivial bundle $\underline{V}$ with fiber V. If the sheaf cohomology groups $H^p(X, S^dE^*)$ are zero for all d and all $p > 0$, and the linear maps*

$$V^*\otimes H^0(X, S^{d-2}E^*) \to H^0(X, S^{d-1}E^*)$$

are all surjective for all $d > 1$, then $H^{d-1}(X, \Lambda^d(\underline{V}/E)^*)$ is isomorphic to the homology at the $V^*\otimes H^0(X, S^{d-1}E^*)$ term of the complex

(17.3.9)

$$\cdots \to \Lambda^2V^*\otimes H^0(X, S^{d-2}E^*) \to V^*\otimes H^0(X, S^{d-1}E^*) \to H^0(X, S^dE^*) \to 0.$$

While the homology of middle term of (17.3.9) may be computable, *a priori* it involves an infinite number of calculations, whereas there are only a finite number of groups $H^{d-1}(X, \Lambda^d(\underline{V}/E)^*)$.

In the next section I relate the generators in degree d of the ideal of a variety Z admitting a Kempf-Weyman desingularization to the sheaf cohomology group $H^{d-1}(X, \Lambda^d(\underline{V}/E)^*)$.

17.3.2. Weyman's "basic theorem". To convert $I_d(Z)/\pi_S(V^* \otimes I_{d-1}(Z))$ to something we can compute, consider the diagram

$$\begin{tikzcd}
\Lambda^d V^* \arrow[r] & \Lambda^{d-1}V^* \otimes H^0(S^1E^*) \arrow[r] & \cdots \arrow[r] & V^* \otimes H^0(S^{d-1}E^*) \arrow[r] & H^0(S^dE^*) \arrow[r] & 0 \\
\Lambda^d V^* \arrow[r] \arrow[u] & \Lambda^{d-1}V^* \otimes V^* \arrow[r] \arrow[u] & \cdots \arrow[r] & V^* \otimes S^{d-1}V^* \arrow[r] \arrow[u] & S^dV^* \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] \arrow[u] & \Lambda^{d-1}V^* \otimes I_1(Z) \arrow[r] \arrow[u] & \cdots \arrow[r] & V^* \otimes I_{d-1}(Z) \arrow[r] \arrow[u] & I_d(Z) \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

The top row was discussed above, the middle row is the Koszul sequence, and the bottom row is the restriction of the Koszul sequence to $I(Z) \subset S^\bullet V^*$, with the last map π_S. The vertical arrows are (17.1.3) tensored with $\Lambda^{d-k}V^*$. The space of generators of the ideal of Z in degree d corresponds to the cokernel of the lower right arrow. Now apply the Snake Lemma (e.g. [**217**, §III.9]) to see it is the homology of the d-th entry in the top sequence, which by Proposition 17.3.1.1 above is $H^{d-1}(\Lambda^d(\underline{V}/E)^*)$. That is:

Under our working hypotheses, the span of generators of the ideal of Z in degree d may be identified with the sheaf cohomology group $H^{d-1}(\Lambda^d(\underline{V}/E)^*)$.

Since we had to use the Snake Lemma and take quotients, we have no canonical way of identifying $H^{d-1}(\Lambda^d(\underline{V}/E)^*)$ with the space of generators in degree d, but in the equivariant setup, at least they are identified as modules.

I summarize the above discussion and include for reference additional information about singularities that is beyond the scope of this discussion:

Theorem 17.3.2.1 ([**333**, Chapter 5]). *Let $Z \subset \mathbb{P}V$ be a variety and suppose there is a smooth projective variety X and a vector bundle $q : E \to X$ that is a subbundle of a trivial bundle $\underline{V} \to X$ with $\underline{V}_z \simeq V$ for $z \in X$ such that the image of the map $\mathbb{P}E \to \mathbb{P}V$ is Z and $\mathbb{P}E \to Z$ is a desingularization of Z. Write $\xi = (\underline{V}/E)^*$.*

If the sheaf cohomology groups $H^i(X, S^dE^*)$ are all zero for $i > 0$ and $d \geq 0$ and if the linear maps $H^0(X, S^dE^*) \otimes V^* \to H^0(X, S^{d+1}E^*)$ are surjective for all $d \geq 0$, then

(1) *$\hat{Z}$ is normal, with rational singularities.*

(2) *The coordinate ring* $\mathbb{C}[\hat{Z}]$ *satisfies* $\mathbb{C}[\hat{Z}]_d \simeq H^0(X, S^d E^*)$.

(3) *The vector space of minimal generators of the ideal of* Z *in degree* d *is isomorphic to* $H^{d-1}(X, \Lambda^d \xi)$ *which is also the homology at the middle term in the complex*

$$\begin{aligned} \cdots \to \Lambda^2 V \otimes H^0(X, S^{d-2}E^*) &\longrightarrow V \otimes H^0(X, S^{d-1}E^*) \\ &\longrightarrow H^0(X, S^d E^*) \to 0. \end{aligned} \tag{17.3.10}$$

(4) *More generally,* $\bigoplus_j H^j(X, \Lambda^{i+j}\xi) \otimes S^{\bullet}V^*$, *with degree of the* j*-th summand shifted by* $-(i+j)$, *is isomorphic to the* i*-th term in the minimal free resolution of* Z.

(5) *If moreover* Z *is a* G*-variety and the desingularization is* G*-equivariant, then the identifications above are as* G*-modules.*

Using these methods, the generators of the ideals of, e.g., $\sigma_r(Seg(\mathbb{P}^1 \times \mathbb{P}B \times \mathbb{P}C))$ and $\sigma_3(Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C))$ have been determined; see [**210**]. The method also gives information about the singularities (e.g., arithmetical Cohen-Macaulay-ness).

17.3.3. The basic theorem applies in many cases of interest. Say a desingularization as above is by a homogeneous bundle E.

Proposition 17.3.3.1 ([**211**]). *If* E *is induced from an irreducible* P*-module, then the sheaf cohomology groups* $H^i(G/P, S^d E^*)$ *are all zero for* $i > 0$ *and the linear maps* $H^0(G/P, S^d E^*) \otimes V^* \to H^0(G/P, S^{d+1}E^*)$ *are surjective for all* $d \geq 0$. *In particular, all the conclusions of Theorem* 17.3.2.1 *apply.*

Proof. Recall from Chapter 16 that an irreducible homogeneous bundle can have nonzero cohomology in at most one degree, but a quotient bundle of a trivial bundle has nonzero sections, thus $H^0(G/P, E^*)$ is a nonzero irreducible module and all other $H^j(G/P, E^*)$ are zero. Let $\mathfrak{f} \subset \mathfrak{p} \subset \mathfrak{g}$ be a semisimple Levi factor, so the weight lattice of $\mathfrak{f}$ is a sublattice of the weight lattice of $\mathfrak{g}$, let $\mathfrak{t}^c$ denote the complement of $\mathfrak{t}_{\mathfrak{f}}$ (the torus of $\mathfrak{f}$) in $\mathfrak{t}_{\mathfrak{g}}$, and let $\mathfrak{g}_0 = \mathfrak{f} + \mathfrak{t}^c$ denote the Levi factor of $\mathfrak{p}$. The bundle E^* is induced from an irreducible $\mathfrak{g}_0$-module U, which is a weight space for $\mathfrak{t}^c$ having nonnegative weight, say $(w_1, \ldots, w_p)$. The bundle $(E^*)^{\otimes d}$ corresponds to a module which is $U^{\otimes d}$ as an $\mathfrak{f}$-module and is a weight space for $\mathfrak{t}^c$ with weight $(dw_1, \ldots, dw_p)$. All the irreducible factors will have highest weights that are dominant for $\mathfrak{f}$ by definition, and therefore dominant for $\mathfrak{g}$. In summary, $S^d E^*$ is completely reducible and each component of $S^d E^*$ has sections and thus is acyclic.

To prove the second assertion, consider the maps

$$V^* \otimes H^0(G/P, S^{r-1}E^*) \to H^0(G/P, S^r E^*).$$

Note that $H^0(G/P, S^jE^*) \subset S^jV^*$. The proof of Proposition 17.3.3.1 will be completed by Lemma 17.3.3.2 below applied to U and each irreducible component of $H^0(G/P, S^rE^*)$. $\square$

With notation as in the proof above and Chapter 16, let $M_{\mathfrak{g}_0}^{\mathfrak{g}}$ denote the subcategory of the category of $\mathfrak{g}_0$-modules generated under direct sum by the irreducible $\mathfrak{g}_0$-modules with highest weight in $\Lambda_{\mathfrak{g}}^+ \subset \Lambda_{\mathfrak{g}_0}^+$ and note that it is closed under tensor product. Let $M_{\mathfrak{g}}$ denote the category of $\mathfrak{g}$-modules. Define an additive functor $\mathcal{F} : M_{\mathfrak{g}_0}^{\mathfrak{g}} \to M_{\mathfrak{g}}$ which takes an irreducible $\mathfrak{g}_0$-module with highest weight λ to the corresponding irreducible $\mathfrak{g}$-module with highest weight λ. Note that if $E^* = G \times_P U$, then $H^0(G/P, S^rE^*) = \mathcal{F}(S^rU)$.

Lemma 17.3.3.2 ([**211**]). *Let $\mathfrak{g}_0 \subset \mathfrak{g}$ and $\mathcal{F}$ be as above. Let U, W be irreducible $\mathfrak{g}_0$-modules. Then*

$$\mathcal{F}(U \otimes W) \subseteq \mathcal{F}(U) \otimes \mathcal{F}(W).$$

Proof. Let $N \subset P$ denote the unipotent radical of P. Any G_0-module W may be considered as a P-module where N acts trivially. Writing $V = \mathcal{F}(W)$ means that V is the G-module $H^0(G/P, G \times_P W^*)^*$ and W is the set of N-invariants of V. The N-invariants of $\mathcal{F}(U)\otimes\mathcal{F}(W)$ contain $U \otimes W$. $\square$

17.4. Subspace varieties

17.4.1. Cases of tensors and symmetric tensors. Let $A_1, \dots, A_n$ be vector spaces of dimensions $\mathbf{a}_1, \dots, \mathbf{a}_n$ and recall from §7.1 the subspace varieties

$$\begin{aligned} &Sub_{\mathbf{b}_1,\dots,\mathbf{b}_n} \\ &\quad := \mathbb{P}\{T \in A_1^* \otimes \cdots \otimes A_n^* \mid \exists A_j' \subseteq A_j^*,\ \dim A_j' = \mathbf{b}_j,\ T \in A_1' \otimes \cdots \otimes A_n'\}. \end{aligned}$$

The basic Theorem 17.3.2.1 can be used to recover the generators of the ideal as modules; it also shows:

Theorem 17.4.1.1. *The subspace varieties $Sub_{\mathbf{b}_1,\dots,\mathbf{b}_n}$ are normal, with rational singularities.*

Proof. Consider the product of Grassmannians

$$B = G(\mathbf{b}_1, A_1^*) \times \cdots \times G(\mathbf{b}_n, A_n^*)$$

and the bundle

$$p : \mathcal{S}_1 \otimes \cdots \otimes \mathcal{S}_n \to B, \tag{17.4.1}$$

where $\mathcal{S}_j$ is the tautological rank $\mathbf{b}_j$ subspace bundle over $G(\mathbf{b}_j, A_j^*)$. Assume that $\mathbf{b}_1 \leq \mathbf{b}_2 \leq \cdots \leq \mathbf{b}_n$. Then the total space $\tilde{Z}$ of $\mathcal{S}_1 \otimes \cdots \otimes \mathcal{S}_n$ maps to

$A_1^*\otimes\cdots\otimes A_n^*$. We let $q:\tilde{Z}\to A_1^*\otimes\cdots\otimes A_n^*$ denote this map, which gives a desingularization of $Sub_{\mathbf{b}_1,\dots,\mathbf{b}_n}$.

Since $E^*=(\mathcal{S}_1\otimes\cdots\otimes\mathcal{S}_n)^*$ is irreducible, we conclude by Proposition 17.3.3.1. □

Now consider $Sub_{\mathbf{a}}(S^dW^*)$ and its desingularization by $S^d\mathcal{S}\to G(\mathbf{a},W^*)$ from §7.1.3.

Using Theorem 17.3.2.1 one may recover the generators of the ideal $I(Sub_{\mathbf{a}}(S^dW))$ and prove:

Proposition 17.4.1.2 ([**333**, §7.2], [**267**])**.**

(1) *The ideal $I(Sub_{\mathbf{a}}(S^dW))$ is the span of all submodules $S_\pi W^*$ in $Sym(S^dW^*)$ for which $\ell(\pi)>\mathbf{a}$.*

(2) *$I(Sub_{\mathbf{a}}(S^dW))$ is generated by $\Lambda^{\mathbf{a}+1}W^*\otimes\Lambda^{\mathbf{a}+1}S^{d-1}W^*$, which may be considered as the $(\mathbf{a}+1)\times(\mathbf{a}+1)$ minors of $\phi_{1,d-1}$.*

(3) *The subvariety $Sub_{\mathbf{a}}(S^dW)$ is normal, Cohen-Macaulay, and it has rational singularities.*

17.4.2. The case of $Sub_r(\Lambda^kV)$. The ideal $I(Sub_r(\Lambda^kV))$ consists of all modules $S_{\overline{\pi}}W$ appearing in $Sym(\Lambda^kV^*)$ with $\ell(\pi)>r$. However, finding the generators of the ideal is not so easy and it is not known in general.

Acknowledgment. I thank A. Boralevi for providing this subsection.

The variety $Sub_r(\Lambda^kV)$ is singular at all points $P\in\Lambda^kV'$, with $V'\subset V$ of dimension strictly smaller than r. A desingularization of $Sub_r(\Lambda^kV)$ is given by the incidence variety

$$\hat{Z}=\{(P,W)\mid P\in W\}\subseteq V\times G(r,V).$$

Recall the tautological sequence

$$0\to\mathcal{S}\to\underline{V}\to\mathcal{Q}\to 0,$$

with $\mathcal{S}$ and $\mathcal{Q}$ the subspace and quotient bundles respectively, of rank r and $\mathbf{v}-r$. Taking exterior powers gives

$$0\to\Lambda^k\mathcal{S}\to\underline{\Lambda^kV}\to\underline{\Lambda^kV}/\Lambda^k\mathcal{S}\to 0.$$

Note that $\hat{Z}$ is the total space of the bundle $\Lambda^k\mathcal{S}$, and that $\Lambda^k\mathcal{S}$ is a subbundle of the trivial bundle $\underline{\Lambda^kV}$ over the Grassmannian $G(r,V)$. Set $\xi=(\underline{\Lambda^kV}/\Lambda^k\mathcal{S})^*$.

By Theorem 17.3.2.1, the minimal generators of the ideal in degree d of the variety $Sub_r(\Lambda^kV)$ are given by

$$H^{d-1}(G(r,V),\Lambda^d\xi).$$

There are several difficulties here. First, in general ξ is far from being irreducible. In fact

$$\operatorname{gr}(\xi) = \Lambda^k \mathcal{Q}^* \oplus (\mathcal{S}^* \otimes \Lambda^{k-1}\mathcal{Q}^*) \oplus \cdots \oplus (\Lambda^{k-1}\mathcal{S}^* \otimes \mathcal{Q}^*).$$

So even after computing the relevant cohomology of the graded bundle $\operatorname{gr}(\xi)$ one must use either spectral sequences or the quiver techniques of [**258**], which are well beyond the scope of this book.

Something can be said for $k = 2$ and $k = 3$. In the case $k = 2$ one gets the skew-symmetric determinantal varieties, defined by the ideal of $(r+2)$-Pfaffians of a generic $\mathbf{v} \times \mathbf{v}$ skew-symmetric matrix, that is, the zero set is $\sigma_r(G(2, V))$.

The next case is $k = 3$. The graded bundle of the bundle $\xi = (\underline{\Lambda^3 V}/\Lambda^3\mathcal{S})^*$ has three pieces:

$$\operatorname{gr}(\xi) = \Lambda^3\mathcal{Q}^* \oplus (\mathcal{S}^* \otimes \Lambda^2\mathcal{Q}^*) \oplus (\Lambda^2\mathcal{S}^* \otimes \mathcal{Q}^*).$$

If moreover $r = \mathbf{v} - 1$, then $\operatorname{rank}\mathcal{Q} = \mathbf{v} - r = 1$, the quotient bundle is a line bundle, and the bundle $\xi = \Lambda^2\mathcal{S}^* \otimes \mathcal{Q}^*$ is irreducible. In this special case O. Porras [**267**] found the whole minimal free resolution for the varieties $Sub_{\mathbf{v}-1}(\Lambda^3\mathbb{C}^{\mathbf{v}})$.

Note that one does not need $k = 3$ for the bundle ξ to be irreducible; this happens any time $r = \mathbf{v} - 1$ independently of k. The problem is the lack of decomposition formulas in general for $k \geq 3$, so to proceed further one would have to fix dimensions, at least partly.

The next case after the one treated by Porras is $r = \mathbf{v} - 2$. What follows is a set of equations for the variety $Sub_5(\Lambda^3\mathbb{C}^7)$. In this case the quotient bundle $\mathcal{Q}$ has rank 2, the subspace bundle $\mathcal{S}$ has rank 5, and the bundle ξ is an extension of two irreducible bundles:

$$\operatorname{gr}(\xi) = (\mathcal{S}^* \otimes \Lambda^2\mathcal{Q}^*) \oplus (\Lambda^2\mathcal{S}^* \otimes \mathcal{Q}^*).$$

Decompose $\Lambda^d(\operatorname{gr}(\xi))$ as a $(SL_{\mathbf{v}-2} \times SL_2)$-module:

$$\Lambda^d(\operatorname{gr}(\xi)) = \bigoplus_{k=0}^{d} \bigoplus_{p_1+p_2=k} \left(S_{2^{p_2},1^{p_1-p_2}}\left(\Lambda^2\mathcal{S}\right) \otimes \Lambda^{d-k}\mathcal{S}\right) \otimes \left(S_{d-k+p_1,d-k+p_2}\mathcal{Q}\right).$$

Using the Borel-Weil-Bott Theorem 16.3.2.3, one computes the cohomology of all irreducible summands. Next one tries to reconstruct the cohomology groups $H^{d-1}(G(5,7), \Lambda^d(\xi))$.

Since $\dim(G(5,7)) = 10$, $d \leq 11$. So fix d and consider all the irreducible summands of $\Lambda^d(\operatorname{gr}\xi)$ having cohomology in the desired degree $d-1$. These are candidates for appearing in $H^{d-1}(\Lambda^d\xi)$. Next check whether or not the modules appearing as cohomology of the irreducible summands also appear in the decomposition of $S^d(\Lambda^3\mathbb{C}^7)$, i.e., whether or not they are also candidates for being modules of polynomials. It remains to determine if the candidate modules that have survived the previous steps are indeed generators of the ideal in degree d of our variety.

Adopt the notation $[a_1, \ldots, a_6] = a_1\omega_1 + \cdots + a_6\omega_6$ in what follows. After the computations, the only nontrivial cases are degree 3 and degree 7. The case of degree 3 is easy: There are no possible cancellations for the two modules $[2,0,0,0,0,0]$ and $[0,0,1,0,0,1]$, so they must appear in $H^2(\Lambda^3\xi)$. In $H^2(\Lambda^3\xi)$ there is also a third module $[0,1,0,0,0,0]$, which does not appear in the decomposition of $S^3(\Lambda^3\mathbb{C}^7)$. Weyman's basic theorem implies that this module cannot appear in the cohomology of the bundle ξ. In fact, among the summands of $\mathrm{gr}(\Lambda^3\xi)$, there is one with nonzero H^1 belonging to the same isotypic component.

The case of degree 7 is a little more involved. In fact, there are no generators of the ideal in this degree because the module appearing in degree 7 is in the ideal generated by the generators in degree 3. In degree 3, the generators are:

$$[2,0,0,0,0,0] \leftrightarrow S_{(3,1^6)}V \subset S^3(\Lambda^3 V),$$
$$[0,0,1,0,0,1] \leftrightarrow S_{(2^3,1^3)}V \subset S^3(\Lambda^3 V).$$

And in degree 7 the potential new module is

$$[0,0,0,0,0,0] \leftrightarrow S_{(3^7)}V \subset S^7(\Lambda^3 V).$$

One calculates that the map

$$\big(S_{(3,1^6)}V \oplus S_{(2^3,1^3)}V\big) \otimes S^4(\Lambda^3 V) \to S^7(\Lambda^3 V)$$

surjects onto $S_{(3^7)}V$. In conclusion:

Proposition 17.4.2.1 (A. Boralevi, L. Oeding)**.** *The ideal of the subspace variety $Sub_5(\Lambda^3\mathbb{C}^7)$ is generated in degree 3 by the irreducible GL_7-modules $S_{(3,1^6)}\mathbb{C}^7$ and $S_{2^3,1^3}\mathbb{C}^7$.*

Hints and answers to selected exercises

Chapter 1

1.2.1.1 $e^1 \otimes (b_1 - b - 2 + 3b_3) + e^2 \otimes (2b_1 + b_3)$. More generally, for any s, t with $st \neq 1$ it equals

$$\frac{1}{1-st}[(e^1 + se^2)\otimes[(b_1 - b - 2 + 3b_3) - t(2b_1 + b_3)] \\ + (e^1 + te^2)\otimes[-t(b_1 - b - 2 + 3b_3) + (2b_1 + b_3)].$$

Chapter 2

2.1(3) Use the parametrization in exercise (2).

2.2.2.1 In the first case one can take the spaces

$$\left\{ \begin{pmatrix} s & 0 \\ t & 0 \end{pmatrix} \middle| \, s, t \in \mathbb{C} \right\}, \qquad \left\{ \begin{pmatrix} 0 & s \\ 0 & t \end{pmatrix} \middle| \, s, t \in \mathbb{C} \right\},$$

and in the second, for the two spaces, take the multiples of the identity matrix and the traceless matrices.

2.2.3(1) Consider the vector $v_1 + \cdots + v_n$.

2.2.3(4) Consider the vector ${}^t(1, 0, \ldots, 0)$.

2.6.6.2 Use Exercises 2.6.3.4 and 2.6.3.5.

2.6.7.1 Say $v_1 = \lambda_2 v_2 + \cdots + \lambda_k v_k$. Then

$$v_1 \wedge v_2 \wedge \cdots \wedge v_k = \lambda_2 v_2 \wedge v_2 \wedge \cdots \wedge v_k \\ + \lambda_3 v_3 \wedge v_2 \wedge \cdots \wedge v_k + \cdots + \lambda_k v_k \wedge v_2 \wedge \cdots \wedge v_k.$$

If the vectors are independent, they may be expanded into a basis, and the resulting vector is one of the basis vectors from the induced basis.

2.6.10(7) The map is $v_1\wedge\cdots\wedge v_k \mapsto \Omega(v_1,\dots,v_k,\cdot,\dots,\cdot)$.

2.6.10(8) Consider $\mathrm{Id}\in\Lambda^nV{\otimes}\Lambda^nV^*$ and the contraction map $\alpha\mapsto\alpha\lrcorner\mathrm{Id}$.

2.6.12(1) If f has rank $\mathbf{v}-1$, we can choose bases such that $f=\alpha^1{\otimes}w_1+\cdots+\alpha^{\mathbf{v}-1}{\otimes}w_{\mathbf{v}-1}$. Now compute $f^{\wedge(\mathbf{v}-1)}$.

2.6.12(6) Choose bases in V,W so that $f=\alpha^1{\otimes}w_1+\cdots+\alpha^n{\otimes}w_n$ and compute both sides explicitly.

2.8.1(2) Consider the composition $\rho_{\begin{array}{|c|c|}\hline 2&3\\\hline 1\\\cline{1-1}\end{array}}\rho_{\begin{array}{|c|c|}\hline 2&3\\\hline 1\\\cline{1-1}\end{array}}$.

2.8.1(8) To see the $GL(V)$ equivariance, note that, e.g., for the first map, we have the commutative diagram

$$
\begin{tikzcd}
v_1\cdots v_d \arrow[r, mapsto] \arrow[d] & \sum_i v_1\cdots\hat{v}_i\cdots v_d{\otimes}v_i \arrow[d] \\
g(v_1)\cdots g(v_d) \arrow[r, mapsto] & \sum_i g(v_1)\cdots\widehat{g(v_i)}\cdots g(v_d){\otimes}g(v_i)
\end{tikzcd}
\tag{17.4.2}
$$

where the vertical arrows are the action of $GL(V)$ and the horizontal, the operator d.

2.8.2.1 The isomorphisms come from linear isomorphisms $V{\otimes}V{\otimes}V\to V{\otimes}V{\otimes}V$.

2.8.2.8 They are different because the first is symmetric in $d-1$ indices and the last is skew-symmetric in $d-1$ indices, so, in particular, their dimensions are different.

2.11.3.2 You should be taking $\frac{1}{4}$ the sum of four numbers, two of which cancel.

Chapter 3

3.1.3.3 The assertion is clear for rank, but if $T(t):A\to B$ is a curve of linear maps that is not injective, then $T(0)$ is not injective either.

3.4.1.2 See §4.3.5.

3.8.2.2 A matrix with Jordan blocks is a limit of diagonalizable matrices.

3.9.1.2 Work by induction.

Chapter 4

4.2.4(3) $I(X)+I(Y)$.

4.3.3.1 Simply take the equations of W and add them to those of X with the variables $x^{k+1},\dots,x^{\mathbf{v}}$ set to zero in the expressions for the equations.

4.3.7.4 Write the coordinates in $S^d\mathbb{C}^2$ as $[z_0, \dots, z_d]$ and observe that $z_i z_j - z_k z_l$ is in the ideal if $i + j = k + l$. Note that this can be written as the set of two by two minors of $\left(\begin{smallmatrix} z_0 & z_1 & \cdots & z_{d-1} \\ z_1 & z_2 & \cdots & z_d \end{smallmatrix}\right)$.

4.3.7.5 $P_j(v) = \langle P_j, v^d \rangle$, so this gives the embedding in coordinates.

4.3.7.8 See [**157**, p. 6].

4.6.1.5 Let $P_1, \dots, P_r$ be generators of $I(X)$, so the tangent space to X at x is the intersection of the linear forms $\ker dP_1|_x, \dots, \ker dP_r|_x$. Now note that $x \in X_{sing}$ if the span of the $dP_j|_x$ drops dimension.

4.6.3(6) $\langle v_{d-1}(x) \rangle \circ \hat{T}_x X$.

4.7.2(1) Let $a^i_1, \dots, a^i_{\mathbf{a}_i}$ be a basis of A_i. Take the point $a^1_1 \otimes \cdots \otimes a^n_1 + \cdots + a^1_r \otimes \cdots \otimes a^n_r$.

4.8(2) The possible Jordan forms are

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}, \quad \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix},$$

which respectively correspond to: intersecting the Segre (which is a quadric surface in $\mathbb{P}^3$) in two distinct points, being a tangent line to the Segre, and being a ruling of the quadric surface, i.e., being a line on the Segre.

4.9.1.3 Say the image had two components. Consider their inverse images.

4.9.1.6 When X is a cone with vertex $\mathbb{P}W$.

4.9.1.7 First note that if $I \subset \mathbb{C}[X]$ is an ideal, then $f^{*-1}(I)$ is an ideal. Now given $x \in X$, let $I = I(x)$, the functions vanishing at x.

4.9.2.3 If $X \subset Y$, then $T_x X$ is a linear subspace of $T_x Y$. Now take $Y = \mathbb{P}V$.

Chapter 5

5.3.2(3) If the embedded tangent line to an algebraic curve is constant, then the curve is a line.

5.3.2(6) Compute the codimensions of $\sigma_r(Seg(\mathbb{P}A \times \mathbb{P}B))$, $\sigma_r(v_2(\mathbb{P}A))$, and $\sigma_r(G(2, A))$.

Chapter 6

6.1.2(2) Use Exercise 6.1.2(1).

6.1.2(8) Let $h(e_s) = f_i$ and all other basis vectors map to zero.

6.4.3(5)

$$\dim S_{a,b}\mathbb{C}^3 = \frac{1}{2}(a+2)(b+1)(a-b+1).$$

6.4.3(8) Recall that $\Lambda^{\mathbf{v}} V$ is a trivial $\mathfrak{sl}(V)$-module.

6.4.5.3 $\dim S_\pi V$.

6.5.4.1 Decompose $(E \otimes F)^{\otimes |\pi|}$ in two different ways.

6.6.2.4 First of all, you know the characters for the trivial [(4)] and alternating [(1111)] representations. Then note that the character of the permutation representation on $\mathbb{C}^4$ is easy to calculate, but the standard representation is just the trivial plus the permutation representation [(31)]. Now use that $[(211)] = [(31)]\otimes[(1111)]$, and finally the character of [(22)] can be obtained using the orthogonality of characters.

6.7.3(5) This is the same as asking if $S_{\pi^*}V \subset S_\mu V \otimes S_d V^*$.

6.8.4.2 Use exercise 6.8.2.3.

6.8.6.1 The vectors of weight (42) are spanned by $(e_1^2)^2(e_2^2)$ and $e_1^2(e_1e_2)^2$. The vectors of weight (222) are spanned by $(e_1)^2(e_2)^2(e_3)^2$, $(e_1e_2)(e_1e_3)(e_2e_3)$, $e_1^2(e_2e_3)^2$, $e_2^2(e_1e_3)^2$, and $e_3^2(e_1e_2)^2$.

6.8.6.2 Without loss of generality, assume that $\dim V = 2$. The weight space of (d,d) in $S^dV\otimes S^dV$ has a basis $(e_1)^d\otimes(e_2)^d$, $(e_1)^{d-1}e_2\otimes e_1(e_2)^{d-1}, \dots,$ $(e_2)^d\otimes(e_1)^d$. Take a linear combination of these vectors with unknown coefficients and determine which (unique up to scale) linear combination is annhilated by the matrix

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

6.10.2(1) $A_1\otimes\cdots\otimes A_n = A_1\otimes(A_2\otimes\cdots\otimes A_n)$ and $Seg(\mathbb{P}A_1\times\cdots\times\mathbb{P}A_n) = Seg(\mathbb{P}A_1\times\mathbb{P}A_{\hat{1}})\cap Seg(\mathbb{P}A_2\times\mathbb{P}A_{\hat{2}})\cap\cdots\cap Seg(\mathbb{P}A_n\times\mathbb{P}A_{\hat{n}})$ where $A_{\hat{j}} = A_1\otimes\cdots\otimes A_{j-1}\otimes A_{j+1}\otimes\cdots\otimes A_n$. Alternatively, use Schur's lemma as is done below for the Grassmannian.

6.10.6.2 Compute their Hilbert functions. For Segre varieties the degree is

$$\frac{(\mathbf{a}_1+\cdots+\mathbf{a}_n-n)!}{(\mathbf{a}_1-1)!\cdots(\mathbf{a}_n-1)!}.$$

6.10.6.6 Use the Pieri formula 6.7.2.1.

Chapter 7

7.1.4.3 Compare dimensions.

7.3.1.2 See Exercise 8.4.3.5.

7.5.1.4 Without loss of generality take $m_1 = 2$ and $m_2 = \cdots = m_r = 1$.

7.6.4.3 Use equation (3.8.5) if you cannot see this invariantly.

7.6.4.5 It remains to show that there is no equation in degree less than nine. Show that the trivial representation does not occur in degree five to conclude.

Chapter 10

10.1.2.3 Say not, let P have degree p and Q degree q with both invariant, and consider $\lambda P^q + \mu Q^p$ giving an infinite number of invariant hypersurfaces of degree pq.

Chapter 12

12.2.4.1 Suppressing the r index, write $s = Xp$, where X is the (unknown) inverse of h. One can then recover the entries of X from the three linear equations we obtain by exploiting the Toeplitz structure of S.

12.4.1.1 $(3v_1 + 2v_2 - v_3)^4 + (v_2 - 5v_3)^4$.

12.5.3.4 *A priori* $k_{\mathcal{S}_A} \leq r$.

12.5.4.1 There are essentially two cases to rule out depending on whether $k_{\mathcal{S}_A}, k_{\mathcal{S}_B} \geq S$, or $k_{\mathcal{S}_A} \geq S > k_{\mathcal{S}_B}$.

Chapter 13

13.5.1.2 Recall that $e_1 \cdots e_n = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} e_{\sigma(1)} \otimes \cdots \otimes e_{\sigma(n)}$ and that the map $E^{\otimes n} \otimes F^{\otimes n} \to (E \otimes F)^{\otimes n}$ sends $e_1 \otimes \cdots \otimes e_n \otimes f_1 \otimes \cdots \otimes f_n$ to $(e_1 \otimes f_1) \otimes \cdots \otimes (e_n \otimes f_n)$.

Chapter 16

16.1.2.1 Consider the eigenspaces of the A_j.

16.1.2.2 More generally, any irreducible representation of a solvable Lie algebra is one-dimensional. This follows from Lie's theorem asserting the existence of weight spaces combined with an induction argument.

16.1.3.2 If (v_i) is a weight basis for V and (w_s) a weight basis for W, then $v_i \otimes w_s$ is a weight basis for $V \otimes W$.

16.2.2.1 Let $P \in S^d V_\lambda^*$, $P(v_\lambda) = \overline{P}(v_\lambda^{\otimes d})$. Now use Schur's lemma.

16.3.2.2 At the identity, for $X \in \mathfrak{g}$, we have $X \mapsto X \bmod \mathfrak{p}$. Now obtain a vector field by left translation.

Chapter 17

17.2.3.1 Choose a basis of V^*, (x_i, e_s), such that the e_s span the kernel of the projection $V^* \to F^*$.

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